# ASYMPTOTICS FOR HIGHER DERIVATIVES OF THE LERCH ZETA-FUNCTION: APPLICATIONS TO THE FORMULAE OF KUMMER, LERCH AND GAUSS

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Abstract. Let $s$ be a complex variables, $z$ a complex parameter, and $a$ and $\lambda$ real parameters with $a > 0$, and write $e(s) = e^{2\pi is}$. The Lerch zeta-function $\phi(s, a, \lambda)$ is defined by the Dirichlet series $\sum_{l=0}^{\infty} e(\lambda l)(a + l)^{-s}$ ($\operatorname{Re} s > 1$), and its meromorphic continuation over the whole $s$-plane; this reduces to the Hurwitz zeta-function $\zeta(s, a)$ if $\lambda$ is an integer, and further to the Riemann zeta-function $\zeta(s) = \zeta(s, 1)$. Note that the domain of the parameter $a$ can be extended through the procedure in [13]. Let $\phi^{(m)}(s, z, \lambda) = (\partial/\partial s)^m \phi(s, z, \lambda)$ for $m = 0, 1, 2, \ldots$ denote any derivative. The aim of this paper is to show that complete asymptotic expansions exist for $\phi^{(m)}(s, a + z, \lambda)$ ($m = 0, 1, \ldots$) when both $z \to 0$ and $z \to \infty$ through $|\arg z| < \pi$ (Theorems 1 and 2), together with the explicit expressions of their remainders (Corollaries 1.1 and 2.2); these can be applied to deduce the classical Fourier series expansions of the log-gamma function $\log \Gamma(s)$ (Corollary 2.3) and the di-gamma function $\psi(s) = (\Gamma'/\Gamma)(s)$ (Corollary 2.4) both for $0 < s < 1$, due to Kummer and Lerch, respectively, as well as to deduce the celebrated closed form evaluation of $\psi(r)$ at any rational point $r$ with $0 < r < 1$ (Corollary 2.5), due to Gauß. Our results in Theorems 1 and 2 further lead us to define and study a generalization of Deninger's $\mathcal{R}_m$-function (Corollaries 1.4–1.6 and 2.6–2.9), which was first introduced by Deninger [3] for extending the log-gamma function into higher orders. The detailed proofs of our results in the present paper will appear, among other things, in the forthcoming article [21].

## 1. Introduction

Throughout the paper, the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, non-negative integers, all integers, real numbers, and complex numbers, respectively, and further $s = \sigma + it$ is a complex variable (with real coordinates $\sigma$ and $t$), $a$ and $\lambda$ are real parameters with $a > 0$, and the notation $e(s) = e^{2\pi is}$ is frequently used. The Lerch zeta-function $\phi(s, a, \lambda)$ is defined by the Dirichlet series

$$\phi(s, a, \lambda) = \sum_{l=0}^{\infty} e(\lambda l)(a + l)^{-s} \qquad (\operatorname{Re} s > 1), \tag{1.1}$$

and its meromorphic continuation over the whole $s$-plane (cf. [30][31]); this reduces if $\lambda \in \mathbb{Z}$ to the Hurwitz zeta-function $\zeta(s, a)$, to the exponential zeta-function $\zeta_\lambda(s) = e(\lambda)\phi(s, 1, \lambda)$ for $\lambda \in \mathbb{R}$, and hence to the Riemann zeta-function $\zeta(s) = \zeta(s, 1) = \zeta_\lambda(s)$ if $\lambda \in \mathbb{Z}$. We note that the domain of the parameter $a$ can be extended to the whole sector $|\arg z| < \pi$ through the procedure in [13].

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It is the principal aim of the present paper to treat asymptotic aspects of the derivatives (of any order) $\phi^{(m)}(s,z,\lambda)=(\partial/\partial s)^m\phi(s,z,\lambda)$ for $m=0,1,2,\ldots$, when $z$ becomes small and large through the sector $|\arg z|<\pi$. Let $\Gamma(s)$ denote the gamma function, and $\psi(s)=(\Gamma'/\Gamma)(s)$ the di-gamma function. We shall then show that complete asymptotic expansions exist for $\phi^{(m)}(s,a+z,\lambda)$ $(m=0,1,2,\ldots)$ as both $z\to 0$ and $z\to\infty$ through $|\arg z|<\pi$ (Theorems 1 and 2), together with the explicit expressions of their remainders (Corollaries 1.1 and 2.2); these can further be applied to deduce the classical Fourier series expansions of $\log\Gamma(s)$ (Corollary 2.3) and of $\psi(s)$ (Corollary 2.4) both on the unit interval, due to Kummer and Lerch, respectively, as well as to deduce the celebrated closed form evaluation of $\psi(r)$ at any rational point $r$ on the unit interval (Corollary 2.5), due to Gauß. Furthermore, our results in Theorems 1 and 2 lead us to define and study a generalization of Deninger's $\mathcal{R}_m$-function (Corollaries 1.4–1.6 and 2.6–2.9), which was first introduced by Deninger [3] for extending $\log\Gamma(s)$ into higher orders. The detailed proofs of our results will appear, among other things, in the forthcoming article [21].

