EXPLICIT TRACE FORMULA OF $\mathrm{SL}_3(\mathbb{Z})$ AND ITS APPLICATIONS

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This is a write up of the author's talk given at the RIMS conference "Automorphic forms, automorphic representations and related topics". All materials presented here are based on joint work with Werner Hoffmann and Satoshi Wakatsuki ([HTW]).

1. Introduction: the trace formula of $\mathrm{SL}_2(\mathbb{Z})$

First let us recall the Selberg trace formula for $\mathrm{SL}_2(\mathbb{Z})$ ([Selb], [HejII]). Let $\mathfrak{H} = \{\tau = x+iy | x \in \mathbb{R},\, y>0\}$ be the Poincaré upper half plane with the hyperbolic metric $\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^2+\mathrm{d}y^2)$ and $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ the associated Laplacian. Let

$$0=\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\cdots$$

be the eigenvalues (counted with multiplicity) of Δ on $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H})$. Then well-known estimate $\lambda_1 \geqslant 1/4$ allows us to introduce the spectral parameters $\rho_j \in \mathbb{R}/\{\pm 1\}$ $(j\geqslant 1)$ by the relation $\lambda_j = \frac{1}{4}+\rho_j^2$ $(j \geqslant 1)$. We set $\rho_0 = i/2$.

Theorem 1.1. *Let $\hat{h}(r)$ is an entire function on $|\mathrm{Im}(r)|<1/2+\delta$ for some $\delta>0$ such that*

(i) $\hat{h}(r)=\hat{h}(-r)$ *and*
(ii) $|\hat{h}(r)| \ll (1+|\mathrm{Re}(r)|)^{-2-\delta}$ *on* $|\mathrm{Im}(r)|<1/2+\delta$.

Define

$$h(u)=\int_{\mathbb{R}}\hat{h}(r)e^{-iur}\,\mathrm{d}r, \quad u\in\mathbb{R}.$$

Set

$$\phi(s)=\frac{\hat{\zeta}(1+2s)}{\hat{\zeta}(1-2s)} \quad \textit{with } \hat{\zeta}(s)=\Gamma_{\mathbb{R}}(s)\zeta(s),$$

$$\psi(s):=\frac{\Gamma'(s)}{\Gamma(s)}=-\gamma_0+\int_0^1\frac{1-x^{s-1}}{1-x}\,\mathrm{d}x,$$

where $\zeta(s)$ is the Riemann zeta function and γ_0 the Euler constant. Then it holds the following identity.

$$\sum_{j=0}^{\infty}\hat{h}(\rho_j)-\frac{1}{4\pi}\int_{\mathbb{R}}\frac{\phi'(it)}{\phi(it)}\hat{h}(t)\,\mathrm{d}t=\frac{1}{12}\int_{\mathbb{R}}\hat{h}(t)\,t\,\tanh(\pi t)\,\mathrm{d}t$$
$$+\sum_{\{\gamma\}\text{ hyperbolic}}\frac{\log \mathrm{N}(\gamma_0)\,h(\log\mathrm{N}(\gamma))}{\mathrm{N}(\gamma)^{1/2}-\mathrm{N}(\gamma)^{-1/2}}$$
$$+\sum_{\{\eta\}\text{ elliptic}}\frac{1}{8m(\eta)\,\sin(\theta(\eta))}\int_{\mathbb{R}}\hat{h}(t)\,\frac{\cosh((2\theta(\eta)-\pi)t)}{\cosh(\pi t)}\mathrm{d}t$$

$$- h(0)\log 2 - \frac{1}{2\pi}\int_{\mathbb{R}} \hat{h}(t)\psi(1+it)\,\mathrm{d}t,$$

where every integral and summation is absolutely convergent.

