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# Extreme Point Characterizations of Closure Spaces *

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## 1. Introduction

Let $X$ be a finite set. We call mapping $\tau: 2^X \to 2^X$ a *closure operator* if $\tau$ satisfies the following conditions.

(C1) $\forall A \subseteq X: A \subseteq \tau(A)$. (Extensionality)

(C2) $\forall A, B \subseteq X: A \subseteq B \Longrightarrow \tau(A) \subseteq \tau(B)$. (Monotonicity)

(C3) $\forall A \subseteq X: \tau(\tau(A)) = \tau(A)$. (Idempotence)

A pair $(X, \tau)$ of a finite set $X$ and a closure operator $\tau: 2^X \to 2^X$ is called a *closure space* (see [3]). A closure space $(X, \tau)$ is a *matroid* if $\tau$ satisfies the following (Steinitz-MacLane) Exchange Axiom:

(EA) $\forall A \subseteq X, \forall q \notin \tau(A): q \in \tau(A \cup p) \Longrightarrow p \in \tau(A \cup q)$

(see Welsh [10] and Oxley [8]). On the other hand, a closure space $(X, \tau)$ is called an *antimatroid* (or *convex geometry*) if $\tau$ satisfies $\tau(\emptyset) = \emptyset$ and the following Antiexchange Axiom:

(AE) $\forall A \subseteq X, \forall p, q \notin \tau(A)$ with $p \neq q: q \in \tau(A \cup p) \Longrightarrow p \notin \tau(A \cup q)$.

See Edelman and Jamison [2] and Korte, Lovász and Schrader [4] for surveys and examples of antimatroids.

The *extreme point operator* $\mathrm{ex}: 2^X \to 2^X$ of a closure space $(X, \tau)$ is defined as $\mathrm{ex}(A) = \{p \mid p \in A, p \notin \tau(A - p)\}$ $(A \subseteq X)$. As the name suggests, the concept of extreme point had first appeared in the context of antimatroid. However, this concept can be applied to general closure spaces. For example, if $(X, \tau)$ is a matroid, $\mathrm{ex}(A)$ is the set of isthmuses of $A$ for each $A \subseteq X$ (see Remark 2.2 below).

We characterize extreme point operators of closure spaces as follows.

**Theorem 1.1**: *A mapping $S: 2^X \to 2^X$ is the extreme point operator of a closure space if and only if $S$ satisfies the following* (Ex1)-(Ex3).

(Ex1) $\forall A \subseteq X: S(A) \subseteq A$. (Intensionality)

(Ex2) $A \subseteq B \subseteq X \Longrightarrow S(B) \cap A \subseteq S(A)$. (Chernoff property)

(Ex3) $\forall A \subseteq X, \forall p, q \notin A: (p \notin S(A \cup p), q \in S(A \cup q)) \Longrightarrow q \in S(A \cup p \cup q)$.

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□

As corollaries of Theorem 1.1, we have following characterizations of the extreme point operators of matroids and antimatroids, respectively.

**Theorem 1.2**: *A mapping $S: 2^X \to 2^X$ is the extreme point operator of a matroid if and only if $S$ satisfies* (Ex1)-(Ex3) *and the following* (Ex4).

(Ex4) $\forall A \subseteq X, \forall p \in X: p \in S(A \cup p) \Longrightarrow S(A \cup p) \supseteq S(A) \cup p$.

□

**Theorem 1.3**: *A mapping $S: 2^X \to 2^X$ is the extreme point operator of an antimatroid if and only if $S$ satisfies* (Ex0)-(Ex2) *and* (Ex5), *where Conditions* (Ex0) *and* (Ex5) *are defined as follows.*

(Ex0) $\forall p \in X: S(\{p\}) = \{p\}$. (Singleton Identity)

(Ex5) $\forall A, B \subseteq X: S(B) \subseteq A \subseteq B \Longrightarrow S(A) \subseteq S(B)$. (Aizerman's Axiom)

□

Conditions (Ex2) and (Ex5) in Theorem 1.3 is easily seen to be equivalent to path-independent condition. However, it is natural in view of Theorem 1.1 to list conditions (Ex0)-(Ex2) and (Ex5) since, as we shall see, Aizerman's Axiom [1] is a strengthening of Condition (Ex3).