## 2. Statement of results: asymptotic expansions

We prepare for describing our results the shifted factorial $(s)_n=\Gamma(s+n)/\Gamma(s)$ with any $n\in\mathbb{Z}$, and the (modified) Stirling polynomial of the first kind, defined for any $j,k\in\mathbb{N}_0$ by

$$\mathfrak{s}_j^k(x)=\frac{1}{j!}\left(\frac{\partial}{\partial z}\right)^k(1-z)^{-x}\{-\log(1-z)\}^j\bigg|_{z=0}. \tag{2.1}$$

The following Theorems 1 and 2 assert complete asymptotic expansions as $z\to 0$ and as $z\to\infty$, respectively, through the sector $|\arg z|<\pi$.

**Theorem 1.** *Let $m\in\mathbb{N}_0$, and $a,\lambda\in\mathbb{R}$ be arbitrary with $a>0$. Then for any integer $K\geq 0$, in the region $\sigma>1-K$ except at $s=1$, we have*

$$\phi^{(m)}(s,a+z,\lambda)=m!\sum_{k=0}^{K-1}\frac{(-1)^k}{k!}\sum_{j=0}^{m}\frac{\mathfrak{s}_{m-j}^k(s)}{j!}\phi^{(j)}(s+k,a,\lambda)z^k + (\rho_K^+)^{(m)}(s,a,\lambda;z) \tag{2.2}$$

*for $|\arg z|<\pi$. Here $\rho_K^+$ is expressed by the Mellin-Barnes type integral (4.2) below, and its mth derivative $(\rho_K^+)^{(m)}=(\partial/\partial s)^m\rho_K^+$ satisfies the estimate*

$$(\rho_K^+)^{(m)}(s,a,\lambda;z)=O(|z|^K) \tag{2.3}$$

*as $z\to 0$ through $|\arg z|\leq\pi-\delta$ with any small $\delta>0$, where the implied O-constant depends at most on $s$, $a$, $\lambda$, $K$ and $\delta$.*

The following expression holds for the case $m=0$ of the remainder in (2.2).

**Corollary 1.1.** *For any $K\geq 1$, in the region $\sigma>1-K$ and in the sector $|\arg z|<\pi$, the Mellin-Barnes type integral in (4.2) is transformed to*

$$\rho_K^+(s,a,\lambda;z)=\frac{(-1)^K(s)_K z^k}{\Gamma(K)}\int_0^1\phi(s+K,a+z\tau,\lambda)(1-\tau)^{K-1}d\tau.$$

*Proof.* To remove the poles of the integrand in (4.2) at $w=k$ $(k=0,1,\ldots,K-1)$, the expression

$$\Gamma(-w)=\frac{(-1)^K\Gamma(-w+K)}{\Gamma(K)}\int_0^1\tau^{w-K}(1-\tau)^{K-1}d\tau,$$

being valid on the path $\operatorname{Re} w = u_K^+$, is inserted in the integrand on the right side of (4.2); this yields the assertion of the corollary upon changing the order of the $w$- and $\tau$-integration, where the resulting inner $w$-integral can be evaluated by substituting the variable $w = w' + K$, and by noting the fact that $\Gamma(s) = \Gamma(s+K)/(s)_K$. □

It can be seen from Corollary 1.1 that $\lim_{K\to+\infty}(\rho_K^+)^{(m)}(s,a,\lambda;z) = 0$ for $|z| < a$; Theorem 1 readily implies the following result.

**Corollary 1.2.** *Let $m$, $a$ and $\lambda$ be as in Theorem 1. Then we have for $|z| < a$ and for any $s \in \mathbb{C}$ except at $s = 1$ the Taylor series expansion*

$$\phi^{(m)}(s,a+z,\lambda) = m!\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{j=0}^{m}\frac{\mathfrak{s}_{m-j}^k(s)}{j!}\phi^{(j)}(s+k,a,\lambda)z^k. \tag{2.4}$$

We remark here in connection with Corollary 1.2 that the monograph of Srivastava-Choi [35] gives a quite systematic presentation of various sums involving the values of zeta and allied functions.