The aim of our research to be reported in this write up is to obtain an explicit trace formula for the rank 2 lattice $\mathrm{SL}_3(\mathbb{Z})$ in a similar style, which can be served as a practical tool for computations related with the automorphic forms of the 5 dimensional space $\mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$. A few remarks on existing works are in order. First, there is a series of publications by D. Wallace whose up shot would be the explicit Selberg trace formula of $\mathrm{SL}_3(\mathbb{Z})$. Unfortunately however, as pointed by D. Miller, there is a serious defect in the proofs. Second, there is a paper by A.B. Venkov [V2], which is supposed to be a first step toward the explicit trace formula of $\mathrm{SL}_3(\mathbb{Z})$ but it seems that the project remain uncompleted. Of course there is a vast work by J. Arthur on the trace formula, which is quite general but is not designed for (numerical) computations. Instead of working out the trace formula of $\mathrm{SL}_3(\mathbb{Z})$ from scratch, we rely on Arthur's invariant trace formula on the adelization of GL_3 to start with.

2. **The trace formula of** $\mathrm{SL}_3(\mathbb{Z})$

The group $\mathrm{SL}_3(\mathbb{R})$ acts on the 5 dimensional real manifold $\mathfrak{H}_5 = \{Z \in \mathrm{M}_3(\mathbb{R}) \mid Z = {}^tZ \gg 0,\ \det(Z) = 1\}$ as

$$\mathfrak{H}_5 \times \mathrm{SL}_3(\mathbb{R}) \ni (Z, g) \quad \to \quad g\,Z\,{}^tg \in \mathfrak{H}_5$$

The action is transitive so that

$$\mathrm{SO}(3) = \mathrm{Stab}_{\mathrm{SL}_3(\mathbb{R})}(1_3), \quad \mathrm{SL}_3(\mathbb{R})/SO(3) \cong \mathfrak{H}_5,$$

where the unit matrix 1_3 is served as a base point of $\mathfrak{H}_5$. The space $\mathfrak{H}_3$ can be endowed with the Riemannian metric stemming from a positive multiple of the Killing form on $\mathfrak{sl}_3(\mathbb{R})$, and hence carries the Laplace-Beltrami operator Δ. It is convenient to use the Iwasawa coordinate $(y_1, y_2, x_1, x_2, x_3)$ of a point $Z = g\,\mathrm{SO}(3) \in \mathfrak{H}_5$ which is determined as

$$g = \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 y_1^{-1} & 0 \\ 0 & 0 & y_2^{-1} \end{bmatrix} \begin{bmatrix} 1 & x_3 & x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} k, \quad (y_1, y_2) \in \mathbb{R}^2_{>0},\ (x_1, x_2, x_3) \in \mathbb{R}^3,\ k \in \mathrm{SO}(3).$$

We use the $\mathrm{SL}_3(\mathbb{R})$-invariant measures $\mathrm{d}\mu$ on $\mathfrak{H}_5$ and the Haar measure on $\mathrm{SL}_3(\mathbb{R})$ which are given as

$$\mathrm{d}\mu(Z) = \frac{\mathrm{d}y_1}{y_1}\frac{\mathrm{d}y_2}{y_2}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \qquad \mathrm{d}g = \mathrm{d}\mu(Z)\,\mathrm{d}k$$

in terms of the Iwasawa coordinates.

For $f \in C_c^\infty(\mathrm{SL}_3(\mathbb{R}) \,/\!\!/\, \mathrm{SO}(3))$, define functions

$$h_f : \mathbb{R}^2 \longrightarrow \mathbb{C}, \quad \widehat{h_f} : \mathbb{C}^2 \longrightarrow \mathbb{C}$$

as

$$h_f(t) := e^{t_1+t_2} \iiint_{\mathbb{R}^3} f\left(\begin{bmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2-t_1} & 0 \\ 0 & 0 & e^{-t_2} \end{bmatrix}\begin{bmatrix} 1 & x_3 & x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix}\right) \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

and

$$\widehat{h_f}(\Lambda) := \iint_{\mathbb{R}^2} h(t)\, e^{i(xt_1+yt_2)}\,\mathrm{d}t, \quad \Lambda = (x, y) \in \mathbb{C}^2.$$

