HIGHER HOMOTOPY ASSOCIATIVITY IN THE HARRIS DECOMPOSITION OF LIE GROUPS

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1. Harris decomposition

If we localize a connected Lie group at a prime p, then it decomposes into a product of small spaces, which is called a mod p decomposition of a Lie group. The direct product factors of a mod p decomposition of a Lie group are well understood, and so the p-local homotopy types of Lie groups are well understood. Recently, several attempts were made to understand the group structure of a p-localized Lie group through its mod p decomposition [1, 4, 5, 6, 7, 9, 10, 11]. In particular, the paper [8] studies how group structures are living in a certain fibration involving Lie groups, and this note is a survey of it.

In [2, 3], Harris showed that the p-localized homotopy groups of a compact connected Lie group admits a direct sum decomposition when a Lie group admits an automorphism of finite order which is prime to p. This result can be easily reinterpret as a mod p decomposition of Lie groups as follows.

Theorem 1.1. *Let (G, H) and p be in the following table.*

(G, H)	$(SU(2n+1), SO(2n+1))$	$(SU(2n), Sp(n))$	$(SO(2n), SO(2n-1))$
p	$p \geq 3$	$p \geq 3$	$p \geq 3$
(G, H)	(E_6, F_4)	$(Spin(8), G_2)$	
p	$p \geq 5$	$p \neq 3$	

Then the fibration $H \to G \to G/H$ splits p-locally so that there is a p-local homotopy equivalence

$$G \simeq_{(p)} H \times G/H. \tag{1.1}$$

2. Result

Now we ask how the group structures of G and H are living in the Harris decomposition (1.1). This is nothing but asking how close to a homomorphism a projection $G_{(p)} \to H_{(p)}$ is. Groups up to homotopy are loop spaces, and so we are asking how close to a loop map a projection $G_{(p)} \to H_{(p)}$ is. It remains to measure a distance between a map between loop spaces and a loop map, and this is typically done by A_n-maps. Recall that an A_n-space for

$n \geq 2$ is an H-space with the $(n-2)$-th higher homotopy associativity. For example, an A_2-space is an H-space, an A_3-space is a homotopy associative H-space, and an A_∞-space is a loop space. A map between A_n-spaces are called an A_n-map if it preserves the A_n-structures. Then one can say that a map between a loop map is close to a loop map when it is an A_n-map as n gets larger. Thus our question is formulated precisely as:

Question 2.1. Let (G, H) be as in Theorem 1.1. For which k and p is a projection $G_{(p)} \to H_{(p)}$ an A_k-map?

Remark 2.2. There are several choices of a projection $G_{(p)} \to H_{(p)}$, but our result holds for any projection whenever it holds for some projection. Then we will not be explicit on a choice of a projection.

Now we state the main theorem of [8].

Theorem 2.3. *Let (G, H), a_k and p be as in the following table.*

(G,H)	$(SU(2n+1), SO(2n+1))$	$(SU(2n), Sp(n))$	$(SO(2n), SO(2n-1))$
a_k	$k(2n+1)$	$2kn-1$	$2(k-1)(n-1)+n$
p	$p \geq 3$	$p \geq 3$	$p \geq 3$
(G,H)	(E_6, F_4)	$(Spin(8), G_2)$	
a_k	$12k-5$	$6k-2$	
p	$p \geq 5$	$p \neq 3$	

Then for $k \geq 2$ the following statements hold:

(1) *for $(G, H) \neq (SO(2n), SO(2n-1))$ the projection $G_{(p)} \to H_{(p)}$ is an A_k-map if and only if $p \geq a_k$;*
(2) *for $(G, H) = (SO(2n), SO(2n-1))$*
   (a) *if $p \geq a_k$ then the projection $G_{(p)} \to H_{(p)}$ is an A_k-map;*
   (b) *if $p < a_k - n + 2$ then the projection $G_{(p)} \to H_{(p)}$ is not an A_k-map.*

There are yet more pairs (G, H) satisfying a mod p decomposition (1.1), and in [8], for such (G, H), a range of p in which a projection $G_{(p)} \to H_{(p)}$ is an A_k-map is also determined.

The proof for a projection $G_{(p)} \to H_{(p)}$ being an A_k-map is done by refining the product decomposition of projective spaces proved in [7], and the proof for a projection $G_{(p)} \to H_{(p)}$ not being an A_k-map is done by a cohomological criterion which is a sort of a higher homotopy associativity version of the following simple lemma.

Lemma 2.4. *Let (G, H) be a connected pair of Lie groups satisfying a mod p decomposition (1.1). Suppose there are maps $f_1 \colon S^{m_1} \to H_{(p)}$ and $f_2 \colon S^{m_2} \to (G/H)_{(p)}$ such that $q_*(\langle h \circ f_1, h \circ f_2 \rangle) \neq 0$, where $q \colon G_{(p)} \to H_{(p)}$ is a projection and $h \colon H_{(p)} \times (G/H)_{(p)} \to G_{(p)}$ is a homotopy equivalence (1.1). Then q is not an H-map.*

Proof. By definition, $q \circ h|_{H_{(p)}} = 1_{H_{(p)}}$ and $q \circ h|_{(G/H)_{(p)}} = *$. Then if q is an H-map,

$$q_*(\langle h \circ f_1, h \circ f_2 \rangle) = \langle q \circ h \circ f_1, q \circ h \circ f_2 \rangle = \langle f_1, * \rangle = 0.$$

which contradicts the assumption. □

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