Irrationality exponents of certain reciprocal sums

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1 Introduction

For any sequence $\{x_n\}$ of positive integers such that $x_n^2 \mid x_{n+1}$ and $x_n^2 \neq x_{n+1}$ for all sufficiently large n and $\varepsilon_n \in \{-1, 1\}$, we define the sum

$$S = 1 + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_n}.$$

In this paper we give the explicit continued fraction expansion of the sum and compute its irrationality exponent, where the irrationality exponent $\mu(\alpha)$ of a real number α is defined by the supremum of the set of numbers μ for which the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{\mu}}$$

has infinitely many rational solutions p/q. Every irrational α has $\mu(\alpha) \geq 2$. If $\mu(\alpha) > 2$, then α is transcendental by Roth's theorem. Our result is as follows (see Theorem 2 in Section 3):

$$\mu(S) = \limsup_{n \to \infty} \frac{\log x_{n+1}}{\log x_n}.$$

For the proof of Theorem 2, we first expand the partial sums

$$S_n = 1 + \sum_{k=1}^{n} \frac{\varepsilon_k}{x_k}$$

in continued fractions in the generic case $x_1 \geq 3$, $x_n^2 \mid x_{n+1}$, and $x_n^2 \neq x_{n+1}$ $(n \geq 1)$ (see Theorem 2 in Section 3), which were given by Hone [4] when $\varepsilon_n = 1$ for all $n \geq 1$. The continued fractions obtained in Theorem 1 have certain symmetric patterns; namely, if the continued fraction expansion of the nth partial sum is written using the standard notation as

$$S_n = [1; a_1, a_2, \ldots, a_{l_n}]$$

with $a_{l_n} \neq 1$, then

$$\begin{aligned} S_{n+1} &= [1; a_1, a_2, \dots, a_{l_{n+1}}] \\ &= [1; a_1, a_2, \dots, a_{l_n}, x_{n+1}/x_n^2 - 1, 1, a_{l_n} - 1, a_{l_n - 1}, \dots, a_1] \end{aligned}$$

if $\varepsilon_{n+1} = 1$, and otherwise,

$$S_{n+1} = [1; a_{l_{n+1}}, \dots, a_2, a_1]$$

(see the formula (11) in Theorem 1). By means of this recursive construction of the continued fraction expansions of S_n, we can compute the irrationality exponent of the sum $S = \lim_{n\to\infty} S_n$ using the following formula (cf., eg., [5, Theorem 1]): The irrationality exponent of the simple continued fraction $\alpha = [a_0; a_1, a_2, \dots]$ with the nth convergent $p_n/q_n = [a_0; a_1, a_2, \dots, a_n]$ is given by

$$\mu(\alpha) = 2 + \limsup_{n\to\infty} \frac{\log a_{n+1}}{\log q_n}. \tag{1}$$

The assumption $x_1 \geq 3$ in Theorem 1 is indispensable, since the minimal partial denominator in the continued fraction expansions of the sums S_n is $x_1 - 2$, which vanishes if $x_1 = 2$. In this degenerate case, we remove these zeros using the formula (6) below and obtain the simple continued fraction expansions, which will be exhibited in Theorem 3 in the final section 4.

2 Continued fraction expansion of the sums

We employ the standard notion for continued fractions:

$$[a_0; a_1, a_2, \dots] = \lim_{n\to\infty} [a_0; a_1, \dots, a_n],$$

where

$$[a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

The numerators p_n and denominators q_n of the nth convergent p_n/q_n satisfy the following relations:

$$\begin{cases} p_{-1} = 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-1}, \\ q_{-1} = 0, \quad q_0 = 1, \quad\ \ q_n = a_n q_{n-1} + q_{n-1}, \end{cases} \tag{2}$$

$$\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \dots, a_2, a_1], \tag{3}$$

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}. \tag{4}$$

We also use the formulas:

$$1-[0;a_1,a_2,\dots,a_n]=[0;1,a_1-1,a_2,\dots,a_n]\,, \tag{5}$$

$$[\dots,a,0,b,\dots]=[\dots,a+b,\dots]\,. \tag{6}$$

A continued fraction $[a_0;a_1,a_2,\dots]$ is said to be simple, if a_0 is an integer and $a_1,a_2,\dots$ are positive integers. We define the length of a finite continued fraction $[a_0;a_1,a_2,\dots,a_n]$ by n.