It is enlightening to note the relation

$$\widehat{h_f}(\Lambda) = \mathrm{Trace}\, I(\Lambda, f),$$

where $I(\Lambda)$ is the minimal principal series $I(\Lambda) := \mathrm{Ind}_B^{\mathrm{SL}_3(\mathbb{R})}(\chi_\Lambda)$ induced from a character χ_Λ of the Borel subgroup of upper-triangular matrices in $\mathrm{SL}_3(\mathbb{R})$ defined as χ_Λ : $\begin{bmatrix} \pm e^{t_1} & * & * \\ 0 & \pm e^{t_2 - t_1} & * \\ 0 & 0 & \pm e^{-t_2} \end{bmatrix} \to \exp(\sqrt{-1}(t_1 x + t_2 y))$. The symmetric group $\mathfrak{S}_3$ (A_2 Weyl group) acts on $\mathbb{C}^3$ by the rule $\sigma\,(t_1, t_2, t_3) = (t_{\sigma^{-1}(1)}, t_{\sigma^{-1}(2)}, t_{\sigma^{-1}(3)})$ preserving the hyperplane

$$\left\{(t_1, t_2, t_3) \in \mathbb{R}^3 | \sum_{j=1}^{3} t_j = 0\right\} = \left\{x\,(\tfrac{2}{3}, \tfrac{-1}{3}, \tfrac{-1}{3}) + y\,(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{-2}{3}) |\, x,\, y \in \mathbb{R}\right\}.$$

Thus $\mathfrak{S}_3$ acts on $\mathbb{C}^2 = \{(x, y) | x, y \in \mathbb{C}\}$. Then the function $\widehat{h_f}$ possessed the Weyl group symmetry

$$\widehat{h_f}(\sigma\Lambda) = \widehat{h_f}(\Lambda), \quad \sigma \in \mathfrak{S}_3,\ \Lambda \in \mathbb{C}^2.$$

Moreover, $\widehat{h_f}$ is a Paley-Wiener function on $\mathbb{C}^2$, i.e., there exists a constant $C > 0$ such that for any $N > 0$ we have the estimate

$$|\widehat{h_f}(\Lambda)| \ll_N (1 + \|\mathrm{Re}(\Lambda)\|)^{-N}\, e^{C\|\mathrm{Im}(\Lambda)\|}.$$

In this article, we are exclusively concerned ourselves with the arithmetic quotient

$$\Gamma\backslash\mathfrak{H}_5 \cong \Gamma\backslash\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3) \quad \text{where } \Gamma = \mathrm{SL}_3(\mathbb{Z})$$

and its L^2-space $L^2(\Gamma\backslash\mathfrak{H}_5)$. By the identification $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3) \cong \mathfrak{H}_5$, we have

$$L^2(\Gamma\backslash\mathfrak{H}_5) \cong L^2(\Gamma\backslash\mathrm{SL}_3(\mathbb{R}))^{\mathrm{SO}(3)},$$

where $L^2(\Gamma\backslash\mathrm{SL}_3(\mathbb{R}))$ is viewed as a unitary representation of $\mathrm{SL}_3(\mathbb{R})$. Any function $f \in C_{\mathrm{c}}^\infty(\mathrm{SL}_3(\mathbb{R}))$ acts on $L^2(\Gamma\backslash\mathrm{SL}_3(\mathbb{R}))$ by

$$[R(f)F]\,(h) = \int_{\mathrm{SL}_3(\mathbb{R})} F(hg)\, f(g)\, \mathrm{d}g, \quad h \in \mathrm{SL}_3(\mathbb{R}).$$

It is known by Langlands that

$$L^2(\Gamma\backslash\mathrm{SL}_3(\mathbb{R})) = L^2_{\mathrm{disc}}(\Gamma\backslash\mathrm{SL}_3(\mathbb{R})) \oplus (\text{Continuous part})$$

and there is a finite multiplicity function $m_\Gamma(\pi) \in \mathbb{N}_0$ on the unitary dual of $\mathrm{SL}_3(\mathbb{R})$ such that

$$L^2_{\mathrm{disc}}(\Gamma\backslash\mathrm{SL}_3(\mathbb{R})) \cong \widehat{\bigoplus}_\pi m_\Gamma(\pi)\, \pi.$$