The rest of this paper is organized as follows. In Section 2, we collect previously known results on extreme point operator of closure spaces and antimatroids. In Section 3, we give a proof of Theorem 1.1. In Section 4, we prove Theorems 1.2 and 1.3. In Section 5, we discuss relationship between Theorem 1.3 and the result of Koshevoy [5].

## 2. Preliminaries

In this section, we collect important lemmas concerning extreme point operators of closure spaces and antimatroids, which will be useful in the subsequent sections.

Extreme point operators of closure spaces can be described as follows.

**Lemma 2.1**: *Suppose that $(X, \tau)$ is a closure space. Then, for each $A \subseteq X$ we have*

$$\mathrm{ex}(A) = \bigcap \{B \mid B \subseteq A, \tau(B) = \tau(A)\}.$$

(Proof) Let $p$ be an extreme point of $A$. Suppose that $B \subseteq A$ and $\tau(B) = \tau(A)$. If $p \notin B$, then since $B \subseteq A - p$, we have $\tau(B) \subseteq \tau(A-p) \subsetneq \tau(A)$, a contradiction. We thus have inclusion $\subseteq$.

Conversely, if $p \in A$ is not an extreme point of $A$, we have $\tau(A-p) = \tau(A)$. Hence, inclusion $\supseteq$ holds. □

Lemma 2.1 is partly due to Edelman and Jamison [2].

For a closure space $(X, \tau)$ and $A \subseteq X$ is called *spanning* if $\tau(A) = X$. For $A \subseteq X$ the *restriction* of $(X, \tau)$ by $A$ is the closure space $(A, \tau_A)$ defined by

$$\tau_A(C) = \tau(C) \cap A \quad (C \subseteq A).$$

**Remark 2.2**: For a matroid $(X,\tau)$ $p \in X$ is called an *isthmus* if $p \in B$ for each spanning set $B$ of $X$. Lemma 2.1 shows that $\mathrm{ex}(A)$ is the set of isthmuses of $(X,\tau_A)$.

The following proposition shows that the extreme point operator of a closure space has an important property called the *Chernoff property* (see Moulin [7]).

**Proposition 2.3** (Chernoff property [9]): *Let $(X,\tau)$ be a closure space. If $A \subseteq B \subseteq X$, we have* $\mathrm{ex}(B) \cap A \subseteq \mathrm{ex}(A)$.

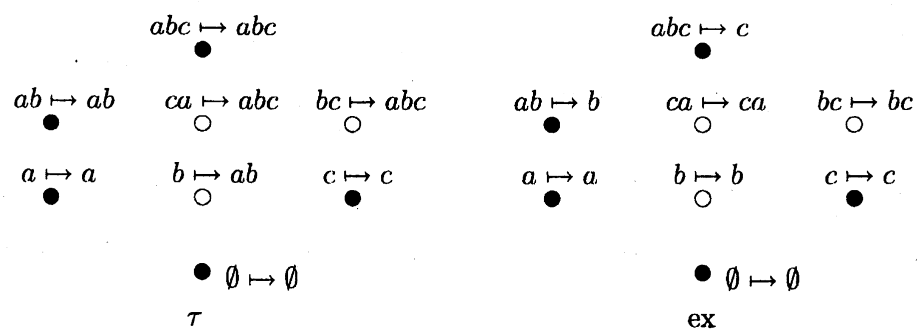
(Proof) If $p \in \mathrm{ex}(B) \cap A$, we have $p \notin \tau(B-p)$. Since $\tau(A-p) \subseteq \tau(B-p)$, we have $p \notin \tau(A-p)$, and hence, we have $p \in \mathrm{ex}(A)$. □

The extreme point operator of a closure space is idempotent as is shown in the following proposition.

**Proposition 2.4** (Idempotency): *Let $(X,\tau)$ be a closure space. We have* $\mathrm{ex}(\mathrm{ex}(A)) = \mathrm{ex}(A)$ *for each* $A \subseteq X$.

(Proof) Since we have $\mathrm{ex}(A) \subseteq A$, it follows from Lemma 2.3 that $\mathrm{ex}(A) = \mathrm{ex}(A) \cap \mathrm{ex}(A) \subseteq \mathrm{ex}(\mathrm{ex}(A))$. □

**Example 2.5**: Consider the closure space $(X,\tau)$ depicted in the left-hand side of the following figure, where $X = \{a,b,c\}$. The associated extreme point operator is shown in the right-hand side.