The case $a = 1$ of Theorem 1 with the relation

$$\phi(s,z,\lambda) - z^{-s} = e(\lambda)\phi(s,1+z,\lambda) \tag{2.5}$$

asserts the following asymptotic expansion as $z \to 0$.

**Corollary 1.3.** *Let $m$, $a$ and $\lambda$ be as in Theorem 1. Then for any integer $K \geq 0$, in the region $\sigma > 1-K$ except at $s = 1$, we have*

$$\begin{aligned}\phi^{(m)}(s,z,\lambda) = z^{-s}(-\log z)^m + m!\sum_{k=0}^{K-1}\frac{(-1)^k}{k!}\sum_{j=0}^{m}\frac{\mathfrak{s}_{m-j}^k(s)}{j!}\\ \times \zeta_\lambda^{(j)}(s+k)z^k + e(\lambda)(\rho_K^+)^{(m)}(s,1,\lambda;z),\end{aligned} \tag{2.6}$$

*where the reminder $e(\lambda)(\rho_K^+)^{(m)}$ satisfies the same estimate as in (2.3) when $z \to 0$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.*

Next let $\delta(\lambda)$ be the symbol which equals 1 or 0, according to $\lambda \in \mathbb{Z}$ or otherwise. Then Apostol [1] introduced the sequence of functions $B_k(x,y)$ $(k \in \mathbb{N}_0)$, defined for any $x, y \in \mathbb{C}$ by the Taylor series expansion

$$\frac{ze^{xz}}{ye^z - 1} = \sum_{k=0}^{\infty}\frac{B_k(x,y)}{k!}z^k$$

centered at $z = 0$; note that

$$B_0(x,y) = \begin{cases} 1 & \text{if } y = 1,\\ 0 & \text{otherwise,}\end{cases} \tag{2.7}$$

and $B_k(x,y)$ reduces to the usual Bernoulli polynomial $B_k(x)$ (cf. [5, 1.13 (2)]) if $y = 1$.

**Theorem 2.** *Let $m$, $a$ and $\lambda$ be as in Theorem 1, and define the polynomials $p_m(s,w)$ and $q_{k,m}(s,w)$ for $m, k \in \mathbb{N}_0$ by*

$$p_m(s;w) = \sum_{j=0}^{m}\frac{\{(s-1)w\}^j}{j!}, \tag{2.8}$$

$$q_{m,k}(s;w) = \sum_{j=0}^{m}\frac{\mathfrak{s}_{m-j}^k(s)}{j!}(-w)^j. \tag{2.9}$$

*Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s = 1$, we have the formula*

$$\begin{aligned}\phi^{(m)}(s, a+z, \lambda) &= \frac{\delta(\lambda)(-1)^m m!}{(s-1)^{m+1}} z^{1-s} p_m(s; \log z) \\ &\quad + m! \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a, e(\lambda))}{(k+1)!} z^{-s-k} q_{m,k}(s; \log z) \\ &\quad + (\rho_K^-)^{(m)}(s, a, \lambda; z),\end{aligned} \tag{2.10}$$

*where $\rho_K^-$ is expressed by the Mellin-Barnes type integral (4.3) below, and its $m$th derivative $(\rho_K^-)^{(m)} = (\partial/\partial s)^m \rho_K^-$ satisfies the estimate*

$$(\rho_K^-)^{(m)}(s, a, \lambda; z) = O\bigl(|z|^{-\sigma-K} \log^m |z|\bigr) \tag{2.11}$$

*as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the implied $O$-constant depends at most on $m$, $s$, $a$, $\lambda$, $K$ and $\delta$.*

The case $a = 1$ of Theorem 2, together with the relations (2.5) and

$$yB_j(1, y) = (-1)^j B_j(0, 1/y) = \begin{cases} B_j(0, y) & \text{if } j \neq 1, \\ B_1(0, y) + 1 & \text{if } j = 1 \end{cases}$$

for any $y \in \mathbb{C} \setminus \{0\}$ (cf. [14, (7.1) and (7.2)]), asserts the following formula.

**Corollary 2.1.** *Let $m$, $\lambda$, $p_m(s; w)$ and $q_{m,k}(s; w)$ be as in Theorem 2. Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s = 1$, we have*

$$\begin{aligned}\phi^{(m)}(s, z, \lambda) &= \frac{\delta(\lambda)e(\lambda)(-1)^m m!}{(s-1)^{m+1}} z^{1-s} p_m(s; \log z) \\ &\quad + m! \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(0, e(\lambda))}{(k+1)!} z^{-s-k} q_{m,k}(s; \log z) \\ &\quad + e(\lambda)(\rho_K^-)^{(m)}(s, 1, \lambda; z),\end{aligned} \tag{2.12}$$

*where the reminder $e(\lambda)(\rho_K^-)^{(m)}$ satisfies the same estimate as in (2.11) when $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.*

It is to be remarked that the case $m = 0$ of Corollary 1.2 and Theorem 2 were first proved (in a unified manner) in terms of Mellin-Barnes type integrals by the author [13, (1.6) and Theorem 1], where the expression (2.14) below for the remainder in (2.10) (with $m = 0$) has been shown at the same time.