Theorem 1. *Let $\{x_n\}$ be a sequence of positive integers such that*

$$x_0=1,\quad x_1\ge 3,\quad x_n^2\mid x_{n+1},\quad z_{n+1}=\frac{x_{n+1}}{x_n^2}\ge 2 \qquad (n\ge 0), \tag{7}$$

and let $\varepsilon_n\in\{-1,1\}$. Then the sums

$$S_n=1+\sum_{k=1}^{n}\frac{\varepsilon_k}{x_k} \tag{8}$$

have the following simple continued fraction expansions:
Case 1. Let $\varepsilon_1=1$. Then

$$S_2=\begin{cases}[1;z_1-1,1,z_2-1,z_1] & \text{if } \ \varepsilon_2=1,\\ [1;z_1,z_2-1,1,z_1-1] & \text{if } \ \varepsilon_2=-1.\end{cases} \tag{9}$$

For $n\ge 2$, if the expansion

$$S_n=[1;a_1,a_2,\dots,a_{l_n}] \tag{10}$$

with $a_{l_n}\ne 1$ and $l_n=3\cdot 2^{n-1}-2$ $(n\ge 2)$ is given, then

$$S_{n+1}=\begin{cases}[1,a_1,\dots,a_{l_n},z_{n+1}-1,1,a_{l_n}-1,a_{l_n-1},\dots,a_1] & \text{if } \ \varepsilon_{n+1}=1,\\ [1,a_1,\dots,a_{l_n-1},a_{l_n}-1,1,z_{n+1}-1,a_{l_n},\dots,a_1] & \text{if } \ \varepsilon_{n+1}=-1\end{cases} \tag{11}$$

with length $l_{n+1}=2l_n+2$.
Case 2. Let $\varepsilon_1=-1$. Then

$$S_n=[0;1,b_1-1,b_2,\dots,b_{l_n}] \tag{12}$$

with $b_{l_n}\ne 1$, where the expansion

$$[1;b_1,b_2,\dots,b_{l_n}]=1-\sum_{k=1}^{n}\frac{\varepsilon_k}{x_k}$$

is given by Case 1.

Corollary 1. *Make the same notations as in Theorem 1. Then*

$$1+\sum_{n=1}^{\infty}\frac{\varepsilon_n}{x_n}=\begin{cases}\lim\limits_{n\to\infty}[1;a_1,a_2,\ldots,a_{l_n}] & \text{if } \varepsilon_1=1,\\ \lim\limits_{n\to\infty}[0;1,b_1-1,b_2,\ldots,b_{l_n}] & \text{if } \varepsilon_1=-1.\end{cases}$$

Theorem 1 follows immediately from the following formulas.

Lemma 1 (cf. [6]). *Let $A, a_1, a_2, \ldots, a_k$ be positive real numbers and let $p_k/q_k = [0; a_1, a_2, \ldots, a_k]$. Assume that $a_k > 1$ and $A > 1$. Then*

$$[0;a_1,a_2,\ldots,a_k,A-1,1,a_k-1,a_{k-1},\ldots,a_2,a_1]=\frac{p_k}{q_k}+\frac{(-1)^k}{Aq_k^2}, \tag{13}$$

$$[0;a_1,a_2,\ldots,a_{k-1},a_k-1,1,A-1,a_k,\ldots,a_2,a_1]=\frac{p_k}{q_k}-\frac{(-1)^k}{Aq_k^2}. \tag{14}$$

Proof of Theorem 1. The expansions (9) can be obtained by direct calculation. Noting that $x_k \mid x_{k+1}$, we have by (8) and (10) $x_n = q_{l_n}$ $(n \geq 1)$.
Case 1. Let $\varepsilon_1 = 1$. Assume first that $\varepsilon_{n+1} = 1$. Applying the formula (13) with $k = l_n$, $A = z_{n+1}$, and $q_{l_n} = x_n$, we get

$$[1;a_1,\ldots,a_{l_n},z_{n+1}-1,1,a_{l_n}-1,a_{l_n-1},\ldots,a_1] \;=\; \frac{p_{l_n}}{q_{l_n}}+\frac{(-1)^{l_n}}{z_{n+1}q_{l_n}^2}=S_n+\frac{1}{x_{n+1}}=S_{n+1}.$$

Similarly, we can prove (11) with $\varepsilon_{n+1} = -1$ using (14).
Case 2. Let $\varepsilon_1 = -1$. The expansion (12) follows from Case 1 and the formula (5), and the proof is completed. □