Antimatroids can be characterized in many ways. Among them is the following due to Edelman and Jamison [2].

For a closure space $(X,\tau)$, a subset $K \subseteq X$ is called *closed* if $\tau(K) = K$.

**Theorem 2.6** (Edelman and Jamison [2]): *Let $(X,\tau)$ be a closure space with $\tau(\emptyset) = \emptyset$. The following conditions are equivalent.*

(a) *$(X,\tau)$ is an antimatroid.*

(b) $\forall A \subseteq X : \tau(A) = \tau(\mathrm{ex}(A))$.

(c) *For each closed set $K$ and $p \notin K$, we have* $p \in \mathrm{ex}(\tau(K \cup p))$.

□

Condition (b) in the above theorem is called the *(finite) Minkowski-Krein-Milman property.*

**Lemma 2.7** (Monjardet and Raderanirina [6, Theorem 2]): *Let $(X, \tau)$ be a closure space. For each $A \subseteq X$, we have* $\mathrm{ex}(\tau(A)) \subseteq \mathrm{ex}(A)$.

(Proof) Let $A \subseteq X$. Since $A \subseteq \tau(A) = \tau(\tau(A))$, we have

$$\{B \mid B \subseteq A, \tau(B) = \tau(A)\} \subseteq \{B \mid B \subseteq \tau(A), \tau(B) = \tau(\tau(A))\}.$$

It follows from Lemma 2.1 that $\mathrm{ex}(\tau(A)) \subseteq \mathrm{ex}(A)$. □

We have the following variant of the Minkowski-Krein-Milman property, where $\tau$ and ex are transposed.

**Lemma 2.8** (Monjardet and Raderanirina [6, Proposition 5]): *A closure space $(X, \tau)$ with $\tau(\emptyset) = \emptyset$ is an antimatroid if and only if for each $A \subseteq X$ we have* $\mathrm{ex}(A) = \mathrm{ex}(\tau(A))$.

(Proof) If $(X, \tau)$ is an antimatroid, then it follows from Theorem 2.6(b) that $\tau(\mathrm{ex}(\tau(A))) = \tau(A)$. Also, we have $\mathrm{ex}(\tau(A)) \subseteq \mathrm{ex}(A) \subseteq A$ by Lemma 2.7. Therefore, we have from Lemma 2.1 that $\mathrm{ex}(A) \subseteq \mathrm{ex}(\tau(A))$.

Conversely, suppose that $(X, \tau)$ is not an antimatroid. Then, by Theorem 2.6(c), there exists a closed $K$ and $p \notin K$ such that $p \notin \mathrm{ex}(\tau(K \cup p))$. However, since we have $p \in \mathrm{ex}(K \cup p)$ by definition of ex, it follows that $\mathrm{ex}(K \cup p) \supsetneq \mathrm{ex}(\tau(K \cup p))$. □

## 3. Extreme point operator of closure spaces

In this section, we give a proof of Theorem 1.1. The following proposition proves the "only if" part of the theorem.

**Proposition 3.1**: *Let $(X, \tau)$ be a closure space and $S: 2^X \to 2^X$ be its extreme point operator. Then, there hold Conditions* (Ex1)–(Ex3).

(Proof) (Ex1) is clear from the definition of extreme point operator. Condition (Ex2) follows from Proposition 2.3.

Let us show (Ex3). Suppose that $p, q \notin A, p \notin S(A \cup p)$ and $q \in S(A \cup q)$. Then, by definition of $S$, we have $p \in \tau(A)$ and $q \notin \tau(A)$. Therefore, we have $\tau(A \cup p) = \tau(A) \not\ni q$, and hence, $q \in S(A \cup p \cup q)$. □

For a mapping $S: 2^X \to 2^X$ define $\tau_S: 2^X \to 2^X$ by

$$\tau_S(A) = A \cup \tilde{A} \quad (A \subseteq X), \tag{3.1}$$

where

$$\tilde{A} = \{q \mid q \notin A, q \notin S(A \cup q)\} \tag{3.2}$$

for each $A \subseteq X$.

