MIXED HODGE STRUCTURE ON FUNDAMENTAL GROUPS AND SULLIVAN MINIMAL MODELS

HISASHI KASUYA

We consider the following "theorem".

"Theorem". *Let M be a compact Kähler manifold and $\pi_1(M,x)$ its fundamental group. There exist mixed Hodge structures on the Malcev completion of $\pi_1(M,x)$.*

There are two ways to construct mixed Hodge structures as this "Theorem". The first way is given by Morgan ([6]) by using the Sullivan 1-minimal model. Consider the Sullivan 1-minimal model $\mathcal{M}^*$ of the de Rham complex $A^*(M)$ of a compact Kähler manifold M. In [6], Morgan constructed a non-unique mixed Hodge structure on $\mathcal{M}^*$. It is known that the dual Lie algebra of $\mathcal{M}^*$ is non-canonically isomorphic to the Malcev Lie algebra of the fundamental group $\pi_1(M,x)$. Hence we obtain a non-canonical mixed Hodge structure on the Malcev completion of $\pi_1(M,x)$.

The second way is given by Hain ([2]) by using iterated integrals. In [2], Hain constructed a mixed Hodge structure on the Malcev completion of $\pi_1(M,x)$ canonically defined by pointed compact kähler manifold (M,x).

We are interested in relation between mixed Hodge structure on Sullivan 1-minimal model $\mathcal{M}^*$ and Hain's mixed Hodge structure on the Malcev completion of $\pi_1(M,x)$. Consider the category $VMHS^u_{\mathbb{R}}(M)$ of unipotent variations of mixed Hodge structures over M and the fiber functor $\epsilon_x : \mathcal{V}MHS^u_{\mathbb{R}}(M) \ni (\mathbf{E},\mathbf{W},\mathbf{F}) \mapsto (\mathbf{E},\mathbf{W},\mathbf{F})_x \in \mathcal{M}HS_{\mathbb{R}}$. For the category $\mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M,x)}, W_*, F^*)$ of mixed Hodge representations of the Malcev completion of $\pi_1(M,x)$ with Hain's mixed Hodge structure associated with (M,x) and the forgetful functor $\tau : \mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M,x)}, W_*, F^*) \to \mathcal{M}HS_{\mathbb{R}}$, in [3], Hain and Zucker proved that the monodromy representation functor defines an equivalence $h_x : VMHS^u_{\mathbb{R}}(M) \to \mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M,x)})$ between tensor categories such that the diagram

$$\begin{array}{ccc} VMHS^u_{\mathbb{R}}(M) & \xrightarrow{\epsilon_x} & \mathcal{M}HS_{\mathbb{R}} \\ \downarrow{\scriptstyle h_x(\cong)} & & \downarrow{\scriptstyle =} \\ (\mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M,x)})), W_*, F^*) & \xrightarrow{\tau} & \mathcal{M}HS_{\mathbb{R}} \end{array}$$

commutes.

Theorem ([5]). *There exists a mixed hodge structure on the Sullivan 1-minimal model $\mathcal{M}^*$ of the de Rham complex $A^*(M)$ of a compact Kähler manifold M such that for the category $\mathrm{Rep}(\mathcal{M}^*, W_*, F^*)$ of mixed Hodge representations of the dual Lie algebra of $\mathcal{M}^*$ corresponding to this mixed hodge structure and the forgetful functor $\sigma : \mathrm{Rep}(\mathcal{M}^*, W_*, F^*) \to \mathcal{M}HS$, we have an equivalence $\Phi_x : \mathrm{Rep}(\mathcal{M}^*, W_*, F^*) \to$*

$VMHS^u_{\mathbb{R}}(M)$ so that the diagram

$$\begin{tikzcd} \mathrm{Rep}(\mathcal{M}^*, W_*, F^*) \arrow[r, "\sigma"] \arrow[d, "\Phi_x(\cong)"] & \mathcal{M}HS_{\mathbb{R}} \arrow[d, "="] \\ VMHS^u_{\mathbb{R}}(M) \arrow[r, "\epsilon_x"] & \mathcal{M}HS_{\mathbb{R}} \end{tikzcd}$$

commutes.

By the theory of Tannaka category, $\mathrm{Rep}(\mathbb{R}\widehat{\pi_1(M,x)}), W_*, F^*)$ and $\mathrm{Rep}(\mathcal{M}^*, W_*, F^*)$ with the functors τ and σ can be non-abelian Hodge structures (see [1]). Via Φ_x and h_x, we can say that two non-abelian Hodge structures equivalent.

References

[1] D. Arapura, The Hodge theoretic fundamental group and its cohomology. The geometry of algebraic cycles, 3–22, Clay Math. Proc., **9**, Amer. Math. Soc., Providence, RI, 2010.

[2] R. M. Hain, The de Rham homotopy theory of complex algebraic varieties. I. K-Theory **1** (1987), no. 3, 271–324.

[3] R. M. Hain, S. Zucker, Unipotent variations of mixed Hodge structure. Invent. Math. **88** (1987), no. 1, 83–124.

[4] H. Kasuya, Techniques of constructions of variations of mixed Hodge structures. Geom. Funct. Anal. **28** (2018), no. 2, 393–442.

[5] H. Kasuya, DGA-Models of variations of mixed Hodge structures. preprint arXiv:1809.03716

[6] J. W. Morgan, The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.* No. **48** (1978), 137–204.

(Hisashi Kasuya) Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Japan.

E-mail address: kasuya@math.sci.osaka-u.ac.jp