Taking SO(3)-invariant vectors, we have

$$L^2_{\mathrm{disc}}(\Gamma\backslash\mathrm{SL}_3(\mathbb{R}))^{\mathrm{SO}(3)} = \widehat{\bigoplus}_{j\in\mathbb{N}_0} \pi_j^{\mathrm{SO}(3)}, \quad \pi_j^{\mathrm{SO}(3)} = \mathbb{C}\, F_j$$

where π_j is a set of irreducible closed subspaces of $L^2_{\mathrm{disc}}(\Gamma\backslash\mathrm{SL}_3(\mathbb{R}))$ with SO(3)-invariant unit vectors F_j. From the theory of zonal spherical functions, we have a unique point $\Lambda_j \in \mathbb{C}^2/\mathfrak{S}_3$ such that $I(\sqrt{-1}\Lambda_j) \to \pi_j$, which determines the eigenvalues in the joint eigenequation :

$$R_{\mathrm{disc}}(f)\, F_j = \widehat{h_f}(\Lambda_j)\, F_j, \quad f \in C_{\mathrm{c}}^\infty(\mathrm{SL}_3(\mathbb{R}) /\!\!/ \mathrm{SO}(3)).$$

From the classification of discrete spectrum of $\mathrm{GL}(n)$ due to Moeglin-Waldspurger ([MoWa]), it follows that the only non-cuspidal eigenfunction is the constant F_0 with $\Lambda_0 = (-\sqrt{-1}, -\sqrt{-1})$, i.e., $F_j\,(j \geqslant 1)$ are all cusp forms.

REMARK : Contrary to the case of $\mathrm{SL}_2(\mathbb{Z})$, the temperedness of the cuspidal spectrum of $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ (i.e., $\Lambda_j \in \mathbb{R}^2$ for $j \geqslant 1$) is not proved up to now. However, it is known that $\Lambda_j\,(j \geqslant 1)$ belongs to the set $(\mathbb{R}^2 \bigcup \mathbb{X}_{\mathrm{non-temp}})/\mathfrak{S}_3$ with

$$\mathbb{X}_{\mathrm{non-temp}} := \mathfrak{S}_3\,\{(x,\,y) \in \mathbb{C}^2 |\, x \in \sqrt{-1}(-1,1),\; x+2y \in \mathbb{R}\,\}.$$

Theorem 2.1. *The trace*

$$\mathrm{tr}\, R_{\mathrm{disc}}(f) \;=\; \sum_{j=0}^{\infty} \widehat{h_f}(\Lambda_j)$$

for any test function $f \in C_{\mathrm{c}}^{\infty}(\mathrm{SL}_3(\mathbb{R}) \,/\!\!/\, \mathrm{SO}(3))$ can be computed in terms of $\hat{h} := \widehat{h_f}$ quite explicitly through the trace formula identity

$$\hat{I}_{\mathrm{spect}}(\hat{h}) = \hat{I}_{\mathrm{geom}}(\hat{h}),$$

with

$$\hat{I}_{\mathrm{spect}}(\hat{h}) = \mathrm{tr}\, R_{\mathrm{disc}}(f) + \tfrac{8}{3}\,\hat{h}(0,0) + \hat{I}_{\mathsf{P}_1}(\hat{h}) + \hat{I}_{\mathsf{P}_0}(\hat{h}),$$

$$\hat{I}_{\mathrm{geom}}(\hat{h}) = \sum_{i=1}^{2} \{\hat{J}^{\mathrm{ss}}_{i,\mathbb{R}}(\hat{h}) + \hat{J}^{\mathrm{ss}}_{i,\mathbb{C}}(\hat{h})\} + \sum_{j=1}^{10} \hat{D}_j(\hat{h}),$$

described in the remaining part of this section.

