On weakly separable polynomials and weakly quasi-separable polynomials in q-skew polynomial rings

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Abstract

The notion of weakly separable extensions and weakly quasi-separable extensions were were introduced by N.Hamaguchi and A.Nakajima as generalizations of separable extensions. and quasi-separable extensions, respectively. In this paper, we shall study weakly separable polynomials and weakly quasi-separable polynomials in q-skew polynomial ring.

1 Introduction

My talk at the conference was based on the paper [12]. The contents of this paper therefore overlaps with the publication.

Let A/B be a ring extension with common identity 1, M an A-A-bimodule, $M^A = \{m \in M \mid zm = mz\ (\forall z \in A)\}$, and z, w arbitrary elements in A. An additive map $\delta : A \to M$ is called a *B-derivation of A to M* if $\delta(zw) = \delta(z)w + z\delta(w)$ and $\delta(B) = \{0\}$. A B-derivation δ of A to M is called *central* if $\delta(z)w = w\delta(z)$, and δ is called *inner* if $\delta(z) = mz - zm$ for some fixed element $m \in M$. We say that A/B is *separable* if the A-A-homomorphism of $A \otimes_B A$ onto A defined by $a \otimes b \mapsto ab$ splits (cf. [3, Definition 2]), or equivalently, every B-derivation of A to N is inner for any A-A-bimodule N (cf. [1, Satz 4.2]). A/B is said to be *quasi-separable* if every central B-derivation of A to N is zero for any A-A-bimodule N. The notion of a quasi-separable extension of commutative rings was introduced by Y.Nakai (cf. [8]), and in the noncommutative case, it was characterized by H.Komatsu (cf. [6]). A/B is called *weakly separable* if every B-derivation of A to A is inner, and A/B is called *weakly quasi-separable* if every central B-derivation of A to A is zero. The notion of weakly separable extension introduced by N.Hamaguchi and A.Nakajim as generalizations of separable extensions and quasi-separable extensions (cf. [2]). Obviously, a separable extension is weakly separable and a quasi-separable extension is weakly quasi-separable. Moreover, K.Komatsu showed that a separable extension is quasi-separable (cf. [6, Theorem 2.4]).

Throughout this article, $C(\Lambda)$ will mean the center of the ring Λ. From now on, let B be a ring, ρ an automorphism of B, D a ρ-derivation (that is, D is an additive endomorphism of B such that $D(\alpha\beta) = D(\alpha)\rho(\beta) + \alpha D(\beta)$ for any α, $\beta \in B$). We set $B^\rho = \{\alpha \in B \mid \rho(\alpha) = \alpha\}$, $B^D = \{\alpha \in B \mid D(\alpha) = 0\}$, and $B^{\rho,D} = B^\rho \cap B^D$. By $B[X;\rho,D]$ we denote the skew polynomial ring in which the multiplication is given by $\alpha X = X\rho(\alpha) + D(\alpha)$ for any $\alpha \in B$. We call $B[X;\rho,D]$ a *q-skew polynomial ring* if there exists a $q \in B^{\rho,D} \cap C(B)$ such that $D\rho(\alpha) = q\rho D(\alpha)$ for any $\alpha \in B$, and we denote it by $B[X;\rho,D]^q$. We write $B[X;\rho] = B[X;\rho,0]$ and $B[X;D] = B[X;1,D]$. Moreover, by $B[X;\rho,D]_{(0)}$ we denote the set of all monic polynomials f in $B[X;\rho,D]$ such that $fB[X;\rho,D] = B[X;\rho,D]f$. For any polynomial $f \in B[X;\rho,D]_{(0)}$, the quotient ring $B[X;\rho,D]/fB[X;\rho,D]$ is a free ring extension of B. A polynomial f in $B[X;\rho,D]_{(0)}$ is called *separable* (resp. *weakly separable* (resp. *weakly quasi-separable*)) in $B[X;\rho,D]$ if $B[X;\rho,D]/fB[X;\rho,D]$ is separable (resp. weakly separable (resp. weakly quasi-separable)) over B.

In the previous paper [11], the author studied weakly separable polynomials in $B[X;\rho]$ and $B[X;D]$. In particular, we showed necessary and sufficient conditions concerning weakly separable polynomials (cf. [11, Theorem 3.2 and Theorem 3.8]). In this paper, we shall give some improvements and generalizations of our results for the q-skew polynomial ring $B[X;\rho,D]^q$ with $q \in B^{\rho,D} \cap C(B)$.

2 Main results

In this section, let $R = B[X;\rho,D]^q$ with $q \in B^{\rho,D} \cap C(B)$, $R_{(0)} = B[X;\rho,D]^q_{(0)}$, and $f = \sum_{i=0}^m X^i a_i \in R_{(0)} \cap B^\rho[X]$ $(a_m = 1, m \geq 1)$. Note that f is in $C(B^{\rho,D})[X]$. We shall use the following conventions:

- $A = R/fR$ (the quotient ring)
- $x = X + fR \in A$ (that is, $\{1, x, x^2, \cdots, x^{m-1}\}$ is a free B-basis of A)
- $I_x =$ an inner derivation of A by x (that is, $I_x(z) = zx - xz$ $(\forall z \in A)$)
- $A_k = \{z \in A \mid \alpha z = z\rho^k(\alpha)\ (\forall \alpha \in B)\}$ $(k \in \mathbb{Z})$