Let $U(\alpha; \gamma; Z)$ denote Kummer's confluent hypergeometric function of the second kind, defined by the integral

$$U(\alpha; \gamma; Z) = \frac{1}{\Gamma(\alpha)\{e(\alpha) - 1\}} \int_{\infty e^{i\varphi}}^{(0+)} e^{-Zw} w^{\alpha-1} (1+w)^{\gamma-\alpha-1} dw$$

for any $\alpha, \gamma \in \mathbb{C}$ and for $|\arg Z + \varphi| < \pi/2$ with any fixed $\varphi \in ]-\pi, \pi[$. Here the path of integration is the loop cranked with an angle $\varphi$ around the origin, which starts from $\infty e^{i\varphi}$, proceeds along the ray from $\infty e^{i\varphi}$ to $\delta e^{i\varphi}$ with a small $\delta > 0$, encircles the origin counter-clockwise, and returns to $\infty e^{i\varphi}$ along the ray, where $\arg w$ varies from $\varphi$ to $\varphi + 2\pi$

along the loop; this allows to prepare the analytic continuation of $U(\alpha;\gamma;Z)$ to the whole sector $|\arg Z| < 3\pi/2$ by rotating appropriately the path of integration. We now set

$$f_{s,K}(Z) = U(1; 2-s-K; Z), \quad g_{s,K}(Z) = U(s+K; s+K; Z) \tag{2.13}$$

both for $|\arg Z| < 3\pi/2$. Then the following expressions are valid for the case $m = 0$ of the remainder in (2.10).

**Corollary 2.2.** *For any $a, \lambda \in [0, 1]$, and in the region $\sigma > -K$ with $K \geq 1$, we have for $|\arg z| < \pi$,*

$$\rho_K^-(s,a,\lambda;z) = \frac{(s)_K z^{1-s-K}}{(2\pi i)^K}\Bigg\{\sum_{l=0}^{\infty}{}' \frac{e(-a(\lambda+l))}{(\lambda+l)^K} f_{s,K}\big(2\pi(\lambda+l)e^{-\pi i/2}z\big) + (-1)^K \sum_{l=0}^{\infty}{}' \frac{e(a(1-\lambda+l))}{(1-\lambda+l)^K} f_{s,K}\big(2\pi(1-\lambda+l)e^{\pi i/2}z\big)\Bigg\}, \tag{2.14}$$

*which is transformed through (2.17) below into*

$$\rho_K^-(s,a,\lambda;z) = (-1)^K(2\pi)^{s-1}(s)_K\Bigg\{e^{\pi i(1-s)/2}\sum_{l=0}^{\infty}{}' \frac{e(-a(\lambda+l))}{(\lambda+l)^{1-s}} g_{s,K}\big(2\pi(\lambda+l)e^{-\pi i/2}z\big) + e^{-\pi i(1-s)/2}\sum_{l=0}^{\infty}{}' \frac{e(a(1-\lambda+l))}{(1-\lambda+l)^{1-s}} g_{s,K}\big(2\pi(1-\lambda+l)e^{\pi i/2}z\big)\Bigg\} \tag{2.15}$$

*for the same $\sigma$, $K$ and $z$ as above.*

*Proof.* We can apply the (slightly extended) functional equation, for any $a, \lambda \in [0, 1]$,

$$\phi(r,a,\lambda) = \frac{\Gamma(1-r)}{(2\pi)^{1-r}}\Bigg\{e^{\pi i(1-r)/2}\sum_{l=0}^{\infty}{}' \frac{e(-a(\lambda+l))}{(\lambda+l)^{1-r}} + e^{-\pi i(1-r)/2}\sum_{l=0}^{\infty}{}' \frac{e(a(1-\lambda+l))}{(1-\lambda+l)^{1-r}}\Bigg\} \qquad (\mathrm{Re}\, r < 0), \tag{2.16}$$

in the argument of [13, Proof of Theorem 1] to deduce (2.14). Next the relation

$$U(\alpha;\gamma;Z) = Z^{1-\gamma}U(\alpha-\gamma+1; 2-\gamma; Z) \tag{2.17}$$

(cf. [5, 6.5 (6)]) shows that $f_{s,K}(Z) = Z^{s+K-1}g_{s,K}(Z)$, which is substituted into the right side of (2.14) to imply the assertion (2.15). □

