On orientations of real algebraic curves determined by spin structures

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1 Introduction

It is a classical problem to decide possible arrangements of ovals of real algebraic curves in the real projective plane $\mathbf{R}P^2$, which is known as the Hilbert's 16'th problem([2]). Harnack was the first to show that the number of components of ovals of a given non-singlar real algebraic curve of degree m is less than or equal to $g(m) = (m-1)(m-2)/2$ ([1]), which is known as Harnack's inequality.

In [4], Rokhlin introduced 'complex orientations' on the algebraic curves of even type, which turned out to be useful for this problem. So searching canonical orientations on a given algebraic curve seems interesting problem.

2 Orientation by Wang

In [5], by using spin structures, Wang gave orientations of fixed point set $E_{\mathbf{R}}$ of a complex conjugate involution $\sigma_E : E \to E$ on a complex vector bundle $\pi : E \to X$ that covers an involution $\sigma : X \to X$ on a closed smooth manifold X. Note that, in case $X = \mathbf{C}A = \{[z_0; z_1; z_2] \in \mathbf{C}P^2 | A(z_0, z_1, z_2) = 0\}$, $E = T\mathbf{C}A$, and σ_E is the differential of the involution $\sigma : X \to X$ given by $\sigma([z_0; z_1; z_2]) = [\overline{z_0}; \overline{z_1}; \overline{z_2}]$, $E_{\mathbf{R}}$ is the total space of real vector bundle of the real algebraic curve $\mathbf{R}A = \{[z_0; z_1; z_2] \in \mathbf{R}P^2 | A(z_0, z_1, z_2) = 0\}$, where A is a real non-singular homogenius polynomial of three variables.

In this section, we see the construction given in [5]. Let $\mathbf{U}(E)$ denote the $\mathbf{U}(r)$-frame bundle for E and $\sigma^{`} : \mathbf{U}(E) \to \mathbf{U}(E)$ its induced involution. Set $P = \mathbf{U}(E) \times_i \mathbf{SO}(2r)$, where $i : \mathbf{U}(r) \to \mathbf{SO}(2r)$ is given by $i(X + \sqrt{-1}Y) = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$. Note that $i(\overline{X + \sqrt{-1}Y}) = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} = T \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} T^{-1}$, where $T = \begin{pmatrix} E & O \\ O & -E \end{pmatrix}$.

Definition 2.1 *We define the involution $\sigma_P : P \to P$ by $\sigma_P([V, M]) = [\sigma_E(V), TMT^{-1}]$ for $V \in \boldsymbol{U}(r)$ and $M \in \boldsymbol{SO}(2r)$.*

Suppose that the principal $\mathbf{SO}(2r)$ bundle $P \to X$ admits a spin structure $\xi \in H^1(P; \mathbf{Z}_2)$, i.e., a double cover $\widetilde{P} \to P$ such that the composite $\widetilde{P} \to P \to X$ is a principal $\mathbf{Spin}(2r)$ bundle whose restriction to a point $* \in X$ is the non-trivial one.

Definition 2.2 *The involution* $\sigma_E : E \to E$ *is compatible with* $\xi \in H^1(P; \boldsymbol{Z}_2)$ *if and only if there exists a bundle automorohism* $\tilde{\sigma}_P : \widetilde{P} \to \widetilde{P}$ *such that* $\tilde{\sigma}_P(xg) = \tilde{\sigma}_P(x)\overline{g}$ *holds for any* $g \in \boldsymbol{Spin}(2r) \subset \boldsymbol{Cl}(\boldsymbol{R}^{2r}) = \boldsymbol{Cl}(\boldsymbol{C}^r)$*, and that the following diagram commutes;*

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\tilde{\sigma}_P} & \widetilde{P} \\ \big\downarrow & & \big\downarrow \\ P & \xrightarrow{\sigma_P} & P. \end{array}$$

We say that σ_P *is compatible with a spin structure* ξ*, and that* $\tilde{\sigma}_P$ *a conjugate lift of* σ_P*.*

With this notation, Wang showed in [5];

Theorem 2.1 *Suppose that* $\sigma_E : E \to E$ *is compatible with a spin structure* $\xi \in H^1(P; \boldsymbol{Z}_2)$*. Then for each conjugate lift* $\tilde{\sigma}_E : P_\xi \to P_\xi$ *there is a canonical orientation of real vector bundle* $E_{\boldsymbol{R}} \to X_{\boldsymbol{R}}$*, where* $X_{\boldsymbol{R}}$ *denotes the fixed point set of the involution* $\sigma : X \to X$*.*

(*Case 1*) First suppose that $E \to X$ is a complex line bundle. Then $P \to X$ is a principal $\mathbf{U}(1)$ bundle. Let P^σ denote the fixed point set of σ_P. Then $P^\sigma \to X_{\mathbf{R}}$ is a principal $\mathbf{Z}_2$ bundle associated with $E_{\mathbf{R}} \to X_{\mathbf{R}}$. By our assumption, we have a spin structure $P_\xi \to P$ and a conjugate morphism $\sigma_\xi : P_\xi \to P_\xi$ such that the following diagram commutes;

$$\begin{array}{ccc} P_\xi & \xrightarrow{\sigma_\xi} & P_\xi \\ \big\downarrow & & \big\downarrow \\ P & \xrightarrow{\sigma_P} & P \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\sigma} & X. \end{array}$$

Since σ_ξ is conjugate, $P_\xi^\sigma \to X_{\mathbf{R}}$ also is principal $\mathbf{Z}_2$ bundle. Let $x \in X_{\mathbf{R}}$, and set $\pi^{-1}(x) = \{a, b\}$. Then we have that $b = -a$. Therefore any fiber of $P_\xi^\sigma \to X_{\mathbf{R}}$ is sent to one point. This implies that P_ξ^σ determines a section of $P^\sigma \to X_{\mathbf{R}}$. Thus we obtain an orientation of $E_{\mathbf{R}} = P^\sigma \times_{\mathbf{Z}_2} \mathbf{R} \to X_{\mathbf{R}}$.