3 Irrationality exponent of the sum

Theorem 2. *Let $\{x_n\}$ be a sequence of positive integers such that*

$$x_n^2 \mid x_{n+1}, \quad x_n^2 \neq x_{n+1} \tag{15}$$

for all sufficiently large n and let $\varepsilon_n \in \{-1, 1\}$. Then the irrationality exponent of the sum

$$S = 1+\sum_{n=1}^{\infty}\frac{\varepsilon_n}{x_n} \tag{16}$$

is given by

$$\mu(S)=\limsup_{n\to\infty}\frac{\log x_{n+1}}{\log x_n}. \tag{17}$$

Corollary 2. *The sum S as in (16) is transcendental, if*

$$\mu(S)=\limsup_{n\to\infty}\frac{\log x_{n+1}}{\log x_n}>2. \tag{18}$$

Corollary 3. *Let $\{x_n\}$ and $\{\varepsilon_n\}$ be as in Theorem 2. Then*

$$\mu\left(1+\sum_{n=1}^{\infty}\frac{\varepsilon_n}{x_n^l}\right)=\limsup_{n\to\infty}\frac{\log x_{n+1}}{\log x_n}\qquad (l=1,2,\ldots).$$

For the proof of Theorem 2, we need the following lemma (cf., eg., [3, Lemma 1]):

Lemma 2. *If α is an irrational number, then*

$$\mu(\alpha)=\mu\left(\frac{a\alpha+b}{c\alpha+d}\right)$$

for any integers a, b, c, and d with $ad-bc\neq 0$.

Proof of Theorem 2. We may assume in view of Lemma 2 that $\{x_n\}$ fulfills (7). So we can apply Theorem 1 to the sum $S=\lim_{n\to\infty}S_n$.
Case 1. Let $\varepsilon_1=1$. Then by (10) and (11), we have

$$\max_{1<k\le l_{n+1}}\frac{\log a_k}{\log q_{k-1}}=\max\left\{\max_{1<k\le l_n}\frac{\log a_k}{\log q_{k-1}},\frac{\log(z_{n+1}-1)}{\log q_{l_n}}\right\}$$

if $\varepsilon_{n+1}=1$. Otherwise, namely, if $\varepsilon_{n+1}=-1$, the last formula holds with the equality replaced by the inequality $\le$. Hence, we obtain

$$\limsup_{k\to\infty}\frac{\log a_k}{\log q_{k-1}}=\limsup_{n\to\infty}\frac{\log(z_{n+1}-1)}{\log x_n}=\limsup_{n\to\infty}\frac{\log x_{n+1}}{\log x_n}-2,$$

and the formula (1) yields (17).
Case 2. Let $\varepsilon_1=-1$. Then

$$\mu(S)=\mu(2-S)=\mu\left(1+\frac{1}{x_1}+\sum_{k=2}^{\infty}\frac{-\varepsilon_k}{x_k}\right)=\limsup_{n\to\infty}\frac{\log x_{n+1}}{\log x_n}$$

by Lemma 2 and Case 1, and the proof is completed. □

4 Continued fraction expansions in the degenerate case

In this section we give the continued fraction expansions of the sums S_n in the case $x_1=2$. We focus on the case $\varepsilon_1=1$, since the other case can be dealt with by using the formula (5). By the formulas (10) and (11) in Theorem 1, partial denominators a_k $(1\le k\le l_{n+1})$ in the expansion of S_{n+1} consist of those in the expansion of S_n, namely, a_k $(1\le k\le l_n)$, plus 1, $z_{n+1}-1(\neq 0)$, and $a_{l_n}-1$. We start with the expansions of S_3 with length $l_3=10$.