We mention here several results relevant to Theorem 2. Meijer's $G$-function (cf. [5, 5.3 (1)]) theoretic interpretation of the formula (2.10) with $m = 0$, as well as of the author's result [16, Theorem 1] on complete asymptotic expansions for Epstein zeta-function, were made by Kuzumaki [29]. Also, the proof of (2.10) with $m = 0$ in [13] is reproduced in the monograph of Chakraborty-Kanemitsu-Tsukada [2, Chap.5.3], in which various alternative proofs of (3.5) below are given. A complete asymptotic expansion, whose shape differs far from that of (2.10) with $\lambda \in \mathbb{Z}$, for $\zeta^{(m)}(s,z)$ $(m = 0, 1, 2, \ldots)$ as $z \to \infty$ through $|\arg z| < \pi$ was obtained more recently by Seri [34] (see also the references therein for various related articles). Matsumoto [33], on the other hand, established complete asymptotic expansions for the extensions of $\zeta(s,z)$ to several variable cases.

## 3. Applications

We proceed in this section to present several applications of Theorems 1 and 2. For this, let $\operatorname{si} x$ and $\operatorname{Ci} x$ denote the sine and cosine integrals, defined respectively by

$$\operatorname{si} x = \int_{+\infty}^{x} \frac{\sin u}{u} du \qquad \text{and} \qquad \operatorname{Ci} x = \int_{+\infty}^{x} \frac{\cos u}{u} du \tag{3.1}$$

for any $x \in ]0, +\infty[$ (cf. [6, 9.8 (1) and (3)]). It is classically known that the evaluations

$$\frac{\partial}{\partial s}\zeta(s,z)\bigg|_{s=0} = \log\left\{\frac{\Gamma(z)}{\sqrt{2\pi}}\right\}, \tag{3.2}$$

$$\left\{\zeta(s,z) - \frac{1}{s-1}\right\}\bigg|_{s=1} = -\frac{\Gamma'}{\Gamma}(z) = -\psi(z) \tag{3.3}$$

hold both for $|\arg z| < \pi$ (cf. [5, 1.10 (9) and (10)]). Then in view of the relation (2.5) with $\lambda \in \mathbb{Z}$, a particular case of the formula (2.10), combined with (2.14) or (2.15), in fact yields the Fourier series expansions (3.5) and (3.7) below, due to Kummer and to Lerch (cf. [5, 1.9.1 (14) and (15)]), respectively. Let $\gamma_0 = -\Gamma'(1)$ denote the 0th Euler constant.

**Corollary 2.3.** *For any $a \in ]0,1[$ and $\lambda \in \{0,1\}$, we have the Fourier series expansion*

$$(\rho_1^-)'(0,a,\lambda;1) = \sum_{l=1}^{\infty} \frac{1}{\pi l}\{-\operatorname{si}(2\pi l)\cos(2\pi a l) + \operatorname{Ci}(2\pi l)\sin(2\pi a l)\}, \tag{3.4}$$

*which with (2.10) and (3.2) implies that*

$$\log\left\{\frac{\Gamma(a)}{\sqrt{2\pi}}\right\} = \sum_{l=1}^{\infty} \frac{\cos(2\pi a l)}{2l} + \sum_{l=1}^{\infty} \frac{1}{\pi l}\{\gamma_0 + \log(2\pi l)\}\sin(2\pi a l). \tag{3.5}$$

**Corollary 2.4.** *For any $a \in ]0,1[$ and $\lambda \in \{0,1\}$, we have the Fourier series expansion*

$$\rho_1^-(1,a,\lambda;1) = B_1(a) - 2\sum_{l=1}^{\infty}\{\operatorname{Ci}(2\pi l)\cos(2\pi a l) + \operatorname{si}(2\pi l)\sin(2\pi a l)\}, \tag{3.6}$$

*which with (2.10) and (3.3) implies that*

$$\begin{aligned}\psi(a)\sin(\pi a) = \sum_{l=1}^{\infty} \log\left(\frac{l}{l+1}\right)\sin\{(2l+1)\pi a\} - \{\gamma_0 + \log(2\pi)\}\sin(\pi a) \\ - \frac{\pi}{2}\cos(\pi a).\end{aligned} \tag{3.7}$$

The case $(m,K) = (1,2)$ of Theorem 2 can be applied to (3.7), upon yielding the following celebrated closed form evaluation due to Gauß (cf. [5, 1.7.3 (29)]).

