Note on radicals of filters in residuated lattices

M.Kondo
Department of Mathematics
Tokyo Denki University, Japan

Leonard Kwuida
Bern University of Applied Sciences
Bern, Switzerland

1 Introduction

One method to investigate properties of logical systems is to prove the completeness theorems using its Lindenbaum-Tarski algebras if they exist, and use algebraic methods through Lindenbaum-Tarski algebras to get various properties of the logical systems. This is also applied to the fuzzy logic proposed by Zadeh. Several interesting logics, such as Multiple-valued logic (MV), Basic logic (BL), Monoidal t-norm based logic (MTL) and so on, are proposed as specific fuzzy logics. As Lindenbaum-Tarski algebras of these logics, MV-algebras, BL-algebras and MTL-algebras and so on, are obtained, and algebraic studies of logics are actively performed. Those algebras are all axiomatic extensions of residuated lattices proposed by Ward and Dilworth [12] in 1939. Also, a logic determined by commutative residuated lattices is proposed in [6]. Hence, it is absolutely necessary to research residuated lattices in order to obtain general and essential properties of fuzzy logics.

In the study of ring theory, especially when considering the quotient rings by radicals (which are ideals), we can get beautiful and prospective results about properties of rings. Let R be a ring and $\mathcal{I}(R)$ be the set of all ideals of R. We define operations $\wedge, \vee, \odot, \rightarrow$ as follows for all $I, J \in \mathcal{I}(R)$:

$$
\begin{aligned}
I \wedge J &= I \cap J, \\
I \vee J &= \{x + y \mid x \in I, y \in J\}, \\
I \odot J &= \left\{\sum a_i b_i \text{ (finite sum)} \mid a_i \in I, b_i \in J\right\}, \\
I \rightarrow J &= \{a \in R \mid aI \subseteq J\}.
\end{aligned}
$$

The structure $(\mathcal{I}(R), \wedge, \vee, \odot, \rightarrow, R, \{0\})$ forms a residuated lattice. Therefore properties of rings are reflected in the residuated lattices through ideals. Thus, introducing the notion of radicals in ring theory to residuated lattices leads to new and interesting results. Since ideals in rings correspond to filters in residuated lattices, the concept of radicals in rings will be applied to filters of residuated lattices.

One of the research directions above has been addressed in [9]. Their authors defined the radicals of filters in BL-algebras and proved some basic results. Since then, properties regarding radicals of filters in BL-algebras are generalized to those of MTL-algebras [10] and of residuated lattices [8]. Unfortunately, there are serious errors in [8] that consequently make their main results not correct.

In this paper, we consider properties of the radicals of filters in residuated lattices in detail. We correctly modify the wrongly proved results of [8]. Moreover, we give a nice description of radicals of filters in residuated lattices in general. A corollary of this description is that for an MTL-algebra X and a filter F of X,

$$rad(F) = \{a \in X \mid a' \to a^n \in F \text{ for all } n \in \mathbb{N}\}.$$

This answers an open problem of [10], where one inclusion has been proved and the other left open.

2 Residuated lattice and filter

In this section we recall some definitions and basic properties on residuated lattices [1, 2, 3, 4, 5, 12].

An algebraic structure $\mathbb{X} := (X, \wedge, \vee, \odot, \to, \mathbf{0}, \mathbf{1})$ is called a bounded integral commutative residuated lattice [1] (simply called *residuated lattice* here) if

1. $(X, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice;
2. $(X, \odot, \mathbf{1})$ is a commutative monoid with unit element $\mathbf{1}$;
3. For all $x, y, z \in X$, $x \odot y \leq z$ if and only if $x \leq y \to z$.

If a residuated lattice satisfies the condition $x \wedge y = (x \to y) \odot x$ (called *divisibility*), then it is called an *$R\ell$-monoid*. An *MTL-algebra* is a residuated lattice satisfying the condition $(x \to y) \vee (y \to x) = \mathbf{1}$ (called *pre-linearity*). Moreover, by a *BL-algebra* we mean an MTL-algebra satisfying the divisibility condition.

Now we can state some properties of residuated lattices. These can be found in [2, 3, 12], except Proposition 1 (8).

