Some Examples of Minimal Groupoids on a Finite Set

Mike Behrisch*
Vienna, Austria

Hajime Machida†
Tokyo, Japan

Abstract

A minimal clone is an atom in the lattice of clones. The classification of minimal clones on a finite set still remains unsolved. A minimal groupoid is a minimal clone generated by a binary idempotent function. In this paper we report some examples of minimal groupoids generated by binary functions which resemble projections.

Keywords: minimal clone, minimal groupoid, binary idempotent function

1 Introduction

In the lattice of clones, an atom is called a minimal clone and a coatom is called a maximal clone. One of the fundamental problems in clone theory is to classify maximal clones as well as minimal clones. In 1970 I. G. Rosenberg [Ro70] published the complete classification of maximal clones on a k-element set for any finite $k > 2$, which solves the case for maximal clones. For minimal clones, however, the problem has not yet been solved. The complete classification for any finite set seems quite a hard task. The minimal clones on a 2-element set have been known since E. Post (1941) and those on a 3-element set were classified by B. Csákány (1983) in [Cs83]. In addition, some partial results were obtained for minimal clones on a 4-element set ([Sz95, Wa00]).

A minimal groupoid is a minimal clone generated by a binary idempotent function. Main purpose of this article is to report our work on minimal groupoids developed in [BM19]. More specifically, after defining the notion of pr-distance for a binary function, we present examples of minimal groupoids generated by binary functions with pr-distance 1 or 2. Furthermore, some examples of minimal groupoids are given whose generators have larger pr-distance.

2 Prerequisites

Let $k > 1$ be an integer and $E_k = \{0, 1, \ldots, k-1\}$. Denote by $\mathcal{O}_k^{(n)}$, $n > 0$, the set of n-variable functions on E_k, i.e., $\mathcal{O}_k^{(n)} = E_k^{E_k^n}$, and by $\mathcal{O}_k$ the set of all functions on E_k, i.e., $\mathcal{O}_k = \bigcup_{n>0} \mathcal{O}_k^{(n)}$. A function e_i^n in $\mathcal{O}_k^{(n)}$, $1 \le i \le n$, is the n-variable i-th *projection* if $e_i^n(x_1, \ldots, x_n) = x_i$ holds for all $x_1, \ldots, x_n \in E_k$. Let $\mathcal{J}_k$ be the set of projections on E_k.

*behrisch@logic.at
†machida.zauber@gmail.com

A subset C of $\mathcal{O}_k$ is a *clone* on E_k if C contains all the projections, i.e., $\mathcal{J}_k \subseteq C$, and is closed under (functional) composition. The set of clones on E_k forms a lattice with respect to set inclusion, which is called the *lattice of clones* and denoted by $\mathcal{L}_k$.

For a clone C on E_k and a subset F of C, F *generates* C if C is the smallest clone containing F. In other words, F generates C if C is the intersection of all clones containing F. When F generates C, we write $C = \langle F \rangle$. If F is a singleton, i.e., $F = \{f\}$ for some $f \in \mathcal{O}_k$, we simply write $\langle f \rangle$ in place of $\langle F \rangle$.

A clone C in $\mathcal{L}_k$ is a *minimal clone* if it is an atom of the lattice $\mathcal{L}_k$. Equivalently, C is a minimal clone if (1) $C \neq \mathcal{J}_k$ and (2) $\mathcal{J}_k \subset C' \subseteq C$ implies $C' = C$ for any C' in $\mathcal{L}_k$.

A minimal clone is generated by a single function, which is not a projection, i.e., $C = \langle f \rangle$ for some $f \in \mathcal{O}_k \setminus \mathcal{J}_k$. A function $f \in \mathcal{O}_k$ is a *minimal function* if f generates a minimal clone and f has the minimum arity among functions generating $\langle f \rangle$.

In [Ro86], I. G. Rosenberg presented what is now called the "Type Theorem" for minimal functions.