**Lemma 3.2**: *Suppose that $S: 2^X \to 2^X$ satisfies Conditions* (Ex1)–(Ex3). *Then, mapping $\tau_S: 2^X \to 2^X$ defined in* (3.1) *is a closure operator.*

(Proof) By its definition, $\tau_S$ satisfies Extensionality (C1). It remains to show Monotonicity (C2) and Idempotence (C3).

We first show (C2). Suppose $A \subseteq B \subseteq X$. Let $p \in \tau_S(A)$. If $p \in B$, then $p \in \tau_S(B)$ and we are done. Suppose $p \notin B$. Invoking (Ex2) to inclusion $A \cup p \subseteq B \cup p$, we have $S(B \cup p) \cap (A \cup p) \subseteq S(A \cup p)$. Since $p \notin S(A \cup p)$, we have $p \notin S(B \cup p)$, and hence, $p \in \tau_S(B)$.

Next we show (C3). Let $A \subseteq X$. It suffices to show that $\widetilde{\tau_S(A)} = \widetilde{A \cup \tilde{A}} = \emptyset$. Suppose that $q \notin A \cup \tilde{A}$.

We prove by induction on $|B|$ that $q \in S(A \cup B \cup q)$ for each $B \subseteq \tilde{A}$. This is trivially true for $B = \emptyset$ since we have $q \in S(A \cup q)$ by definition of $\tilde{A}$. Suppose $\emptyset \neq B \subseteq \tilde{A}$ and let $p \in B$. We have $p, q \notin A \cup (B - p)$. By the induction hypothesis, we have $q \in S(A \cup (B - p) \cup q)$. Since $A \cup p \subseteq A \cup B$, we have by (Ex2) that

$$S(A \cup B) \cap (A \cup p) \subseteq S(A \cup p).$$

Since $p \in \tilde{A}$, we have $p \notin S(A \cup p)$, and hence, $p \notin S(A \cup B) = S(A \cup (B - p) \cup p)$. By (Ex3), we have $q \in S(A \cup (B - p) \cup p \cup q) = S(A \cup B \cup q)$.

We have $q \in S(A \cup \tilde{A} \cup q)$ in particular. Since $q \notin A \cup \tilde{A}$ is arbitrary, we have $\widetilde{\tau_S(A)} = \emptyset$. This completes the proof of the present lemma. □

Note that the set $\mathcal{L}$ of closed subsets of closure space $(X, \tau_S)$ is given by

$$\mathcal{L} = \{A \mid A \subseteq X, \forall p \in X - A: p \in S(A \cup p)\}. \tag{3.3}$$

by definition (3.1) of $\tau_S$.

The next theorem proves the "if" part of Theorem 1.1.

**Theorem 3.3**: *Suppose that a mapping* $S: 2^X \to 2^X$ *satisfies* (Ex1)–(Ex3). *Then,* $(X, \tau_S)$ *defined by* (3.1) *is a closure space with its extreme point operator being* $S$.

(Proof) Lemma 3.2 shows that $(X, \tau_S)$ is a closure space.

Let $\mathrm{ex}: 2^X \to 2^X$ be the extreme point operator of $(X, \tau_S)$. We shall show $\mathrm{ex}(A) = S(A)$ for each $A \subseteq X$. Suppose $A \subseteq X$.

Let $p \in \mathrm{ex}(A)$. We have $p \in A$ and $p \notin \tau_S(A - p)$. By definition of $\tau_S$, we have $p \in S((A - p) \cup p) = S(A)$. Conversely, let $p \in S(A)$. Then, by definition of $\tau_S$, we have $p \notin \tau_S(A - p)$. Since $p \in A$ due to (Ex1), we conclude that $p \in \mathrm{ex}(A)$. □

## 4. Extreme point operators of matroids and antimatroids

In this section, we prove Theorems 1.2 and 1.3.

We first prove Theorem 1.2 concerning extreme point operators of matroids.

