Fractional Operations on Quadratic Fields *

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1 Abstract

We study fractional operations on a quadratic field F over K. A fractional operation is described by a pair of elements of F. Two fractional operations are (linearly) equivalent, if they are transformed by a bijective linear map. We give a complete classification of fractional operations on a quadratic field modulo the equivalence.

2 Fractional operations on K-algebras

Let K be a field and A be a K-algebra, and set $A^* = A \setminus \{0\}$. A K-*right fractional operation* (*right fraction* for short) on A is a mapping $* : A \times A \to A$ such that for every $a \in A$ the mapping $x \mapsto x * a$ is K-linear, that is,

$$(kx + \ell y) * a = k(x * a) + \ell(y * a) \tag{1}$$

for $k, \ell \in K$ and $x, y \in A$, and it satisfies the reduction law, that is,

$$(xa) * (ya) = x * y \tag{2}$$

for all $x, y \in A$ and $a \in A^*$. The mapping $*$ is a *left fraction* if it satisfies (1), (2) and

$$(ax) * (ay) = x * y \tag{3}$$

for all $x, y \in R$ and $a \in A^*$. When A is commutative, there is no distinction between left and right fractions, and so we simply call it a *fraction* on A.

The mapping $*_0 : A \to A$ defined by

$$x *_0 y = 0 \tag{4}$$

for $(x, y) \in A \times A$ is a left and right fraction on A, and is called the *zero fraction* or the *trivial fraction* on A.

*This is a preliminary report and a final version will appear elsewhere.

Proposition 2.1. *Suppose that* $A \neq \mathbb{Z}_2$. *If* $*$ *is a fraction on* A, *then*

$$x * 0 = 0$$

for all $x \in A$.

Proof. First, suppose that $\mathrm{char}(K) \neq 2$. Then, by (1) and (2) for $x \in A$ we have

$$x * 0 = (2x) * (2 \cdot 0) = 2x * 0 = 2(x * 0).$$

It follows that $x * 0 = 0$.

On the other hand, suppose that $\mathrm{char}(K) = 2$. First, we have

$$0 * 0 = (0 + 0) * 0 = 2(0 * 0) = 0.$$

Next, let $x \in A^*$, then we have

$$x * 0 = (x \cdot 1) * (x \cdot 0) = 1 * 0. \tag{5}$$

Suppose moreover $x \neq 1$ (there is such x because $F \neq \mathbb{Z}_2$), then by (5) we have

$$1 * 0 = (1 + x) * 0 = 1 * 0 + x * 0 = 2(1 * 0) = 0.$$

So we find that $x * 0 = 0$ for any $x \in A$. □

The assumption $A \neq \mathbb{Z}_2$ in Proposition 2.1 is necessary, as we have the following example.

Example 2.2. Suppose that $K = F = \mathbb{Z}_2 = \{0, 1\}$. We have exactly four fractional operations on $\mathbb{Z}_2$ Given by the operation tables:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

When A is a division algebra, the mapping $*_n$ defined by

$$x *_n y = \begin{cases} xy^{-1} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases} \tag{6}$$

for $(x, y) \in A \times A$ is a right fraction on A, and is called the *natural right fraction* on A.

Proposition 2.3. *If* A *is a division algebra and* $*$ *is a right fraction on* A, *then,* $*$ *is the natural right fraction if only if*

$$x * 1 = x \quad \textit{for all} \quad x \in A. \tag{7}$$

Proof. If $*$ is the natural right fraction, clearly (7) holds. Conversely, if (7) holds, then for any $(x, y) \in R \times R^*$ we have

$$x * y = (xy^{-1}y) * y = (xy^{-1}) * 1 = xy^{-1}$$

by (2). □

The *left natural fraction* is defined in a similar manner (replace xy^{-1} by $y^{-1}x$ in (6)), and we get a similar result to Proposition 2.3.

Two right (or left) fractions $*$ and $*'$ on A are (*linearly*) *equivalent*, if there is a bijective K-linear mapping $f : A \to A$ such that

$$f(x * y) = f(x) *' f(y) \tag{8}$$

holds for any $(x, y) \in A \times A$. We call f satisfying (8) a *linear mapping giving the equivalence* $* \cong *'$.