**Corollary 2.5.** *For any $p, q \in \mathbb{Z}$ with $0 < p < q$, we have*

$$\psi\left(\frac{p}{q}\right) = -\gamma_0 - \log q - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) + \sum_{r=1}^{\lfloor q/2 \rfloor}{}' \cos\left(\frac{2\pi p r}{q}\right)\log\left\{2 - 2\cos\left(\frac{2\pi r}{q}\right)\right\}, \tag{3.8}$$

*where the primed summation symbol on the right side indicates that the last term is to be halved if $q$ is even.*

We proceed to state the last assertions. The function

$$\mathcal{R}_{m,0}(z) = (-1)^{m+1}\left(\frac{\partial}{\partial s}\right)^m \zeta(s,z)\Big|_{s=0} \qquad (m=1,2,\ldots)$$

was first introduced and studied in detail by Deninger [3], for the purpose of obtaining a better understanding of the Kronecker limit formula for real quadratic fields. We introduce in this respect the generalized Deninger function $\mathcal{R}_{m,n}(z,\lambda)$ for any $m \in \mathbb{N}_0$, $n \in \{1\} \cup (-\mathbb{N}_0)$ and $\lambda \in \mathbb{R}$, defined by

$$\mathcal{R}_{m,1}(z,\lambda) = (-1)^{m+1}\left(\frac{\partial}{\partial s}\right)^m \left\{\phi(s,z,\lambda) - \frac{\delta(\lambda)}{s-1}\right\}\Big|_{s=1} \tag{3.9}$$

for $n = 1$, and for any $n \in -\mathbb{N}_0$,

$$\mathcal{R}_{m,n}(z,\lambda) = (-1)^{m+1}\left(\frac{\partial}{\partial s}\right)^m \phi(s,z,\lambda)\Big|_{s=n}, \tag{3.10}$$

both in $|\arg z| < \pi$. Then Theorems 1 and 2 readily imply the following Corollaries 1.4–1.6 and 2.6–2.9, respectively.

**Corollary 1.4.** *For any $m \in \mathbb{N}_0$ and for any $a, \lambda \in \mathbb{R}$ with $a > 0$, we have the formulae*:

i) *for any integer $K \geq 1$,*

$$\begin{aligned}\mathcal{R}_{m,1}(a+z,\lambda) &= \mathcal{R}_{m,1}(a,\lambda) + (-1)^{m+1} m! \sum_{k=1}^{K-1} \frac{(-z)^k}{k!} \\ &\quad\times \sum_{j=0}^{m} \frac{\mathfrak{s}_{m-j}^k(1)}{j!}\phi^{(j)}(k+1,a,\lambda) + O(|z|^K);\end{aligned} \tag{3.11}$$

ii) *for any $n \in \mathbb{N}_0$ and for any integer $K \geq n+2$,*

$$\begin{aligned}&\mathcal{R}_{m,-n}(a+z,\lambda) \\ &= (-1)^{m+1} m!\Bigg[\sum_{k=0}^{K-1}\frac{(-z)^k}{k!}\sum_{j=0}^{m}\frac{(-1)^{j+1}\mathfrak{s}_{m-j}^k(-n)}{j!}\mathcal{R}_{j,k-n}(a,\lambda) \\ &\quad + \frac{(-z)^{n+1}}{(n+1)!}\left\{\delta(\lambda)\mathfrak{s}_m^n(-n) + \sum_{j=1}^{m}\frac{(-1)^j\mathfrak{s}_{m-j}^n(-n)}{(j-1)!}\mathcal{R}_{j-1,1}(a,\lambda)\right\} \\ &\quad + \sum_{k=n+2}^{K-1}\frac{(-z)^k}{k!}\sum_{j=0}^{m}\frac{\mathfrak{s}_{m-j}^k(-n)}{j!}\phi^{(j)}(k-n,a,\lambda)\Bigg] + O(|z|^K),\end{aligned} \tag{3.12}$$

*both as $z \to 0$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the implied $O$-constants depend at most on $a$, $\lambda$, $K$, $m$, $n$ and $\delta$.*

The limit case $K \to +\infty$ of (3.11) with $m = 1$ implies the following Taylor series expansion, which is a slight extension of [5, 1.17(5)], since $\psi(z) = \mathcal{R}_{1,1}(z,\lambda)$ holds, by (3.3) and (3.9), for $|\arg z| < \pi$ if $\lambda \in \mathbb{Z}$.