2.1. **The spectral side.**

$$\hat{I}_{\mathrm{spect}}(\hat{h}) \;=\; \sum_{j=0}^{\infty} \hat{h}(\Lambda_j) + \frac{8}{3}\hat{h}(0,0) + \hat{I}_{\mathsf{P}_1}(\hat{h}) + \hat{I}_{\mathsf{P}_0}(\hat{h})$$

with

$$\hat{I}_{\mathsf{P}_1}(\hat{h}) = \frac{12}{\pi} \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{Z'(it,\varphi_n)}{Z(it,\varphi_n)}\,\hat{h}(-t-r_n, 2r_n)\,\mathrm{d}t + \frac{6}{\pi}\int_{\mathbb{R}} \frac{\phi_0'(it)}{\phi_0(it)}\,\hat{h}(-t,0)\,\mathrm{d}t,$$

$$\hat{I}_{\mathsf{P}_0}(\hat{h}) = \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \left\{ \frac{\phi_0'(it_1)}{\phi_0(it_1)}\frac{\phi_0'(it_2)}{\phi_0(it_2)} + \frac{\phi_0'(it_2)}{\phi_0(it_2)}\frac{\phi_0'(i(t_1+t_2))}{\phi_0(i(t_1+t_2))} + \frac{\phi_0'(i(t_1+t_2))}{\phi_0(i(t_1+t_2))}\frac{\phi_0'(it_1)}{\phi_0(it_1)} \right\} \hat{h}(-t_1,-t_2)\,\mathrm{d}t_1\mathrm{d}t_2,$$

where $\{\varphi_n\}_{n=0}^{\infty}$ is an ONB of $L^2_{\mathrm{d}}(\mathrm{GL}_2(\mathbb{Z})\mathbb{R}_+\backslash \mathrm{GL}_2(\mathbb{R})/O(2))$ consisting of even Hecke-Maass forms on $\mathrm{SL}_2(\mathbb{Z})$ such that $\Delta\varphi_n = (\frac{1}{4} + r_n^2)\varphi_n$, and

$$Z(s,\varphi_n) = \frac{\hat{L}(1-s,\varphi_n)}{\hat{L}(1+s,\varphi_n)} \qquad\qquad \phi_0(s) = \frac{\hat{\zeta}(1-s)}{\hat{\zeta}(1+s)}$$

with $\hat{\zeta}(s) = \prod_{p<\infty}(1-p^{-s})^{-1} \times \Gamma_{\mathbb{R}}(s)$ and

$$\hat{L}(s,\varphi_n) = \prod_{p<\infty} (1 - c_{\varphi_n}(p)p^{-s} + p^{-2s})^{-1}\,\Gamma_{\mathbb{R}}(s+ir_n)\Gamma_{\mathbb{R}}(s-ir_n),$$

where $c_{\varphi_n}(p)$ is the p-the normalized Fourier coefficient of φ_n:

$$\varphi_n(\tau) = \sum_{j\in\mathbb{Z}-\{0\}} c_{\varphi_n}(j)\, y^{1/2} K_{ir_n}(2\pi|j|y)\, e^{2\pi ijx}, \quad c_{\varphi_n}(1) = 1.$$

We set

$$\begin{aligned} \mathsf{G} &= \mathrm{GL}(3), \\ \mathsf{P}_0 = \mathsf{M}_0\mathsf{N}_0 &= \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\} \quad \text{(minimal parabolic)}, \\ \mathsf{P}_1 = \mathsf{M}_1\mathsf{N}_1 &= \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\} \quad \text{(maximal parabolic)}. \end{aligned}$$

2.2. **The geometric side.** First recall that two points $\gamma, \gamma \in \mathsf{G}(\mathbb{Q})$ are defined to be $\mathscr{O}$-equivalent when the semi-simple part of their Jordan decompositions are $\mathsf{G}(\mathbb{Q})$-conjugate. The $\mathscr{O}$-equivalence classes are given by the Jordan canonical form easily.