Theorem 2.1 *Let $k \geq 2$. Any minimal function on E_k is of one of the following five types:*

(1) unary function
(2) binary idempotent function
(3) ternary majority function
(4) ternary minority function
(5) semiprojection

We review the definitions of the terms used above: For $f \in \mathcal{O}_k^{(2)}$, f is *idempotent* if $f(x,x) \approx x$. For $g \in \mathcal{O}_k^{(3)}$, g is a *majority* (resp., *minority*) function if $g(x,x,y) \approx g(x,y,x) \approx g(y,x,x) \approx x$ (resp., y). Also, $h \in \mathcal{O}_k^{(n)}$ is a *semiprojection* if there is i ($1 \leq i \leq n$) such that $h(x_1, \ldots, x_n) \approx x_i$ whenever $|\{x_1, \ldots, x_n\}| < n$. (Here, $g(x,x,y) \approx x$, for example, means $g(x,x,y) = x$ for all $x, y \in E_k$.)

It is a well-known fact, and plays an important rôle in this article, that $\mathcal{L}_k$ is atomic, i.e., every clone in $\mathcal{L}_k \setminus \{\mathcal{J}_k\}$ contains a minimal clone.

For f and g in $\mathcal{O}_k^{(n)}$, f is *conjugate* to g if f is obtained from g by renaming the elements in E_k. When f and g are *conjugate* to each other, $\langle f \rangle$ is a minimal clone if and only if $\langle g \rangle$ is a minimal clone. Furthermore, f is the *dual* of g if $f(x,y) = g(y,x)$ for all $x, y \in E_k$. If f is the dual of g then, clearly, $\langle f \rangle = \langle g \rangle$.

We shall call a clone C a *groupoid* if C is generated by a binary function, that is, if $C = \langle f \rangle$ for some $f \in \mathcal{O}_k^{(2)}$. A *minimal groupoid* is a minimal clone generated by a binary minimal function.

3 Pr-Distance

Intending to measure the "distance" of a binary function from the projections, we introduced a mapping $\delta : \mathcal{O}_k^{(2)} \longrightarrow \{0, \ldots, k^2\}$ for $k > 1$ as follows ([BM19]). For a binary function f in $\mathcal{O}_k^{(2)}$, let

$$\begin{aligned} \delta_1(f) &= k^2 - \#\{\, (i,j) \in E_k^2 \mid f(i,j) = i \,\}, \\ \delta_2(f) &= k^2 - \#\{\, (i,j) \in E_k^2 \mid f(i,j) = j \,\} \end{aligned}$$

and

$$\delta(f) \;=\; \min\{\,\delta_1(f),\ \delta_2(f)\,\}.$$

Thus, $\delta_1(f)$ (resp., $\delta_2(f)$) is the Hamming distance of f from the projection e_1^2 (resp., e_2^2). We shall call $\delta(f)$ the *pr-distance* of f. Evidently, $\delta(f) = 0$ if and only if f is a projection.

4 Minimal Groupoids on E_3

In 1983, B. Csákány ([Cs83]) determined all minimal clones on the 3-element set E_3. The total number of minimal clones on E_3 is 84. Among them, the number of minimal clones generated by binary idempotent functions, i.e., minimal groupoids, is 48.

It turns out that binary minimal functions on E_3 can be classified into three classes: Commutative functions and two types of non-commutative functions, those with $\delta(f) = 1$ and those with $\delta(f) = 2$. (Here, $f \in \mathcal{O}_k^{(2)}$ is *commutative* if $f(x, y) \approx f(y, x)$ holds.) The number of minimal groupoids generated by commutative binary functions is 12 and the number of minimal groupoids generated by non-commutative binary functions f with $\delta(f) = 1$ (resp., $\delta(f) = 2$) is 12 (resp., 24).