3 Fractions on quadratic fields

Let K be a field and F be a quadratic extension of K, that is, $F = K[\zeta]$, where $d = \zeta^2 \in K$ and $\zeta \notin K$. Then any element $x \in F$ is uniquely expressed as

$$x = a + b\,\zeta$$

with $a, b \in K$. We set $\Re(x) = a$ and $\Im(x) = b$.

Proposition 3.1. *For any* $\alpha, \beta \in F$*, the operation* $*$ *given as*

$$x * y = \begin{cases} \Re(x/y)\,\alpha + \Im(x/y)\,\beta & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases} \tag{9}$$

for $(x, y) \in F \times F^*$ *is a fraction on* F*. Conversely, any fraction* $*$ *on* F *is given as* (9) *for some* $\alpha, \beta \in F$*.*

Proof. We easily see that the operation given as (9) is a fraction on F. Conversely, let $*$ be a fraction on F. Let $x \in F$ and $y \in F^* = F \setminus \{0\}$, and let $\alpha = 1 * 1$ and $\beta = \zeta * 1$. Then, we have

$$\begin{aligned} x * y &= (x/y) * 1 = \Re(x/y) * 1 + \Im(x/y)\,\zeta * 1 \\ &= \Re(x/y)\,\alpha + \Im(x/y)\,\beta. \end{aligned} \tag{10}$$

By (10) together with Proposition 2.1, we have (9). □

Let $*_{(\alpha,\beta)}$ denote the operation $*$ given by (9). In particular, $*_{(0,0)}$ is the trivial operation; $x *_{(0,0)} y = 0$ for $(x, y) \in F \times F^*$, and $*_{(1,\zeta)}$ is the natural fraction; $x *_{(1,\zeta)} y = x/y$ for $(x, y) \in F \times F^*$.

The following gives a condition for fractions given as (9) are equivalent.

Proposition 3.2. *Let* $\alpha, \beta, \alpha', \beta' \in \mathbb{C}$*. The fractions* $*_{(\alpha,\beta)}$ *and* $*_{(\alpha'.\beta')}$ *are equivalent if and only if there is a bijective* K*-linear mapping* $f : F \to F$ *such that*

$$f\left(\alpha\,\Re(x/y) + \beta\,\Im(x/y)\right) = \alpha'\,\Re(f(x)/f(y)) + \beta'\,\Im(f(x)/f(y)) \tag{11}$$

holds for any $(x, y) \in F \times F^*$*.*

Proof. Let $f : F \to F$ be a bijective K-linear mapping satisfying (11). By (9), the lefthand side of (8) is equal to the lefthand side of (11) and the righthand side of (8) is equal to the righthand side of (11). □

A fraction $*$ on F is *non-degenerate* if it is surjective, that is, $F * F = F$, otherwise, it is *degenerate*. The trivial fraction is degenerate and the natural fraction is non-degenerate. Clearly, a non-degenerate fraction and a degenerate fraction never be equivalent.

Let $\alpha, \beta \in F$. Since $*_{(\alpha,\beta)}$ is given as (9) and

$$\{\mathfrak{R}(x/y) \,|\, x \in F, y \in F^*\} = \{\mathfrak{I}((x/y) \,|\, x \in F, y \in F^*\} = K,$$

we have

Proposition 3.3. *A fraction $*_{(\alpha,\beta)}$ is non-degenerate if and only if α and β are linearly independent over K.*

4 Main theorem and the sufficiency

For $\alpha = a + b\zeta \in F$ $(a, b \in K)$, let $\overline{\alpha} = a - b\zeta$ be its conjugate. For $(\alpha, \beta), (\alpha', \beta') \in F \times F^*$, we write $(\alpha, \beta) \sim (\alpha', \beta')$ if

$$\alpha\beta' = \beta\alpha' \ \text{ or } \ \overline{\alpha}\beta' = -\overline{\beta}\alpha'.$$

We state our main theorem which characterizes fractions on quadratic fields up to equivalence.