Proposition 1. *Let $\mathbb{X}$ be a residuated lattice, $x, y, z \in X$ and $m, n \in \mathbb{N}$. The following properties hold:*

1. $x \odot y \leq x \wedge y$,
2. $x \leq y \iff x \to y = \mathbf{1}$,
3. $x \odot (x \to y) \leq y$,

4. $x \leq y \implies \begin{cases} x \odot z \leq y \odot z, \\ z \to x \leq z \to y, \\ y \to z \leq x \to z, \end{cases}$

5. $\mathbf{1} \to x = x$,

6. $x \vee y = 1 \implies x \odot y = x \wedge y$,

7. $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$,

8. $(x \vee y)^{m+n} \leq x^m \vee y^n$.

For $x \in X$ we set $x' = x \to \mathbf{0}$, which is a negation in a sense.

Proposition 2. *Let $\mathbb{X}$ be a residuated lattice, $x, y \in X$. The following properties hold:*

1. $\mathbf{0}' = \mathbf{1}$, $\mathbf{1}' = \mathbf{0}$ *and* $x \odot x' = \mathbf{0}$,

2. $x \leq y \implies x' \geq y'$,

3. $x \leq x''$ *and* $x''' = x'$,

4. $(x \vee y)' = x' \wedge y'$,

5. $x \to y \leq y' \to x'$ *and* $x' \to y' = y'' \to x''$,

6. $(x'')^m \leq (x^m)''$,

7. $(x \odot y)' = x \to y'$,

8. $x \vee x' = 1 \implies x'' = x$ *and* $x^2 = x$.

Following [3, 4, 5], we define filters of residuated lattices as follows: Let $\mathbb{X}$ be a residuated lattice. A non-empty subset $F \subseteq X$ is called a *filter* of $\mathbb{X}$ if for all $x, y \in X$

(F1) $x, y \in F$ implies $x \odot y \in F$;

(F2) $x \in F$ and $x \leq y$ imply $y \in F$.

A filter P of $\mathbb{X}$ is called *prime* if $x \vee y \in P$ implies $x \in P$ or $y \in P$ for all $x, y \in X$. A filter is called *maximal* if there is no proper filter containing it.

For a non-empty subset $S \subseteq X$, we denote by $[S)$ the filter generated by S. In particular, we write $[x)$ for the filter generated by a singleton $\{x\}$. There is a concrete description of such filters. See for example in [1, 3, 4] and [5].

Proposition 3. *Let $\mathbb{X}$ be a residuated lattice and $\emptyset \neq S \subseteq X$. Then*

$$[S) = \{x \mid \exists n \in \mathbb{N}, \exists s_i \in S, 1 \leq i \leq n, s_1 \odot \cdots \odot s_n \leq x\}.$$

Corollary 1. *If F is a filter of residuated lattice $\mathbb{X}$ and $a \in X$, then*

$$[F \cup \{a\}) = \{x \mid \exists u \in F,\ \exists n \in \mathbb{N}, u \odot a^n \leq x\}.$$

We denote by $\mathcal{F}(X)$ the class of all filters of $\mathbb{X}$.

3 Radicals of filters

We define a radical of a filter according to [8, 9] and [10]. Let $\mathbb{X}$ be a residuated lattice and F be a filter of $\mathbb{X}$. We denote by $Spec(\mathbb{X})$ (resp. $Max(\mathbb{X})$) the class of all prime (resp. maximal) filters of $\mathbb{X}$. It is easy to show that every maximal filter is a prime filter, i.e. $Max(\mathbb{X}) \subseteq Spec(\mathbb{X})$. By $Spec_F(\mathbb{X})$ (resp. $Max_F(\mathbb{X})$) we denote the set of prime (resp. maximal) filters containing F.

$$Spec_F(\mathbb{X}) = \{P \in Spec(\mathbb{X}) \mid F \subseteq P\};$$
$$Max_F(\mathbb{X}) = \{M \in Max(\mathbb{X}) \mid F \subseteq M\}.$$

Now we can define the *radical* of a filter F as follows:

$$rad(F) := \bigcap \{M \in Max(\mathbb{X}) \mid F \subseteq M\} = \bigcap Max_F(\mathbb{X}).$$

Following our notation, the usual radical $Rad(X)$ which is defined as the intersection of all maximal filters, is $rad(\{\mathbf{1}\})$, since

$$Rad(\mathbb{X}) := \bigcap \{M \mid M \in Max(\mathbb{X})\} = rad(\{\mathbf{1}\}).$$

Now we are ready to prove our first main result, which is a characterization theorem of radicals $rad(F)$ of filters.