**Corollary 1.5.** *For any real $a > 0$, in the disk $|z| < a$, we have*

$$\psi(a+z) = \psi(a) + \sum_{k=1}^{\infty}(-1)^{k+1}\zeta(k+1,a)z^k. \tag{3.13}$$

The generalized Euler-Stieltjes constants $\gamma_m(z)$ ($m \in \mathbb{N}_0$) are defined by the Laurent series expansion

$$\zeta(s,z) = \frac{1}{s-1} + \sum_{m=0}^{\infty} \gamma_m(z)(s-1)^m \qquad (0 < |s-1| < 1) \tag{3.14}$$

(cf. [7, 1.8(1.122)]), which shows with (3.9) that

$$\gamma_m(z) = \frac{(-1)^{m+1}}{m!} \mathcal{R}_{m,1}(z,\lambda) \qquad (m = 0, 1, \ldots) \tag{3.15}$$

for $|\arg z| < \pi$ if $\lambda \in \mathbb{Z}$; this asserts upon (3.11) the following asymptotic expansion as $z \to 0$.

**Corollary 1.6.** *Let $a$ and $m$ be as in Theorem 1. Then for any integer $K \geq 0$, we have*

$$\gamma_m(a+z) = \gamma_m(a) + \sum_{k=1}^{K-1} \frac{(-z)^k}{k!} \sum_{j=0}^{m} \frac{\mathfrak{s}_{m-j}^k(1)}{j!} \zeta^{(j)}(k+1,a) + O(|z|^K) \tag{3.16}$$

*as $z \to 0$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.*

Theorem 2 yields the following asymptotic expansion as $z \to \infty$.

**Corollary 2.6.** *Let $a$, $\lambda$, $m$ and $n$ be as in Corollary 1.4. Then for any integer $K \geq 0$, we have the formulae*:

i) *for any integer $K \geq 0$,*

$$\begin{aligned} &\mathcal{R}_{m,1}(a+z,\lambda) \\ &\quad = \frac{\delta(\lambda)\log^{m+1} z}{m+1} - (-1)^m m! \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a, e(\lambda))}{(k+1)!} z^{-k-1} q_{m,k}(1; \log z) \\ &\quad\quad + O(|z|^{-K-1} \log^m |z|); \end{aligned} \tag{3.17}$$

ii) *for any $n \in \mathbb{N}_0$ and for any integer $K \geq n+1$,*

$$\begin{aligned} \mathcal{R}_{m,-n}(a+z,\lambda) = &-\frac{\delta(\lambda)(-1)^m m!}{(n+1)^{m+1}} z^{n+1} p_m(-n; \log z) \\ &- (-1)^m m! \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a, e(\lambda))}{(k+1)!} z^{n-k} q_{m,k}(-n; \log z) \\ &+ O(|z|^{n-K} \log^m |z|), \end{aligned} \tag{3.18}$$

*both as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the implied $O$-constants depend at most on $a$, $\lambda$, $m$, $n$, $K$ and $\delta$.*

We obtain from (3.17) the following asymptotic expansion as $z \to \infty$, in view of (3.15).

**Corollary 2.7.** *Let $a$ and $m$ be as in Theorem 2. Then for any integer $K \geq 0$, we have*

$$\begin{aligned} \gamma_m(a+z) = &\frac{(-\log z)^{m+1}}{(m+1)!} + \sum_{k=0}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a)}{(k+1)!} z^{-k-1} q_{m,k}(1; \log z) \\ &+ O(|z|^{-K-1} \log^m |z|) \end{aligned} \tag{3.19}$$

*as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.*

The case $(m,n)=(1,0)$ of (3.18) further implies upon (3.2) the following (shifted) variant of Stirling's formula (cf. [5, 1.18(12)]).

**Corollary 2.8.** *For any integer $K \geq 0$, we have*

$$\log \Gamma(a+z) = \Bigl(a+z-\frac{1}{2}\Bigr)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{K-1} \frac{(-1)^{k+1}B_{k+1}(a)}{k(k+1)} z^{-k} + O(|z|^{-K}\log|z|)$$

*as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$.*

The final corollary asserts the limit formulae for $\mathcal{R}_{m,n}(z,\lambda)$, which are also the consequences of Theorem 2.

**Corollary 2.9.** *For any $m \in \mathbb{N}_0$ and in $|\arg z| < \pi$, we have*

$$\mathcal{R}_{m,1}(z,\lambda) = \lim_{L\to+\infty}\biggl\{\frac{\delta(\lambda)e(\lambda L)\log^{m+1}L}{m+1} - \sum_{l=0}^{L-1}\frac{e(\lambda l)\log^m(z+l)}{z+l}\biggr\} \tag{3.20}$$

*for $n=1$, and for any $n \in \mathbb{N}_0$,*

$$\begin{aligned}\mathcal{R}_{m,-n}(z,\lambda) = \lim_{L\to+\infty}\Biggl[&(-1)^m m! e(\lambda L)\biggl\{\frac{L^{m+1}}{(n+1)^{m+1}}p_m(-n;\log L) \\ &- \sum_{k=0}^{n}\frac{(-1)^{k+1}B_{k+1}(z,e(\lambda))}{(k+1)!}L^{n-k}q_{m,k}(-n;\log L)\biggr\} \\ &- \sum_{l=0}^{L-1} e(\lambda l)(z+l)^n \log^m(z+l)\Biggr].\end{aligned} \tag{3.21}$$

*Remark.* The case $\lambda \in \mathbb{Z}$ of (3.20) readily implies upon (3.15) the classical limit formula for $\gamma_m(z)$ with $m=0,1,2,\ldots$ (cf. [7, 1.8 (1.123)].