5 Examples of Minimal Groupoids on E_k

We shall consider minimal groupoids on E_k, $k > 2$, generated by non-commutative binary functions f with $\delta(f) = 1$ in Subsection 5.1 and $\delta(f) = 2$ in Subsection 5.2.

Before going further, we shall give a simple, but useful, sufficient condition for $f \in \mathcal{O}_k^{(2)}$ to be a minimal function. In the sequel, $f(x, y)$ will be denoted by xy when f is understood.

Lemma 5.1 ([BM19]) *Let $f \in \mathcal{O}_k^{(2)} \setminus \mathcal{J}_k$ be an idempotent function. If all the terms*

$$x(xy),\ x(yx),\ (xy)x,\ (xy)y,\ (xy)(yx)$$

are equivalent to xy or yx (as a function) then $\langle\, f\,\rangle$ is a minimal clone.

The proof is by induction on the depth of a term over f. Note that $y(yx)$, $y(xy)$, $(yx)y$, $(yx)x$, $(yx)(xy)$ are the duals of $x(xy)$, $x(yx)$, $(xy)x$, $(xy)y$, $(xy)(yx)$, respectively.

5.1 The Case: $\delta(f) = 1$

In the last paragraph of Section 4, it is stated that the number of minimal groupoids on E_3 generated by binary idempotent functions f with $\delta(f) = 1$ is 12. Since there are 6 pairs $(x, y) \in E_3^2$ with $x \neq y$, this fact implies that every binary idempotent function f on E_3 having $\delta(f) = 1$ is a generator of a minimal clone. One can ask, then, whether this property generalizes to arbitrary $k > 1$? The answer turns out to be 'yes', as shown below. Note that the contents of this subsection have already appeared in [BM19].

We start with two binary functions p_1 and p_2 in $\mathcal{O}_k^{(2)}$, $k > 2$, defined by

$$p_a(x, y) \;=\; \begin{cases} a & \text{if } (x, y) = (0, 1) \\ x & \text{otherwise} \end{cases}$$

for $a = 1, 2$. Obviously, p_a is non-commutative and $\delta(p_a) = 1$ for each $a = 1, 2$. The top part of the Cayley table of p_a is shown below.

p_1 :

$x\backslash y$	0	1	2	3	$\cdots$	$k-1$
0	0	1	0	0	$\cdots$	0
1	1	1	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

p_2 :

$x\backslash y$	0	1	2	3	$\cdots$	$k-1$
0	0	2	0	0	$\cdots$	0
1	1	1	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

Lemma 5.2 *For each $a = 1, 2$, p_a is a minimal function.*

Sketch of the proof First, consider p_1. (Here, xy denotes $p_1(x, y)$.) It is easy to see that all terms

$$x(xy),\ x(yx),\ (xy)x,\ (xy)y,\ (xy)(yx)$$

are equivalent to xy. Hence, p_1 is a minimal function by Lemma 5.1.

Next, take p_2 and denote $p_2(x, y)$ by xy. In this case, Lemma 5.1 is not applicable as $x(xy)$ is equivalent to x, but not to xy nor yx.

Suppose that p_2 is not a minimal function. Then, since the lattice $\mathcal{L}_k$ of clones is atomic, there must exist a minimal function $g \in \mathcal{O}_k$ satisfying $\langle g \rangle \subset \langle p_2 \rangle$ where the inclusion is strict. Rosenberg's type theorem (Theorem 2.1) asserts that g must be one of the following: (1) unary function, (2) binary idempotent function, (3) ternary majority function, (4) ternary minority function and (5) semiprojection.

We can verify, however, that none of these five cases are possible. Cases (1) to (4) are easy while Case (5) requires more careful inspection. (Refer to [BM19] for the detailed discussion.) This proves that p_2 is, in fact, a minimal function. □

The following lemma says that the functions p_1 and p_2 represent, in a sense, all binary idempotent functions f with $\delta(f) = 1$.