Example 1. *The continued fraction expansions of* S_3 *with* $\varepsilon_1 = 1$.

$$\begin{array}{lll} [1; z_1-1, 1, z_2-1, z_1, z_3-1, 1, z_1-1, z_2-1, 1, z_1-1] & \textit{if} & (\varepsilon_2, \varepsilon_3) = (1,1), \\ [1; z_1-1, 1, z_2-1, z_1-1, 1, z_3-1, z_1, z_2-1, 1, z_1-1] & \textit{if} & (\varepsilon_2, \varepsilon_3) = (1,-1), \\ [1; z_1, z_2-1, 1, z_1-1, z_3-1, 1, z_1-2, 1, z_2-1, z_1] & \textit{if} & (\varepsilon_2, \varepsilon_3) = (-1,1), \\ [1; z_1, z_2-1, 1, z_1-2, 1, z_3-1, z_1-1, 1, z_2-1, z_1] & \textit{if} & (\varepsilon_2, \varepsilon_3) = (-1,-1). \end{array}$$

Example 1 implies that, if $\varepsilon_2 = -1$, there is only one zero $z_1 - 2$ in the expansions of S_3 and $a_1 = a_{l_n} = z_1$ for all $n \geq 3$. Hence, since $a_{l_n} - 1 = z_1 - 1 \neq 0$, all zeros appearing in the expansion of S are generated from the zero $z_1 - 2$ in S_3 by the recursive procedure from (10) to (11). On the other hand, if $\varepsilon_2 = 1$, there is no zero in the expansions of S_3. To study this case more precisely, we observe:

Example 2. *The continued fraction expansions of* S_4 *with* $(\varepsilon_1, \varepsilon_2) = (1,1)$.

$$\begin{array}{lll} [1; a_1, \dots, a_9, z_1-1, z_4-1, 1, z_1-2, a_9, \dots, a_1] & \textit{if} & \varepsilon_4 = 1, \\ [1; a_1, \dots, a_9, z_1-2, 1, z_4-1, z_1-1, a_9, \dots, a_1] & \textit{if} & \varepsilon_4 = -1 \end{array}$$

with length $l_4 = 22$, *where the expansions* $S_3 = [1; a_1, \dots, a_{10}]$ *are given in Example 1 with* $\varepsilon_2 = 1$.

Example 1 and 2 with (11) imply that, if $\varepsilon_2 = 1$, there is only one zero $z_1 - 2$ in the expansions of S_4 and $a_1 = a_{l_n} = z_1 - 1$ for all $n \geq 3$. Hence each of the expansions of S_{n+1} $(n \geq 4)$ contains zeros which come from that of S_n plus one new zero $a_{l_n} - 1 = z_1 - 2$.

In this way, we can locate all zeros, namely, $z_1 - 2$, in the expansions of S and remove them using the formula (6). Rewriting the continued fractions of the form $[\dots, 1, z_1 - 1]$ as $[\dots, 2]$, we obtain:

Theorem 3. *Let* $\{x_n\}$ *be a sequence of positive integers such that*

$$x_1 = 2, \quad x_n^2 | x_{n+1}, \quad z_{n+1} = \frac{x_{n+1}}{x_n^2} \geq 2 \ \ (n \geq 1),$$

and let $\varepsilon_n \in \{-1, 1\}$. *Then the sums*

$$T_n = 1 + \sum_{k=1}^{n} \frac{\varepsilon_k}{x_k}$$

have the following simple continued fraction expansions:
Case 1.1. Let $(\varepsilon_1, \varepsilon_2) = (1,1)$. *Then*

$$T_3 = \begin{cases} [1; 1, 1, z_2-1, 2, z_3-1, 1, 1, 1, z_2-1, 2] & \textit{if} \quad \varepsilon_3 = 1, \\ [1; 1, 1, z_2-1, 1, 1, z_3-1, 2, z_2-1, 2] & \textit{if} \quad \varepsilon_3 = -1 \end{cases}$$

with length 9. *For* $n \geq 3$, *if the expansion* $T_n = [1; 1, 1, a_3, \ldots, a_{t_n-1}, 2]$ *with* $t_n = 5{\cdot}2^{n-2} - 1$ *is given, then*

$$T_{n+1} = \begin{cases} [1; 1, 1, a_3, \ldots, a_{t_n-1}, 1, 1, z_{n+1} - 1, 2, a_{t_n-2}, \ldots, a_3, 2] & \text{if} \quad \varepsilon_{n+1} = 1, \\ [1; 1, 1, a_3, \ldots, a_{t_n-2}, 2, z_{n+1} - 1, 1, 1, a_{t_n-1}, \ldots, a_3, 2] & \text{if} \quad \varepsilon_{n+1} = -1 \end{cases}$$

with length $t_{n+1} = 2t_n + 1$.
Case 1.2. Let $(\varepsilon_1, \varepsilon_2) = (1, -1)$. *Then*

