# TOPOLOGY OF THE SPACE ON WHICH CELLULAR AUTOMATA WORKS

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ABSTRACT. We investigate :

(A) Topology of the product of discrete topological spaces

and

(B) Homeomorphism of two topologies ($\mathrm{T_p}$) and ($\mathrm{T_m}$) of cell space $C = S^{\mathbb{Z}^n}$ of a cellular automaton $\mathcal{A} = (\mathbb{Z}, S, N, f)$ .

[1]

## 1. PRODUCT OF DISCRETE TOPOLPGICAL SPACES

**Definition.**

- $X_i$ : **discrete topological space for** $i$ **in** $I$

  $\mathcal{O}_i$ : **the system of open sets of** $X_i$ ,
- $X = \prod_{i \in I} X_i$ : **the product of** $X_i$ **of weak topology**

  $\mathcal{O}$ : **the system of open sets of** $X$,

  **and so**
- $\mathcal{O}_0 = \{\prod_{j \in J} O_j \times \prod_{i \in I \setminus J} X_i \mid O_j \in \mathcal{O}_j,\ J \subseteq I,\ |J| < \infty\}$

  : **an open basis of** $\mathcal{O}$.

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**Now we state our theorem.**

**Theorem A.** *For $X_i$'s descrete the following are equivalent:*

**(a)** $X = \prod_{i \in I} X_i$ *is discrete.*

**(b)** $| \{ i \in I \ | \ |X_i| \geq 2 \} | < \infty$,

*that is,* $|X_i| = 1$ *for almost all* $i$ *in* $I$.

**Corollary.** *The product of discrete topological spaces is discrete if and only if it is homeomorphic to a product of a finite number of discrete topological spaces. In particular, if $S$ is a discrete topological space containing at least two elements, then its infinite product is not discrete.*

**Proof.**

(a) $\Rightarrow$ $X \ni {}^\forall x$ **: open,** **(since** $X$ **is discrete)**

$\Rightarrow$ $X \ni {}^\forall x = \cup_{\lambda \in \Lambda} O_\lambda$ **for some** $O_\lambda$**'s** $\in \mathcal{O}_0$,

**(for** $\mathcal{O}_0$ **is an open basis of** $\mathcal{O}$**)**

$= O_\lambda$ **for any** $\lambda$ **in** $\Lambda$,

**(since** $|\{x\}| = 1$**)**

$\Rightarrow$ **setting**

$$O_\lambda = \prod_{j \in J} x_j \ \times \ \prod_{i \in I \setminus J} X_i \text{ for some } J, \ |J| < \infty$$

**and**

$$x = \prod_{j \in J} x_j \ \times \ \prod_{i \in I \setminus J} x_i,$$

**we have**

$\Rightarrow$ $X_i = x_i$ **for** $i$ **in** $I \setminus J$ **with** $|J| < \infty$,

$\Rightarrow$ (b).

(b) $\Rightarrow$ **For**

$$J = \{ i \in I \ | \ |X_i| \geq 2 \}$$

**we have**

$$|J| < \infty$$

$\Rightarrow$ **Choose**

$$X \ni {}^\forall x = \prod_{j \in J} x_j \ \times \prod_{i \in I \setminus J} x_i, \quad x_i \in X_i, \quad x_j \in X_j,$$

**Then by the definition of** $J$

$${}^\forall i \in I \setminus J, \quad x_i = X_i$$

**and so**

$$X \ni {}^{\forall}x = \prod_{j \in J} x_j \ \times \prod_{i \in I \setminus J} X_i \in \mathcal{O}_0, \quad x_j \in X_j$$

$\Rightarrow$ **(a)**

**Q.E.D.**

## 2. The topology of the configuration set $C$ of a cellular automaton $\mathcal{A}$

- $\mathcal{A} = (\mathbb{Z}, S, N, f)$ **: a cellular automaton**

  $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$ **: rational integers,**
- $\mathcal{Z}^n = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ **: cell space**
- $S = \{s_1, s_2, \cdots, s_m\}$ **: states**
- $f : S^l \to S$ **: local map**

**Further we define**

- $C = S^{\mathbb{Z}^n} = \mathrm{Map}(\mathbb{Z}^n, \mathrm{S})$ **: configurations**

**where we give**

- $S$ **: the discrete topology**

**Then** $C = S^{\mathbb{Z}^n}$ **has two topologies** $(\mathbf{T}_p)$ **and** $(\mathbf{T}_m)$ **following :**

$(\mathbf{T}_m)$

**For**

${}^{\forall}c \in C = S^{\mathbb{Z}^n}$,

${}^{\forall}\epsilon = 2^{-\lambda}, \quad \lambda \in \mathbb{N}$

**define**

$$U_\epsilon(c) = \{c' \in C \mid d(c, c') < \epsilon\ \}$$

**: the** $\epsilon$**-neighbourhood of** $c$

**where**

$d(c,c') = 2^{-min\{\delta(0,i)\ \mid\ c(i)\neq c'(i),\ i\in I\}}\quad for\quad c,c'\quad in\quad C,$

$\delta(0,i) = \sqrt{{i_1}^2 + {i_2}^2 + \cdots + {i_n}^2}$ **for** $i = (i_1, i_2, \cdots, i_n)$ **in** $\mathbb{Z}^n$. .