## 4. Outline of the proofs

We shall show in this section the outline of the proofs of Theorems 1 and 2.

The common starting point of the proofs of Theorems 1 and 2 is the Mellin-Barnes type integral formula

$$\phi(s,a+z,\lambda) = \frac{1}{2\pi i}\int_{(u)}\frac{\Gamma(s+w)\Gamma(-w)}{\Gamma(s)}\phi(s+w,a,\lambda)z^w dw \tag{4.1}$$

for $\sigma > 1$ in the sector $|\arg z| < \pi$, where $u$ is a constant satisfying $1-\sigma < u < 0$; this was first shown by the author [13, (2.6)].

*Outline of the proof of Theorem 1.* Suppose temporarily that $\sigma > 1$. Let $u_K^+$ for any integer $K \geq 0$ be a constant satisfying $K-1 < u_K^+ < K$. Then the path in (4.1) can be moved from $(u)$ to $(u_K^+)$, upon passing over the poles of the integrand at $w=k$ $(k=0,1,\ldots,K-1)$; this yields the case $m=0$ of (2.2) with

$$\rho_K^+(s,a,\lambda;z) = \frac{1}{2\pi i}\int_{(u_K^+)}\frac{\Gamma(s+w)\Gamma(-w)}{\Gamma(s)}\phi(s+w,a,\lambda)z^w dw. \tag{4.2}$$

The temporary restriction on $\sigma$ can be relaxed at this stage to $\sigma > 1 - K$, under which $u_K^+$ can be taken as $\max(K-1, 1-\sigma) < u_K^+ < K$, and the path $(u_K^+)$ separates the poles of the integrand at $w = 1-s-k$ $(k = 0, 1, \ldots)$ and at $w = k$ $(k = 0, 1, \ldots, K-1)$, from those at $w = k$ $(k = K, K+1, \ldots)$. We now differentiate $m$-times the resulting formula, to obtain the expression in (2.2).

The remaining estimate (2.3) is derived by moving further the path in (4.2) from $(u_K^+)$ to $(u_{K+1}^+)$, and then by the $m$-times differentiation of the resulting equality. □

*Outline of the proof of Theorem 2.* Let $u_K^-$ for any integer $K \geq 0$ be a constant satisfying $-\sigma - K < u_K^- < -\sigma - K + 1$. Then the path of integration in (4.1) can be moved from $(u)$ to $(u_K^-)$, upon passing over the poles of the integrand at $w = -s-k$ $(k = -1, 0, 1, \ldots, K-1)$. Collecting the residues of the relevant poles, we obtain the case $m = 0$ of (2.10) with

$$\rho_K^-(s, a, \lambda; z) = \frac{1}{2\pi i} \int_{(u_K^-)} \frac{\Gamma(s+w)\Gamma(-w)}{\Gamma(s)} \phi(s+w, a, \lambda) z^w dw, \tag{4.3}$$

where the residues are computed by

$$\mathrm{Res}_{s=1}\, \phi(s, a, \lambda) = B_0(a, e(\lambda)) = \delta(\lambda),$$
$$\phi(-k, a, \lambda) = -\frac{B_{k+1}(a, e(\lambda))}{k+1} \qquad (k \in \mathbb{N}_0)$$

(cf. [1][13]). Here the temporary restriction on $\sigma$ can be relaxed at this stage into $\sigma > -K$, under which $u_K^-$ is taken as $-\sigma - K < u_K^- < \min(-\sigma - K + 1, 0)$, and the path $(u_K^-)$ separates the poles of the integrand at $w = k$ $(k = 0, 1, \ldots)$ and at $w = -s-k$ $(k = -1, 0, 1, \ldots, K-1)$, from those at $w = -s-k$ $(k = K, K+1, \ldots)$. The $m$-times differentiation of the resulting formula therefore gives the expression in (2.10).

The remaining estimate (2.11) is derived by moving further the path in (4.3) from $(u_K^-)$ to $(u_{K+1}^-)$, and then by the $m$-times differentiation of the resulting equality. □

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