Theorem 1. *Let $\mathbb{X}$ be a residuated lattice and $F \in \mathcal{F}(\mathbb{X})$. Then*

$$rad(F) = \{x \in X \mid \forall n \in \mathbb{N}\ \exists m \in \mathbb{N} \text{ s.t } (((x^n)')^m)' \in F\}.$$

Proof. We set $\Gamma = \{x \in X \mid \forall n \in \mathbb{N}\, \exists m \in \mathbb{N} \text{ s.t } (((x^n)')^m)' \in F\}$. First we show that $\Gamma \subseteq rad(F)$. Otherwise, there exists $x \in \Gamma$ such that $x \notin rad(F)$. There exists a maximal filter M such that $F \subseteq M$ and $x \notin M$. By Proposition **??** (3) we have $(x^n)' \in M$ for some $n \in \mathbb{N}$. Since $x \in \Gamma$ there exists $m \in \mathbb{N}$ such that $(((x^n)')^m)' \in F \subseteq M$. Since M is a filter and $(x^n)' \in M$, we also get $((x^n)')^m \in M$. Thus, $(((x^n)')^m)'$ and $((x^n)')^m$ are in the maximal filter M. But this is a contradiction. Hence, $\Gamma \subseteq rad(F)$.
Conversely, suppose that $x \notin \Gamma$. Then there is $n \in \mathbb{N}$ such that $(((x^n)')^m)' \notin F$ for all $m \in \mathbb{N}$. For this n, the filter generated by $F \cup \{(x^n)'\}$ is not equal to X. In fact, if $[F \cup \{(x^n)'\}) = X$ then by Corollary 1 there should exist $u \in F$ and $k \in \mathbb{N}$ such that $\mathbf{0} = u \odot ((x^n)')^k$. Thus $u \leq ((x^n)')^k \to \mathbf{0}$, and $(((x^n)')^k)' \in F$. This is a contradiction. Therefore, $[F \cup \{(x^n)'\}) \neq X$. Then, there exists a maximal filter M such that $[F \cup \{(x^n)'\}) \subseteq M$. If $x \in M$ then $x^n \in M$ and $(x^n)' \in M$. This is a contradiction. Thus, $x \notin M$. Hence, M is a maximal filter with $x \notin M$ and $F \subseteq M$. This means that $x \notin rad(F)$ and $rad(F) \subseteq \Gamma$. □

Proposition 4. *Let F be a filter of a residuated lattice $\mathbb{X}$ and $x \in X$. If $x' \to (x^n)'' \in F$ for all $n \in \mathbb{N}$, then $x \in rad(F)$.*

Corollary 2. *Let F be a filter of a residuated lattice $\mathbb{X}$ and $x \in X$. If $x' \to x^n \in F$ for all $n \in \mathbb{N}$, then $x \in rad(F)$.*

Conversely,

Proposition 5. *Let F be a filter of a residuated lattice $\mathbb{X}$ and $x \in X$. If $x \in rad(F)$ then for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $(x')^m \to x^n \in F$.*

Remark 1. *In [10] (Theorem 3.5), the authors proved that*

> *For F a filter F of an MTL-algebra $\mathbb{X}$ and $x \in X$, $x \in rad(F)$ implies that $x' \to x^n \in F$ for all $n \in \mathbb{N}$.*

After proving that they stated the following question as open problem:

> *Under what suitable conditions is the converse of the above theorem true?*

Our result in Corollary 2 answers this question, even in a more general sense. The converse holds without any additional condition, not only for MTL-algebras, but for residuated lattices in general. Therefore, $rad(F)$ in an MTL-algebras $\mathbb{X}$ can also be described by

$$rad(F) = \{x \in X \mid x' \to x^n \in F \text{ for all } n \in \mathbb{N}\}.$$