Lemma 5.3 *Let $f \in \mathcal{O}_k^{(2)}$ be idempotent with $\delta(f) = 1$. Then f, or its dual, is conjugate to p_1 or p_2.*

Combining Lemmata 5.2 and 5.3 we obtain:

Proposition 5.4 *Let $f \in \mathcal{O}_k^{(2)}$ be idempotent. If $\delta(f) = 1$ then f is a minimal function.*

Thus the case of f with pr-distance 1 is completely settled.

5.2 The Case: $\delta(f) = 2$

Let us define a binary function $q_{ab} \in \mathcal{O}_k^{(2)}$ for a, $b \in E_k \setminus \{0\}$ in the following way.

$$q_{ab}(x, y) \ = \ \begin{cases} a & \text{if } \ (x, y) = (0, 1) \\ b & \text{if } \ (x, y) = (0, 2) \\ 0 & \text{if } \ x = 0 \ \text{ and } \ y \in E_k \setminus \{1, 2\} \\ x & \text{otherwise} \end{cases}$$

Clearly, q_{ab} is idempotent and $\delta(q_{ab}) = 2$.

In particular, we shall focus on four functions q_{11}, q_{12}, q_{33} and q_{34}. The top two rows of the Cayley table of each of the four functions is shown below.

q_{11} :

$x\backslash y$	0	1	2	3	$\cdots$	$k-1$
0	0	1	1	0	$\cdots$	0
1	1	1	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

q_{12} :

$x\backslash y$	0	1	2	3	$\cdots$	$k-1$
0	0	1	2	0	$\cdots$	0
1	1	1	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

q_{33} :

$x\backslash y$	0	1	2	3	$\cdots$	$k-1$
0	0	3	3	0	$\cdots$	0
1	1	1	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

q_{34} :

$x\backslash y$	0	1	2	3	$\cdots$	$k-1$
0	0	3	4	0	$\cdots$	0
1	1	1	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

Lemma 5.5 *All of q_{11}, q_{12}, q_{33} and q_{34} are minimal functions.*

Sketch of the proof For q_{11} and q_{12}, it is easy to see that

$$x(xy) = x(yx) = (xy)x = (xy)y = (xy)(yx) = xy$$

holds, from which the results follow from Lemma 5.1. On the other hand, for q_{33} and q_{34}, we have

$$x(xy) = x, \quad x(yx) = (xy)x = (xy)y = (xy)(yx) = xy$$

and Lemma 5.1 is not applicable. However, for these functions the similar line of argument used for p_2 in the proof of Lemma 5.2 can be applied to verify that there does not exist a minimal function $g \in \mathcal{O}_k$ satisfying $\langle g \rangle \subset \langle q_{33} \rangle$ (or, $\langle g \rangle \subset \langle q_{34} \rangle$). Thus, q_{33} and q_{34} are proved to be minimal. □

Next, we consider functions q_{ab} for $(a, b) = (1, 3)$, $(2, 1)$ and $(2, 3)$.

Lemma 5.6 *Each of q_{13}, q_{21} and q_{23} is not a minimal function.*

Proof For q_{13} and q_{23}, $x(xy)$ has pr-distance 1 and $\langle x(xy) \rangle \subset \langle xy \rangle$. For q_{21}, $x(xy) = q_{12}$ and, again, $\langle x(xy) \rangle \subset \langle xy \rangle$. □

Let W be the set of binary idempotent functions $f \in \mathcal{O}_k^{(2)}$ satisfying

(1) $\delta(f) = 2$ and (2) $(\exists u \in E_k)\ (\forall x \in E_k \setminus \{u\})\ (\forall\, y \in E_k)\ [\ f(x, y) = x\ ]$.

Thus $f \in W$ has two 'singular values' and they sit only on the u-th row in its Cayley table. Furthermore, let V be the subset of W defined by

$$V = \{\, q_{ab} \,|\, (a, b) = (1, 1),\ (1, 2),\ (1, 3),\ (2, 1),\ (2, 3),\ (3, 3),\ (3, 4)\}.$$

Lemma 5.7 *(1) Any two functions in V are not mutually conjugate to each other.*
(2) For any $f \in \mathcal{O}_k^{(2)}$, if $f \in W$ then f is conjugate to some function in V.