(*Case 2*) When the complex dimension of the complex vector bundle $E \to X$ is greater than 1, we apply the meathod of Case 1 to the complex line bundle $det_{\mathbf{C}}E \to X$.

3 Construction of orientations by using pin structures

Let $L \to X$ be a complex line bundle and $\sigma_L : L \to L$ a complex congugate involution such that the following diagram commutes;

$$\begin{array}{ccc} L & \xrightarrow{\sigma_L} & L \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\sigma} & X. \end{array}$$

Let $\mathbf{U}(1) \to \mathbf{U}(L) \to X$, $\mathbf{SO}(2) \to \mathbf{SO}(L) \to X$, and $\mathbf{O}(2) \to \mathbf{O}(L) \to X$ denote the principal bundles associated with $L \to X$.

Then for each $e \in \mathbf{U}(L)$, set $\sigma(e) = \alpha e$, where $\alpha = a + \sqrt{-1}b \in \mathbf{U}(1)$. Then we have that $\sigma(\sqrt{-1}e) = -\sqrt{-1}\sigma(e) = (b - \sqrt{-1}a)e$. Thus we obtain that, by setting $M_\alpha = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, $\sigma(e, \sqrt{-1}e) = (e, \sqrt{-1}e)M_\alpha = (e, \sqrt{-1}e)R_\alpha T$, where $R_\alpha = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Define $i_P : P = \mathbf{U}(L) \times_\rho \mathbf{SO}(2) \to \mathbf{SO}(L)$ by $i_P([e, A]) = (e, \sqrt{-1}e)A$. Then we have the following commutative diagram, where R_T denotes the right multiplication by T;

$$\begin{tikzcd} P \arrow[r, "\sigma_U"] \arrow[d, "i_P"'] & P \arrow[d, "i_P"'] \\ \mathbf{SO}(L) \arrow[r, "R_T \circ \sigma_U"] & \mathbf{SO}(L). \end{tikzcd}$$

Therefore the following diagram commutes between the induced homomorphisms of cohomology;

$$\begin{tikzcd} H^1(P; \mathbf{Z}_2) & H^1(P; \mathbf{Z}_2) \arrow[l, "\sigma_U^*"'] \\ H^1(\mathbf{SO}(L); \mathbf{Z}_2) \arrow[u, "i_P^*"] & H^1(\mathbf{SO}(L); \mathbf{Z}_2) \arrow[l, "\sigma_L^* \circ R_T^*"'] \arrow[u, "i_P^*"] \end{tikzcd}$$

Definition 3.1 *For a spin structure* $\xi \in H^1(\boldsymbol{SO}(L); \boldsymbol{Z}_2)$, *set* $\tilde{\xi} = \xi \oplus R_T^*(\xi) \in H^1(\boldsymbol{SO}(L); \boldsymbol{Z}_2) \oplus H^1(\boldsymbol{SO}(L)T; \boldsymbol{Z}_2) = H^1(\boldsymbol{O}(L); \boldsymbol{Z}_2)$.

With this definition, we have the following;

Proposition 3.1 $i_P^*(\xi) \in H^1(P; \boldsymbol{Z}_2)$ *is compatible with* σ_U *if and only if* $\tilde{\xi} \in H^1(\boldsymbol{O}(L); \boldsymbol{Z}_2)$ *is a pin sturucture that is preserved by* σ.

By a similar method in [3], we obtain a section $\tilde{d\sigma} \in \Gamma(Ad(\mathbf{Pin}(L)|_{X_\mathbf{R}})) \subset \Gamma(\mathbf{Pin})(X)|_{X_\mathbf{R}} \times_{Ad} \mathbf{Cl}(2) \cong \Gamma(\overset{*}{\wedge} L|_F)$ via $\tilde{\xi}$.

Fix a point $x \in X_\mathbf{R}$. Then we my assume that $L_x = \mathbf{R}\langle u, w\rangle$ and that $\sigma_L|_x$ is the reflection determined by v. Thus we have that $\tilde{d\sigma}|_x = \pm v$.

Whether this section coinsides with the section given in Sectoin 2 or not should be investigated.

References

[1] Harnack. Uber vieltheiligkeit der ebenen algebraischen curven. *Math. Ann.*, 10:189–99, 1876.

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[4] Vladimir Abramovich Rokhlin. Complex orientations of real algebraic curves. *Functional Analysis and its Applications*, 8(4):331–334, 1974.

[5] Shuguang Wang. Orientability of real parts and spin structures. *JP J. Geom. Topol*, 7(1):159–174, 2007.