(Proof of Theorem 1.2) Suppose that $S: 2^X \to 2^X$ is the extreme point operator of a matroid $(X, \tau)$. Let $A \subseteq X$, $p \notin A$ and $p \in S(A \cup p)$. We have to show that $S(A \cup p) \supseteq S(A) \cup p$. Let $q \in S(A)$ and suppose, on the contrary, that $q \notin S(A \cup p)$. Then, by definition of $S$, we have $q \notin \tau(A - q)$ and $q \in \tau(A - q \cup p)$. It follows from Exchange Axiom that $p \in \tau(A - q \cup q) = \tau(A)$. This means that $p \notin S(A \cup p)$, a contradiction.

Conversely, suppose $S: 2^X \to 2^X$ satisfies (Ex1)–(Ex4). We know from Theorem 1.1 that $S$ is the extreme point operator of a closure space $(X, \tau)$. Hence, it suffices to show that $\tau$ satisfies Exchange Axiom (EA).

Suppose that $p \in \tau(A \cup q) - \tau(A)$. Since $p \notin \tau(A)$, we have $p \notin A$ and $p \in S(A \cup p)$. Then, we have, by (Ex4), that $S(A \cup p) \supseteq S(A) \cup p$. Suppose, on the contrary, that we have $q \in S(A \cup p \cup q)$. Then,

$$S(A \cup p \cup q) \supseteq S(A \cup p) \cup q \supseteq S(A) \cup p \cup q.$$

However, since $p \in \tau(A \cup q)$, we have $p \notin S(A \cup p \cup q)$, a contradiction. Therefore, we have $q \notin S(A \cup p \cup q)$, and hence, $q \in \tau(A \cup p)$. □

Next, we consider extreme point operators of antimatroids.

**Proposition 4.1** (see Mounlin [7]): *Condition* (Ex2) *is equivalent to any one of the following four conditions, provided that* (Ex1) *holds.*

(Ex2a) $\forall A, B \subseteq X: S(A \cup B) \subseteq S(S(A) \cup B)$.

(Ex2b) $\forall A, B \subseteq X: S(A \cup B) \subseteq S(S(A) \cup S(B))$.

(Ex2c) $\forall A, B \subseteq X: S(A \cup B) \subseteq S(A) \cup S(B)$.

(Ex2d) $\forall A, B \subseteq X: S(A \cup B) \subseteq S(A) \cup B$.

□

The following lemma shows that Condition (Ex3) is a weakening of Aizerman's Axiom (Ex5).

**Lemma 4.2**: *Condition* (Ex5) *implies Condition* (Ex3), *provided that Conditions* (Ex1)–(Ex2) *hold.*

(Proof) Suppose that a mapping $S: 2^X \to 2^X$ satisfies Conditions (Ex1), (Ex2) and (Ex5). Let us consider $A \subseteq X$ and $p, q \notin A$ such that $p \notin S(A \cup p)$ and $q \in S(A \cup q)$. Then, it follows from Proposition 4.1 and (Ex2) that

$$S(A \cup p \cup q) \subseteq S(A \cup p) \cup q \subseteq A \cup q.$$

Applying (Ex5) to the inclusions

$$S(A \cup p \cup q) \subseteq A \cup q \subseteq A \cup p \cup q,$$

we have $q \in S(A \cup q) \subseteq S(A \cup p \cup q)$. □

**Theorem 4.3**: *Suppose that* $S: 2^X \to 2^X$ *satisfies* (Ex0)–(Ex2) *and* (Ex5). *Then,* $(X, \tau_S)$ *is an antimatroid with its extreme point operator being* $S$.

(Proof) We have from Lemma 4.2 and Theorem 3.3 that $(X, \tau_S)$ is a closure space and that $S$ is the extreme point operator of $(X, \tau_S)$. Therefore, it suffices to show that $(X, \tau_S)$ is an antimatroid. We show that $(X, \tau_S)$ satisfies the condition in Lemma 2.8.

Let $A \subseteq X$ be arbitrary. We have from Theorem 3.3 and Lemma 2.7 that

$$S(\tau_S(A)) \subseteq S(A) \subseteq \tau_S(A). \tag{4.1}$$

Applying Aizerman's Axiom (Ex5) to (4.1), we have $S(A) = S(S(A)) \subseteq S(\tau_S(A))$, where the equation follows from Proposition 2.4. Since we have $\tau_S(\emptyset) = \emptyset$ by (Ex0), it follows from Lemma 2.8 that $(X, \tau_S)$ is an antimatroid. □

(Proof of Theorem 1.3) The "if" part of the theorem follows from Theorem 4.3.