Theorem 4.1. *For $(\alpha, \beta), (\alpha', \beta') \in (\mathbb{C} \times \mathbb{C}) \setminus \{(0, 0)\}$, the fractions $*_{(\alpha,\beta)}$ and $*_{(\alpha',\beta')}$ are equivalent if and only if $(\alpha, \beta) \sim (\alpha', \beta')$.*

In this section we prove the sufficiency that $(\alpha, \beta) \sim (\alpha', \beta')$ implies $*_{(\alpha,\beta)} \cong *_{(\alpha',\beta')}$.

First suppose that

$$\alpha\beta' = \beta\alpha'. \tag{12}$$

If $\alpha \neq 0$, then $\alpha' \neq 0$ by (12). Let $\gamma = \alpha'/\alpha$, and define a K-linear mapping $f : F \to F$ by

$$f(x) = \gamma \cdot x$$

for $x \in F$. Then, we have

$$\begin{aligned} f\left(\alpha\,\mathfrak{R}(x/y) \,+\, \beta\,\mathfrak{I}(x/y)\right) &= \gamma\left(\alpha\,\mathfrak{R}(x/y) \,+\, \beta\,\mathfrak{I}(x/y)\right) \\ &= \alpha'\,\mathfrak{R}(x/y) + \beta'\,\mathfrak{I}(x/y) \\ &= \alpha'\,\mathfrak{R}(f(x)/f(y)) \,+\, \beta'\,\mathfrak{I}(f(x)/f(y)) \end{aligned}$$

for any $(x, y) \in F \times F^*$. Thus, f satisfies (11) and gives the equivalence

$$*_{(\alpha,\beta)} \cong *_{(\alpha',\beta')}.$$

If $\alpha = 0$, then $\beta \neq 0$, $\alpha' = 0$ by (12) and $\beta' \neq 0$. Set $\gamma = \beta'/\beta$, and define a K-linear mapping f by

$$f(x) = \gamma \cdot x$$

for $x \in F$. Then, f gives the equivalence $*_{(0,\beta)} \cong *_{(0,\beta')}$.

Next, suppose that

$$\overline{\alpha}\beta' = -\overline{\beta}\alpha'. \tag{13}$$

If $\alpha \neq 0$, then $\alpha' \neq 0$ by (13). Let $\gamma = \alpha'/\overline{\alpha}$, and define a K-linear mapping $f : F \to F$ by

$$f(x) = \gamma \cdot \overline{x}$$

for $x \in F$. Then, we have

$$\begin{aligned} f\left(\alpha\,\Re(x/y) \;+\; \beta\,\Im(x/y)\right) &= \gamma\left(\overline{\alpha}\,\Re(x/y) \;+\; \overline{\beta}\,\Im(x/y)\right) \\ &= \alpha'\Re(x/y) \;-\; \beta'\,\Im(x/y) \\ &= \alpha'\Re(\overline{x}/\overline{y}) \;+\; \beta'\,\Im(\overline{x}/\overline{y}) \\ &= \alpha'\,\Re(f(x)/f(y)) + \beta'\,\Im(f(x)/f(y)) \end{aligned}$$

for any $(x, y) \in F \times F^*$. Thus, f satisfies (11) and gives the equivalence

$$*_{(\alpha,\beta)} \cong *_{(\alpha',\beta')}.$$

If $\alpha = 0$, then $\beta \neq 0$ and $\beta' \neq 0$. Let $\gamma = \beta'/\overline{\beta}$ and define a K-linear mapping $f : F \to F$ by $f(x) = \gamma \cdot x$. Then, f gives the equivalence $*_{(0,\beta)} \cong *_{(0,\beta')}$.

5 Non-degenerate fractions

In this section we shall prove that the necessity of the condition $(\alpha, \beta) \sim (\alpha', \beta')$ in Theorem 4.1 for non-degenerate operations.

Let $*_{(\alpha,\beta)}$ and $*_{(\alpha',\beta')}$ be fractions with $\alpha, \beta, \alpha', \beta' \in F$. Suppose that they are equivalent and f is a K-linear mapping giving the equivalence. By Lemma 3.2, f satisfies (11), and replacing x by x/y and y by 1 in (11), we have

$$f\left(\alpha\,\Re(x/y) \;+\; \beta\,\Im(x/y)\right) = \alpha'\,\Re(f(x/y)/c) \;+\; \beta'\,\Im(f(x/y)/c) \tag{14}$$

for $(x, y) \in F \times F^*$, where $c = f(1)$. Because the left-hand sides of (11) and (14) are equal, we have

$$\alpha'\,\Re(f(x)/f(y)) + \beta'\,\Im(f(x)/f(y)) = \alpha'\,\Re(f(x/y)/c) \;+\; \beta'\,\Im(f(x/y)/c). \tag{15}$$