Note that A_0 is the centralizer of B in A. Moreover, we define polynomials $Y_j \in R$ $(0 \leq j \leq m-1)$ as follows:

$$
\begin{aligned}
Y_0 &= X^{m-1} + X^{m-2}a_{m-1} + \cdots + Xa_2 + a_1, \\
Y_1 &= X^{m-2} + X^{m-3}a_{m-1} + \cdots + Xa_3 + a_2, \\
&\cdots\cdots \\
Y_j &= X^{m-j-1} + X^{m-j-2}a_{m-1} + \cdots + Xa_{j+2} + a_{j+1}, \\
&\cdots\cdots \\
Y_{m-2} &= X + a_{m-1}, \\
Y_{m-1} &= 1.
\end{aligned}
$$

Note that $XY_j = Y_{j-1} - a_j$ $(1 \leq j \leq m-1)$ and $XY_0 = f - a_0$. These polynomials Y_j $(0 \leq j \leq m-1)$ were introduced by Y.Miyashita to characterize separable polynomials in $B[X;\rho,D]$ (cf. [7]). Now let $y_j = Y_j + fR \in A$ $(0 \le j \le m-1)$ and we define a map $\tau : A \to A$ as follows:

$$\tau(z) = \sum_{j=0}^{m-1} y_j z x^j \quad (z \in A).$$

Obviously, τ is an endomorphism of A as a $C(A)$-$C(A)$-bimodule.

We can see that if δ is a B-derivation of A then $\delta(x) \in A_1 \cap \mathrm{Ker}(\tau)$, and if $u \in A_1 \cap \mathrm{Ker}(\tau)$ then there exists a B-derivation δ of A such that $\delta(x) = u$. Then we shall state the following theorem which is generalizations of [11, Theorem 3.2] and [11, Theorem 3.8].

Theorem 2.1. *f is weakly separable in R if and only if*

$$A_1 \cap \mathrm{Ker}(\tau) \subset I_x(A_0).$$

In virtue of Theorem 2.1, we have the following which is an improvement of [11, Theorem 3.10].

Theorem 2.2. *Assume that $R = B[X;D]$ and let $f = \sum_{i=0}^{m} X^i a_i$ $(a_m = 1, m \ge 1)$ be in $R_{(0)}$.*

(1) *f is weakly separable in R if and only if the following sequence of $C(A)$-$C(A)$-homomorphisms is exact:*

$$0 \longrightarrow C(A) \overset{\text{injection}}{\longrightarrow} A_0 \overset{I_x}{\longrightarrow} A_0 \overset{\tau}{\longrightarrow} C(A).$$

(2) *f is separable in R if and only if the following sequence of $C(A)$-$C(A)$-homomorphisms is exact:*

$$0 \longrightarrow C(A) \overset{\text{injection}}{\longrightarrow} A_0 \overset{I_x}{\longrightarrow} A_0 \overset{\tau}{\longrightarrow} C(A) \longrightarrow 0.$$

At the end of this section, we shall mention briefly on weakly quasi-separable polynomials in $B[X;\rho,D]^q_{(0)}$. We can define an automorphism ρ^* of R and a ρ^*-derivation D^* of A as follows:

$$\rho^*\left(\sum_j X^j c_j\right) = \sum_j X^j \rho(c_j),\ D^*\left(\sum_j X^j c_j\right) = \sum_j X^j D(c_j) \quad (c_j \in B).$$

Since $f \in B^{\rho,D}[X]$, it is easy to see that $\rho^*(fR) \subset fR$ and $D^*(fR) \subset fR$. Hence there is an automorphism $\tilde{\rho}$ of A and a $\tilde{\rho}$-derivation $\tilde{D}$ of A which is naturally induced by ρ^* and D^*, respectively. More precisely, $\tilde{\rho}$ and $\tilde{D}$ are defined by

$$\tilde{\rho}\left(\sum_{j=0}^{m-1} x^j c_j\right) = \sum_{j=0}^{m-1} x^j \rho(c_j),\ \tilde{D}\left(\sum_{j=0}^{m-1} x^j c_j\right) = \sum_{j=0}^{m-1} x^j D(c_j) \quad (c_j \in B).$$

We put here $A^{\tilde{\rho}} = \{z \in A \,|\, \tilde{\rho}(z) = z\}$, $A^{\tilde{D}} = \{z \in A \,|\, \tilde{D}(z) = 0\}$, and $A^{\tilde{\rho},\tilde{D}} = A^{\tilde{\rho}} \cap A^{\tilde{D}}$. Clearly, $A^{\tilde{\rho},\tilde{D}} \subset \mathrm{Ker}(I_x)$. We can see that $A^{\tilde{\rho},\tilde{D}} = \mathrm{Ker}(I_x)$ if $\mathrm{Ker}(I_x) \subset A^{\tilde{\rho}}$. Then we have the following.

Proposition 2.3. *Assume that* $\mathrm{Ker}(I_x) \subset A^{\tilde{\rho}}$. *If* f *is weakly separable in* $C(B^{\rho,D})[X]$, *then* f *is weakly quasi-separable in* R.

In virtue of Proposition 3.1 and [11, Theorem 2.2], we have the following.

Corollary 2.4. f *is weakly quasi-separable in* $B[X; D]$ *if* f *is weakly separable in* $C(B^D)[X]$.

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