In other words, functions in V represent all functions in W. Combining Lemma 5.9 to Lemma 5.7 we obtain the following, which appeared in [BM19] in a slightly different form.

Proposition 5.8 *For any $f \in W$, f is a minimal function if and only if it is conjugate to one of q_{11}, q_{12}, q_{33} or q_{34}.*

5.3 The Case: $\delta(f) \geq 3$

In the case of three-element set E_3, it was shown that all non-commutative minimal functions have pr-distance 1 or 2. Does this property generalize to any k-element set E_k for $k > 3$? The answer is 'no'. In fact, there are many non-commutative minimal functions $f \in \mathcal{O}_k^{(2)}$, $k > 3$, whose pr-distance exceeds 2, i.e., $\delta(f) \geq 3$. Here we present two such examples.

For $k > 1$, let $r_a \in \mathcal{O}_k^{(2)}$, $a = 0, 1$, be defined by

$$r_a(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ a & \text{if } (x,y) = (0,1) \\ 1 & \text{if } x = 0 \text{ and } y \in E_k \setminus \{0,1\} \\ x & \text{otherwise.} \end{cases}$$

The top part of the Cayley table of r_a, $a = 0, 1$, is shown below.

r_0 :

$x\backslash y$	0	1	2	3	$\cdots$	$k-1$
0	0	0	1	1	$\cdots$	1
1	1	1	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

r_1 :

$x\backslash y$	0	1	2	3	$\cdots$	$k-1$
0	0	1	1	1	$\cdots$	1
1	1	1	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

Evidently, $\delta(r_0) = k - 2$ and $\delta(r_1) = k - 1$.

Lemma 5.9 *Both of r_0 and r_1 are minimal functions.*

Sketch of the proof For r_1, we have

$$x(xy) = x(yx) = (xy)x = (xy)y = (xy)(yx) = xy,$$

and the result follows from Lemma 5.1. For r_0, we have

$$x(xy) = x, \quad x(yx) = (xy)x = (xy)y = (xy)(yx) = xy.$$

Then the similar argument used for p_2 in the proof of Lemma 5.2 can be applied again to prove that q_0 is a minimal function. □

References

[BM19] Behrisch, M. and Machida, H., An approach toward classification of minimal groupoids on a finite set, *Proc. 49th ISMVL*, IEEE, 2019, 164–169.

[Cs83] Csákány, B., All minimal clones on the three element set, *Acta Cybernet.*, **6**, 1983, 227–238.

[Cs83a] Csákány, B., Three-element groupoids with minimal clones, *Acta Sci. Math.*, (Szeged) **45**, 1983, 111–117.

[LP96] Lévai, L. and Pálfy, P. P., On binary minimal clones, *Acta Cybernet.*, **12**, 1996, 279–294.

[MR93] Machida, H. and Rosenberg, I. G., Essentially minimal groupoids, in *"Algebras and Orders"*, Kluwer Academic Publishers, 1993, 287–316.

[Ro70] Rosenberg, I. G., Über die funktionale Vollständigkeit in den mehrwertigen Logiken, *Rozpravy Československé Akad. Věd. Řada Mat. Přírod. Věd*, **80**, 1970, 3–93.

[Ro86] Rosenberg, I. G., Minimal clones I: The five types, *Colloq. Math. Soc. J. Bolyai*, **43**, North Holland, 1986, 405–427.

[Sz95] Szczepara, B., Minimal clones generated by groupoids, Ph.D. Thesis, Université de Montréal, 1995.

[Wa00] Waldhauser, T., Minimal clones generated by majority operations, *Algebra Universalis*, **44**, 2000, 15–26.