**($\mathbf{T}_p$)**

**For**

$^\forall c \in C = S^{\mathbb{Z}^n}$,

$^\forall J \subseteq \mathbb{Z}^n, \quad |J| < \infty$

**define**

$V_J(c) = (\prod_{j\in J} c(j)) \ \times S^{\mathbb{Z}^n \setminus J}$

**: the $J$-neighbourhood of** $c$

**Then, since $S$ is discrete, we have $c_j \in \mathcal{O}_j$ for any $j \in J$. This enable us to take**

$\{V_J(c) \mid c \in C, J \subseteq \mathbb{Z}^n, \quad |J| < \infty\}$

**as $O_0$ an open basis.**

**Then, we have :**

**Theorem B.**

**($\mathbf{T}_p$) $\simeq$ ($\mathbf{T}_m$) : homeomorphic**

**Proof. (a) We show ($\mathbf{T}_p$) $\leq$ ($\mathbf{T}_m$), i.e.,** $^\exists U_\epsilon(c) \subseteq {}^\forall V_J(c)$

**For**

$^\forall V_J(c) = \prod_{j\in J} c(j) \ \times S^{\mathbb{Z}^n \setminus J}$

**with**

$^\forall c \in C = S^{\mathbb{Z}^n}$,

$^\forall J \subseteq S^{\mathbb{Z}^n}$ and $|\mathrm{J}| < \infty$

**choose** $\lambda \in \mathbb{N}$

**such that (A)** $^\forall j \in J$, $\quad \delta(0,j) < \lambda$

**and set**

$^\forall \epsilon = 2^{-\lambda}$

**Then,**

$$c' \in U_\epsilon(c) \;\Rightarrow\; d(c,c') < \epsilon = 2^{-\lambda}, \;\textbf{ where}$$

$$d(c,c') = 2^{-min\{\delta(0,i)\ \mid\ c(i)\neq c'(i)\}} \quad \text{for} \quad \text{c, c}' \quad \text{in} \quad \text{C},$$

$$\Rightarrow\; \lambda < \min\{\delta(0,\mathrm{i}) \mid \mathrm{c(i)} \neq \mathrm{c'(i)}\}$$

$\Rightarrow$ **By (A)**

**if** $j \in J$**, we have** $\delta(0,j) < \lambda$

**and so** $c(j) = c'(j)$

$$\Rightarrow\; c' \in V_J(c) = \prod_{j\in J} c(j) \;\times S^{\mathbb{Z}^n \setminus J}$$

**Thus,**

$U_\epsilon(c) \subseteq V_J(c)$ **and so** $(\mathbf{T}_p) \le (\mathbf{T}_m)$**.**

**(b) Next we show** $(\mathbf{T}_m) \le (\mathbf{T}_p)$**, i.e. ,** $^{\exists}V_J(c) \subseteq {}^{\forall}U_\epsilon(c)$**.**

**For**

$^{\forall}U_\epsilon(c)$

**with**

$c \in C = S^{\mathbb{Z}^n}$**,**

$\epsilon = 2^{-\lambda}, \quad \lambda \in \mathbb{N}$

**let**

**(B)** $J = \{j \in I \mid \delta(0,j) \le \lambda\}$

**Then, since**

$V_J(c) = \prod_{j\in J} c(j) \;\times S^{\mathbb{Z}^n \setminus J}$**,**

**we have**

$$c' \in V_J(c) \;\Rightarrow\; {}^{\forall}j \in J, \quad c(j) = c'(j)$$

$\Rightarrow$ **by (B) if** $\delta(0,j) \le \lambda$ **, we have** $j \in J$

**and so** $c(j) = c'(j)$

$$\Rightarrow\; \lambda < \min\{\delta(0,\mathrm{j}) \mid \mathrm{c(j)} \neq \mathrm{c'(j)}\}$$

$$\Rightarrow\; 2^{-min\{\delta(0,j)\ \mid\ c(j)\neq c'(j)\}} < 2^{-\lambda} = \epsilon$$

$$\Rightarrow\; d(c,c') < \epsilon$$

$$\Rightarrow\; c' \in U_\epsilon(c)$$

**Thus**

$V_J(c) \subseteq U_\epsilon(c)$ **and so** $(\mathbf{T}_m) \le (\mathbf{T}_p)$**.**

**Q.E.D.**

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