The next results aim to describe the radicals when the underlying set is linearly ordered. An element $x \in X$ is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $x^n = 0$. In this case we say that x has a finite order. It is obvious that if x is nilpotent then $x \notin rad(F)$ for all proper filter F by the definition of $rad(F)$. Conversely,

Proposition 6. *Let F be a filter of a residuated lattice $\mathbb{X}$ and $x \in X$. If X is linear, then $x \notin rad(F)$ implies that x is nilpotent.*

Proof. Let $\mathbb{X}$ be a linearly ordered residuated lattice. Since $x \notin rad(F)$, there exists $n \in \mathbb{N}$ such that $x' \to x^n \notin F$ and hence $x' \not\leq x^n$. Since $\mathbb{X}$ is linear, we have $x^n \leq x'$ and thus $x^{n+1} = \mathbf{0}$, that is, x is nilpotent. □

Corollary 3. *If $\mathbb{X}$ is a linearly ordered residuated lattice, then, for every filter F of $\mathbb{X}$, we have*

$$rad(F) = \{x \in X \mid \mathrm{ord}(x) = \infty\}.$$

After describing radicals of filters, we will now have a close look at rad as an operator on filters.

Proposition 7. *Let $\mathbb{X}$ be a residuated lattice and F and G be proper filters of $\mathbb{X}$. Then the following hold:*

1. *$F \subseteq rad(F)$.*
2. *$F \subseteq G$ implies $rad(F) \subseteq rad(G)$.*
3. *$rad(rad(F)) = rad(F)$.*
4. *$rad(F) \wedge rad(G) = rad(F \wedge G)$.*

5. $rad(\bigwedge_\lambda rad(F_\lambda)) = \bigwedge_\lambda rad(F_\lambda)$

6. $rad(F) \vee rad(G) \subseteq rad(F \vee G)$.

7. $rad(\bigvee_\lambda F_\lambda) = rad(\bigvee_\lambda rad(F_\lambda))$.

8. $rad(F \to G) \subseteq rad(F) \to rad(G)$

Proof. We only prove (3) and (5).
(3) It is enough to show that $rad(rad(F)) \subseteq rad(F)$. If $x \notin rad(F)$ then there exists a maximal filter M such that $F \subseteq M$ but $x \notin M$. Since $F \subseteq M$ and M is a maximal filter, we get $rad(F) \subseteq M$. Thus, there is a maximal filter M such that $rad(F) \subseteq M$ but $x \notin M$. This means that $x \notin rad(rad(F))$. Therefore, $rad(rad(F)) \subseteq rad(F)$.
(5) It is obvious that $rad(\bigwedge_\lambda rad(F_\lambda)) \subseteq \bigwedge_\lambda rad(F_\lambda)$. Conversely, if $x \in rad\,(\bigwedge_\lambda rad(F_\lambda))$, then for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $(((x^n)')^m)' \in \bigwedge_\lambda rad(F_\lambda)$ Since $\bigwedge_\lambda rad(F_\lambda) \subseteq rad(F_\mu)$ for all μ, we have $(((x^n)')^m)' \in rad(F_\mu)$ and thus $x \in rad(rad(F_\mu)) = rad(F_\mu)$ for all μ. This means that $x \in \bigwedge_\lambda rad(F_\lambda)$. It follows that $rad(\bigwedge_\lambda rad(F_\lambda)) \subseteq \bigwedge_\lambda rad(F_\lambda)$. Therefore, $rad(\bigwedge_\lambda rad(F_\lambda)) = \bigwedge_\lambda rad(F_\lambda)$.

□

Remark 2. *The properties (1), (2) and (3) in Proposition 7 mean that rad is a closure operator on $\mathcal{F}(\mathbb{X})$. Therefore (4), (5), (6) and (7) can be obtained from the general theory of closure operators [7]. Note that $X \in \mathcal{F}(\mathbb{X})$ and $rad(X) = X$.*

The following results, first proved for BL-algebras in [9], and then for MTL-algebras in [10], also hold for residuated lattices.