By Proposition 3.3, α' and β' are linearly independent over K, and hence $\Re(f(x)/f(y)) = \Re(f(x/y)/c)$ and $\Im(f(x)/f(y)) = \Im(f(x/y)/c)$ by (15). Therefore, we have

$$c \cdot \frac{f(x)}{f(y)} = f\Big(\frac{x}{y}\Big). \tag{16}$$

Replacing x by xy in (16), we get

$$c \cdot f(xy) = f(x)f(y). \tag{17}$$

Define the mapping $h : F \to F$ by

$$h(x) = \frac{f(x)}{c}$$

for $x \in F$. By (17) we have

$$h(xy) = h(x)h(y)$$

for any $x, y \in F$. Moreover, for $k \in K$ we have

$$h(k) = \frac{f(k)}{c} = \frac{kf(1)}{c} = k.$$

Therefore, h is an automorphism of F over K, and so either $h(x) = x$ or $h(x) = \overline{x}$.

In the first case

$$f(x) = c \cdot x \tag{18}$$

for $x \in F$. Letting $x = 1$ and $y = 1$ in (14) and using (18), we have

$$c \cdot \alpha = f(\alpha) = f\big(\alpha\mathfrak{R}(1) + \beta\mathfrak{I}(1)\big) = \alpha'\mathfrak{R}\big(f(1)/c\big) + \beta'\mathfrak{I}\big(f(1)/c\big) = \alpha'.$$

Letting $x = \zeta$ and $y = 1$ in (14) and using (18), we have

$$c\beta = f(\beta) = f(\alpha\mathfrak{R}(\zeta)+\beta\mathfrak{I}(\zeta) = \alpha'\mathfrak{R}\big(f(\zeta)/c\big)+\beta'\mathfrak{I}\big(f(\zeta)/c\big) = \alpha'\mathfrak{R}(\zeta)+\beta'\mathfrak{I}(\zeta) = \beta'.$$

Hence,

$$\alpha\beta' = \beta\alpha'. \tag{19}$$

In the second case

$$f(x) = c \cdot \overline{x}$$

for $x \in F$. Letting $x = 1$ and $y = 1$ in (14) an using (18), we have

$$c \cdot \overline{\alpha} = f(\alpha) = \alpha'.$$

Letting $x = \zeta$ and $y = 1$ in (14) and using (18), we have

$$c \cdot \overline{\beta} = f(\beta) = -\beta'.$$

Hence,

$$\overline{\alpha}\beta' = -\overline{\beta}\alpha'. \tag{20}$$

Therefore, either (19) or (20) holds, that is, $(\alpha, \beta) \sim (\alpha', \beta')$, as desired.

Example 5.1. For $\alpha, \beta \in F$, the fraction $*_{(\alpha,\beta)}$ is isomorphic to the natural fraction $*_{(1,\zeta)}$ if and only if $\alpha \neq 0$ and $\beta/\alpha = \zeta$.

6 Degenerate fractions

In this section we treat degenerate fractions. Let $\alpha, \beta, \alpha', \beta' \in \mathbb{C} \setminus \{0\}$. Suppose that the operations $*_{(\alpha,\beta)}$ and $*_{(\alpha',\beta')}$ are degenerate and they are equivalent. We shall show that $(\alpha, \beta) \cong (\alpha', \beta')$

Because α and β (and α' and β') are linearly dependent over K by Proposition 3.3, we can write

$$\alpha = k \cdot \beta \text{ and } \alpha' = k' \cdot \beta' \tag{21}$$

for some $k, k' \in K^* = K \setminus \{0\}$. Because $(\alpha, \beta) \sim (k, 1)$ and $(\alpha', \beta') \sim (k', 1)$ by (21), we see $*_{(\alpha,\beta)} \cong *_{(k,1)}$ and $*_{(\alpha',\beta')} \cong *_{(k',1)}$ by the results in Section 4, and hence $*_{(k,1)}$ is equivalent to $*_{(k',1)}$. Let f be a K-linear map giving this equivalence.