$$T_3 = \begin{cases} [1; 2, z_2 - 1, 1, 1, z_3 - 1, 2, z_2 - 1, 2] & \text{if} \quad \varepsilon_3 = 1, \\ [1; 2, z_2 - 1, 2, z_3 - 1, 1, 1, z_2 - 1, 2] & \text{if} \quad \varepsilon_3 = -1 \end{cases}$$

with length 8. *For* $n \geq 3$, *if the expansion* $T_n = [1; 2, a_2, \ldots, a_{t_n-1}, 2]$ *with length* $t_n - 1$ *is given, then*

$$T_{n+1} = \begin{cases} [1; 2, a_2, \ldots, a_{t_n-1}, 2, z_{n+1} - 1, 1, 1, a_{t_n-2}, \ldots, a_2, 2] & \text{if} \quad \varepsilon_{n+1} = 1, \\ [1; 2, a_2, \ldots, a_{t_n-2}, 1, 1, z_{n+1} - 1, 2, a_{t_n-1}, \ldots, a_2, 2] & \text{if} \quad \varepsilon_{n+1} = -1 \end{cases}$$

with length $t_{n+1} - 1$.
Case 2. Let $\varepsilon_1 = -1$. *Then*

$$T_n = \begin{cases} [0; 2, b_2, \ldots, b_{t_n}] & \text{if} \quad \varepsilon_2 = -1, \\ [0; 1, 1, b_2, \ldots, b_{t_n-1}] & \text{if} \quad \varepsilon_2 = 1, \end{cases}$$

where the expansion

$$1 - \sum_{k=1}^{n} \frac{\varepsilon_k}{x_k} = [1; b_1, b_2, \ldots, b_{u_n}]$$

with $u_n = t_n$ *if* $\varepsilon_2 = -1$, $= t_{n-1}$ *if* $\varepsilon_2 = 1$ *is given by Case 1.1 or 1.2.*

After the conference Amou kindly sent the last named author his joint paper [1] with Bugeaud, in which our Theorem 2 was already generalized (see [1, Lemma 3]). Recently, the authors proved the following theorems which improves the result in [1].

Theorem ([2, Theorem 1]). *Let* $\{x_n\}$ *be a sequence of rational numbers greater than one and let* $\varepsilon_n \in \{1, 1\}$ *with* $\varepsilon_1 = 1$. *Put* $z_1 = x_1$, $z_{n+1} = x_{n+1}x_n^{-2}$ $(n \geq 1)$ *and define*

$$\delta_1 = den\, z_1, \qquad \delta_{n+1} = \delta_n^2 den\, z_{n+1} \qquad (n \geq 1). \tag{19}$$

Assume that the following two conditions hold:

(i) $z_n \geq 1$ *for all sufficiently large* n,

(ii) $\log \delta_{n+1} = o(\log x_n)$.

Then the irrationality exponent of the number

$$S = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_n}$$

is equal to

$$\tau = \limsup_{n\to\infty} \frac{\log x_{n+1}}{\log x_n}.$$

Theorem ([2, Theorem 2])**.** *Make the same notations as in Theorem 2. Put $x_n = t_n/s_n$ with $t_n, s_n \in \mathsf{Z}_{>0}$. Assume that the following two conditions hold:*

(i)′ $s_n^2 | s_{n+1}, t_n^2 | t_{n+1}$ *for all sufficiently large n,*

(ii)′ $\log s_{n+1} = o(\log t_n)$.

Then the irrationality exponent of the number S is equal to

$$\tau = \limsup_{n\to\infty} \frac{\log t_{n+1}}{\log t_n}.$$

References

[1] M. Amou and Y. Bugeaud, *Expansions in integer basis and exponents of Diophantine approximation*, London Math. Soc. **81** (2010), 297–316.

[2] D. Duverney, T. Kurosawa, and I. Shiokawa, *Irrationality exponents of certain sums of rational numbers*, preprint.

[3] D. Duverney and I. Shiokawa, *Irrationality exponents of numbers related with Cahen's constant*, Mh. Math. (to appear).

[4] A. N. W. Hone, *On the continued fraction expansion of certain Engel series*, J. Number theory **164** (2016), 269–281.

[5] J. Sondow, *Irrationality measures, irrationality basis, and a theorem of Jarnik*, 2004, `http://arxiv.org/abs/math/0406300v1`.

[6] A. J. van der Poorten and J. Shallit, *Folded continued fractions*, J. Number Theory **40** (1992), 237–259.

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