Let us show the "only if" part. Let $S$ be the extreme point operator of an antimatroid $(X, \tau)$. Since an antimatroid is a closure space, we have (Ex1)–(Ex2) by Proposition 3.1. Also, since $\tau(\emptyset) = \emptyset$, we have $S(\{p\}) = \{p\}$ for each $p \in X$.

To show (Ex5), let us suppose $S(B) \subseteq A \subseteq B$. Then, it follows from the monotonicity of $\tau$ and Theorem 2.6(b) that

$$\tau(B) = \tau(S(B)) \subseteq \tau(A) \subseteq \tau(B),$$

and hence, we have $S(A) = S(B)$ by Lemma 2.8. □

## 5. Concluding remarks

A *choice function* on $X$ is a mapping $S: 2^X \to 2^X$ satisfying the following two conditions (see Moulin [7]).

(Ex1) $S(A) \subseteq A \quad (A \subseteq X)$. (Intensionality)

(NE) $S(A) \neq \emptyset \quad (\emptyset \neq A \subseteq X)$. (Nonemptiness)

Koshevoy [5] characterized extreme point operators of antimatroids as path-independent choice functions as follows.

**Theorem 5.1** (Koshevoy [5]): *A mapping $S: 2^X \to 2^X$ is the extreme point operator of an antimatroid if and only if $S$ satisfies* (Ex1), (NE) *and the following* (PI).

(PI) $\forall A, B \subseteq X: S(A \cup B) = S(S(A) \cup S(B))$. (Path Independence)

□

Path-independent property (PI) decomposes into Chernoff property (Ex2) and Aizerman's Axiom (Ex5) as the following lemma shows.

**Lemma 5.2** (Aizerman and Malishevski [1]; see also Moulin [7]): *Condition* (PI) *is equivalent to Conditions* (Ex2) *and* (Ex5), *provided that* (Ex1) *holds.* □

The following proposition shows the equivalence of Theorem 1.3 and Theorem 5.1.

**Proposition 5.3**: *The set of Conditions* (Ex0), (Ex1), (Ex2) *and* (Ex5) *is equivalent to that of Conditions* (Ex1), (NE) *and* (PI).

(Proof) Suppose that $S: 2^X \to 2^X$ satisfies (Ex1), (NE) and (PI). Then, by Lemma 5.2, $S$ satisfies (Ex2) and (Ex5). Also, Conditions (Ex1) and (NE) implies (Ex0).

Conversely, suppose that $S$ satisfies (Ex0), (Ex1), (Ex2) and (Ex5). Then, by Lemma 5.2, we have (PI). It remains to show that $S: 2^X \to 2^X$ satisfies (NE). Suppose, on the contrary, that for some $A \neq \emptyset$ we have $S(A) = \emptyset$. Let $p \in A$. Then, we have $S(A) \subseteq \{p\} \subseteq A$. It follows from (Ex5) that $S(\{p\}) \subseteq S(A) = \emptyset$. This contradicts (Ex0). □

Koshevoy proved the "if" part of Theorem 5.1 as follows. He showed that, given a choice function $S: 2^X \to 2^X$ satisfying (PI), the mapping defined by

$$\bar{S}(A) = \bigcup\{B \mid B \subseteq X, S(B) = S(A)\} \quad (A \subseteq X). \tag{5.1}$$

is a closure operator and that $S$ is the extreme point operator of $(X, \bar{S})$. This approach does not work for proving Theorem 1.1 since $\bar{S}$ may not be a closure operator. (Consider the extreme point operator given in Example 2.5. We have $\bar{\mathrm{ex}}(\{c\}) = \{a, b, c\} \not\subseteq \{c, a\} = \bar{\mathrm{ex}}(\{c, a\})$.)

However, if $S: 2^X \to 2^X$ satisfies the conditions in Theorem 1.3 (or equivalently, those in Theorem 5.1), then we have $\tau_S = \bar{S}$.

**Proposition 5.4**: *Suppose that mapping $S: 2^X \to 2^X$ satisfies Conditions* (Ex0)–(Ex2) *and* (Ex5). *Then, we have $\tau_S = \bar{S}$, where $\tau_S$ and $\bar{S}$ are, respectively, defined by* (3.1) *and* (5.1).