Proposition 8. *Let F be a proper filter of a residuated lattice $\mathbb{X}$. Then*

1. *If $x, y \in rad(F)$, then there exists $m \in \mathbb{N}$ such that $(x')^m \to y \in F$;*

2. *If $x, y \in rad(F)$, then there exists $m \in \mathbb{N}$ such that $((x')^m \odot y')' \in F$;*

3. *If $\mathbb{X}$ is linearly ordered and $x \in rad(F)$, then there exists $k \in \mathbb{N}$ such that $(x')^k \leq x$.*

The next results examine the intersection of radicals with Boolean elements. By $B(\mathbb{X})$, we mean the set of complemented elements of $\mathbb{X}$, that is,

$$B(X) = \{x \in X \mid \exists y \in X;\ x \wedge y = \mathbf{0}, x \vee y = \mathbf{1}\}.$$

It is easy to show that $B(\mathbb{X}) = \{x \in X \,|\, x \vee x' = \mathbf{1}\}$.

Proposition 9. *Let F be a filter of residuated lattice $\mathbb{X}$. Then $rad(F) \cap B(\mathbb{X}) = F \cap B(\mathbb{X})$.*

Corollary 4. $rad(\{\mathbf{1}\}) \cap B(\mathbb{X}) = \{\mathbf{1}\}$.

The last results examine the preservation of radicals under homomorphisms. Let $\mathbb{X}$ and $\mathbb{Y}$ be residuated lattices. A map $f : X \to Y$ is called a *homomorphism* if $f(\mathbf{0}) = \mathbf{0}$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in X$, where $* \in \{\wedge, \vee, \odot, \to\}$. We denote by $\ker f$ the kernel of f, defined by $\ker f = \{x \in X \mid f(x) = 1\}$.

Proposition 10. *Let $\mathbb{X}$ and $\mathbb{Y}$ be residuated lattices and $f : X \to Y$ be a homomorphism. Then $\ker f$ is a filter and $rad(\ker f) = f^{-1}(rad(\{\mathbf{1}\})$.*

We show a more general result about homomorphic images of radicals of filters. Note that $\ker f = f^{-1}(\{1\})$.

Theorem 2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be residuated lattices, $f : X \to Y$ be a homomorphism and $G \in \mathcal{F}(\mathbb{Y})$. Then we have*

$$f^{-1}(rad(G)) = rad(f^{-1}(G)).$$

Theorem 3. *Let $\mathbb{X}$ and $\mathbb{Y}$ be residuated lattices, $f : X \to Y$ be a surjective homomorphism and $F \in \mathcal{F}(\mathbb{X})$. If $\ker f \subseteq F$ then we have*

$$f(rad(F)) = rad(f(F)).$$

Proof. We first show that $f(F)$ is a filter of $\mathbb{Y}$.

It is obvious that $\mathbf{1} \in f(F)$. Suppose that $a, a \to b \in f(F)$. There exist $x \in F$ and $y \in F$ such that $f(x) = a$ and $f(y) = a \to b$. Since f is surjective, there exists $u \in X$ such that $f(u) = b$. Then we have $f(y) = a \to b = f(x) \to f(u) = f(x \to u)$ and $y \to (x \to u) \in \ker f \subseteq F$. It follows from $x, y \in F$ that $u \in F$ and thus $b = f(u) \in f(F)$. Therefore, $f(F)$ is a filter of $\mathbb{Y}$.

Now we can prove that $f(rad(F)) = rad(f(F))$. Let a be in $rad(f(F))$. Since f is surjective, there is $x \in X$ such that $a = f(x)$. By Theorem 1, for all $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with $(((f(x)^n)')^m)' \in f(F)$. But

$$f((((x^n)')^m)') = ((((f(x))^n)')^m)' \in f(F)$$

implies $((((x^n)')^m)') \in F$ because $\ker(f) \subseteq F$. Again by Theorem 1 we get $x \in rad(F)$, and $a = f(x) \in f(rad(F))$. Therefore, $rad(f(F)) \subseteq f(rad(F))$. The converse can also be proved similarly. □

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Department of Mathematics
Tokyo Denki University
Senju Asahi-cho 5, Adachi, Tokyo 120-8551
JAPAN
E-mail address: mkondo@mail.dendai.ac.jp

東京電機大学・数学系列　近藤 通朗

Bern University of Applied Sciences
Bern, Switzerland
E-mail address: leonard.kwuida@bfh.ch

Bern University of Applied Sciences　Leonard Kwuida