By Proposition 3.2 we have

$$f\,(k \cdot \Re(x/y) + \Im(x/y)) = k' \cdot \Re(f(x)/f(y)) \; + \; \Im(f(x)/f(y)) \tag{22}$$

for $(x, y) \in F \times F^*$. Because the right-hand side of (22) is in K and $k \cdot \Re(x/y) + \Im(x/y)$ ranges over the whole K, we see that $f(x) \in K$ for all $x \in K$. Since f is K-linear,

$$f(x) = c \cdot x \tag{23}$$

for $x \in K$, where $c = f(1) \in K^*$. Letting $x = y = 1$ in (22), we have

$$ck = f(k) = k'. \tag{24}$$

By (23) and (24), (22) becomes

$$c\,(k \cdot \Re(x/y) + \Im(x/y)) = ck \cdot \Re(f(x)/f(y)) \; + \; \Im(f(x)/f(y)). \tag{25}$$

Now, let $f(\zeta) = a + b\zeta$ with $a, b \in K$. Letting $x = \zeta$ and $y = 1$ in (25), we have

$$c = \frac{cka + b}{c}.$$

Hence,

$$g_1 = c^2 - cka - b = 0. \tag{26}$$

Letting $x = 1$ and $y = \zeta$ in (25) (recall that $\zeta^2 = d$), we have

$$\begin{aligned} \frac{c}{d} &= c\,(k \cdot \Re(\zeta/d) + \Im(\zeta/d)) = ck\Re(c/(a + b\zeta)) + \Im(c/(a + b\zeta)) \\ &= \frac{c(cka - b)}{a^2 - db^2}. \end{aligned}$$

Hence

$$g_2 = a^2 - db^2 - d(cka - b) = 0. \tag{27}$$

Finally, let $x = 1$ and $y = 1 + \zeta$. Then, the left-hand side of (25) equals

$$c\left(k \cdot \Re\left(\tfrac{1}{1+\zeta}\right) + \Im\left(\tfrac{1}{1+\zeta}\right)\right) = c\left(k \cdot \Re\left(\tfrac{1-\zeta}{1-d}\right) + \Im\left(\tfrac{1-\zeta}{1-d}\right)\right) = \frac{c(k-1)}{1-d},$$

and the right-hand side of (25) equals

$$ck\mathfrak{R}\left(\tfrac{c}{c+a+b\zeta}\right)+\mathfrak{I}\left(\tfrac{c}{c+a+b\zeta}\right)=\frac{c(ck(c+a)-b)}{(c+a)^2-db^2}.$$

Hence, we get

$$g_3=(k-1)((c+a)^2-db^2)-(1-d)(ck(c+a)-b)=0. \tag{28}$$

By (26), (27) and (28) we have

$$(dk-1)g_1+(k-1)g_2-g_3=2ac(1-dk^2)=0. \tag{29}$$

Note that $c\neq 0$, and $dk^2\neq 1$ because $\zeta=\sqrt{d}$ is not in K. So, if $\mathrm{char}(K)\neq 2$, it follows from (29) that $a=0$. Hence, by (27) we have

$$db(b-1)=0.$$

Because $d\neq 0$ and $b\neq 0$ (otherwise f is not injective), we get $b=1$. Thus by (26) we obtain $c^2=1$.

Next, suppose $\mathrm{char}(K)=2$, then we have

$$dg_1+g_2=a^2+(b+c)^2d=0.$$

Because $\zeta=\sqrt{d}\notin K$, we see that $a=b+c=0$. So, by (26) we have

$$c(c+1)=0,$$

and because $c\neq 0$ we get $c=1$.

Thus, in any case we see $c=\pm 1$. Hence, we have $k'=\pm k$ by (24). Hence $(k,1)\sim(k',1)$, and so $(\alpha,\beta)\sim(\alpha',\beta')$ as desired.

We have proved that $*_{(\alpha,\beta)}\cong *_{(\alpha',\beta')}$ implies $(\alpha,\beta)\sim(\alpha',\beta')$ for non-degenerate fractions in the previous section and for degenerate fractions in this section. The proof of Theorem 4.1 is complete.

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