To show Proposition 5.4, we need the following lemma.

**Lemma 5.5**: *Let $(X, \tau_1)$ and $(X, \tau_2)$ be closure spaces with their extreme point operators being* $\mathrm{ex}_1$ *and* $\mathrm{ex}_2$, *respectively. If $\tau_1 \neq \tau_2$, then* $\mathrm{ex}_1 \neq \mathrm{ex}_2$.

(Proof) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the set of closed subsets of $(X, \tau_1)$ and $(X, \tau_2)$, respectively. Since a closure operator is uniquely determined by its closed sets, we have $\mathcal{L}_1 \neq \mathcal{L}_2$. Suppose, say, $\mathcal{L}_1 \not\subseteq \mathcal{L}_2$ and let $A \in \mathcal{L}_1 - \mathcal{L}_2$. Then, we have $\tau_1(A) = A$ and $\tau_2(A) \supsetneq A$. Let $p \in \tau_2(A) - A$. We have $p \in \mathrm{ex}_1(A \cup p)$ and $p \notin \mathrm{ex}_2(A \cup p)$ by definition of extreme point operator. Therefore, we have $\mathrm{ex}_1 \neq \mathrm{ex}_2$. □

(Proof of Proposition 5.4) Suppose that mapping $S: 2^X \to 2^X$ satisfies (Ex0)–(Ex2) and (Ex5). Then, we have from Theorem 4.3 that $(X, \tau_S)$ is an antimatroid with its extreme point operator being $S$. However, $S$ is also the extreme point operator of $(X, \bar{S})$ by Theorem 5.1 and we must have $\tau_S = \bar{S}$ by Lemma 5.5. □

If $S: 2^X \to 2^X$ is the extreme point operator of an antimatroid $(X, \tau)$, then the representation of the mapping $\bar{S} = \tau$ in (5.1) looks nice in comparison with the representation of $S$ in (2.1) since, in this case, we have

$$\mathrm{ex}(A) = \bigcap \{B \mid B \subseteq X, \tau(B) = \tau(A)\}. \tag{5.2}$$

We close this paper posing a question about rationalizability of extreme point operators of closure spaces. It is known that a choice function $S: 2^X \to 2^X$ is path-independent if and only if it is *pseudo-rationalizable*, i.e. there exists linear orders $\preceq_1, \cdots, \preceq_n$ on $X$ such that

$$S(A) = \bigcup_{i=1}^{n} \max\nolimits_{\preceq_i}(A) \quad (A \subseteq X) \tag{5.3}$$

([1]; see also [7]), where for a partial order $\preceq$ on $X$ $\max_{\preceq}: 2^X \to 2^X$ is defined as

$$\max\nolimits_{\preceq}(A) = \{p \mid p \in A, \nexists q \in A: p \prec q\} \quad (A \subseteq X). \tag{5.4}$$

Is it possible to "rationalize" functions satisfying (Ex1)-(Ex3) in some sense?

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## References

[1] M. A. Aizerman and A. V. Malishevski: General theory of best variants choice: some aspects. *IEEE Transactions on Automatic Control* **26** (1981) 1030–1040.

[2] P. Edelman and R. Jamison: Theory of convex geometries. *Geometriae Dedicata* **19** (1985) 247–270.

[3] G. Grätzer: *General lattice theory* (Academic Press, New York, 1978).

[4] B. Korte, L. Lovász, and B. Schrader: *Greedoids* (Springer, Berlin, 1991).

[5] G. A. Koshevoy: Choice functions and abstract convex geometry. *Mathematical Social Science* **38** (1999) 35–44.

[6] B. Monjardet and V. Raderanirina: The duality between the anti-exchange closure operators and the path independent choice operators on a finite set. *Mathematical Social Sciences* **41** (2001) 131–150.

[7] H. Moulin: Choice functions over a finite set: A summary. *Social Choice and Welfare* **2** (1985) 147–160.

[8] J. G. Oxley: *Matroid theory* (Oxford University Press, New York, 1992).

[9] J. L. Pfaltz: Closure lattices. *Discrete Mathematics* **154** (1996) 217–236.

[10] D. J. A. Welsh: *Matroid theory* (Academic Press, London, 1976).