(1) $\{\eta\}_{\mathsf{G}(\mathbb{Q})}$ (with $\eta \in \mathsf{G}(\mathbb{Q})$ such that $\mathbb{Q}(\eta)/\mathbb{Q}$ is a cubic field)
(2) $\{\delta = \left[\begin{smallmatrix} a & \\ & \delta_0 \end{smallmatrix}\right]\}_{\mathsf{G}(\mathbb{Q})}$ (with $a \in \mathbb{Q}^\times$, $\delta_0 \in \mathrm{GL}_2(\mathbb{Q})$ such that $\mathbb{Q}(\delta_0)$ is a quadratic field)
(3) $\left\{ \left[\begin{smallmatrix} a & & \\ & b & \\ & & c \end{smallmatrix}\right] \right\}_{\mathsf{G}(\mathbb{Q})}$ $(a, b, c \in \mathbb{Q}^\times$ are distinct)
(4) $\left\{ \left[\begin{smallmatrix} b & & \\ & a & \\ & & a \end{smallmatrix}\right] \right\}_{\mathsf{G}(\mathbb{Q})} \cup \left\{ \left[\begin{smallmatrix} b & & \\ & a & 1 \\ & & a \end{smallmatrix}\right] \right\}_{\mathsf{G}(\mathbb{Q})}$ $(a, b \in \mathbb{Q}^\times$ are distinct)
(5) $\left\{ \left[\begin{smallmatrix} z & & \\ & z & \\ & & z \end{smallmatrix}\right] \right\}_{\mathsf{G}(\mathbb{Q})} \cup \left\{ \left[\begin{smallmatrix} z & & \\ & z & 1 \\ & & z \end{smallmatrix}\right] \right\}_{\mathsf{G}(\mathbb{Q})} \cup \left\{ \left[\begin{smallmatrix} z & 1 & \\ & z & 1 \\ & & z \end{smallmatrix}\right] \right\}_{\mathsf{G}(\mathbb{Q})}$ $(z \in \mathbb{Q}^\times)$,

where $\{\gamma\}_{\mathsf{G}(\mathbb{Q})}$ denotes the $\mathsf{G}(\mathbb{Q})$-conjugacy class of $\gamma \in \mathsf{G}(\mathbb{Q})$.

REMARK : The elements in (3) do not contribute to $\mathrm{SL}_3(\mathbb{Z})$-trace formula, because $a, b, c \in \mathbb{Z}^\times = \{\pm 1\}$ can not be distinct.

The geometric side I (regular semisimple-terms)

(A) $\mathsf{G}(\mathbb{Q})$**-elliptic terms** :

$$\hat{J}^{\mathrm{ss}}_{3,\mathbb{R}}(\hat{h}) = \sum_{\{\eta\}} \frac{\mathrm{vol}(\Gamma_\eta\backslash\mathrm{SL}_3(\mathbb{R})_\eta)}{|D(\Phi_\eta)|}\, \hat{J}^{\eta}_{3,\mathbb{R}}(\hat{h}), \qquad \hat{J}^{\mathrm{ss}}_{3,\mathbb{C}}(\hat{h}) = \sum_{\{\eta'\}} \frac{\mathrm{vol}(\Gamma_\eta\backslash\mathrm{SL}_3(\mathbb{R})_\eta)}{|D(\Phi_\eta)|}\, \hat{J}^{\eta'}_{3,\mathbb{C}}(\hat{h}).$$

The terms $\hat{J}^{\eta}_{3,\mathbb{R}}(\hat{h})$ and $\hat{J}^{\eta}_{3,\mathbb{C}}(\hat{h})$ are determined by Harish-Chandra and R. Herb (in a more general setting). $\{\eta\}$ (resp. $\{\eta'\}$) runs over all the Γ-conjugacy classes in Γ whose characteristic polynomial Φ_η (resp. $\Phi_{\eta'}$) is $\mathbb{Q}$-irreducible and has no complex roots (resp. has two complex roots).

(B) $\mathsf{M}_1(\mathbb{Q})$**-elliptic terms** :

$$\hat{J}^{\mathrm{ss}}_{2,\mathbb{R}}(\hat{h}) = \sum_{\delta} \frac{\mathrm{vol}(\mathrm{GL}_2(\mathbb{Z})_\delta\backslash\mathrm{GL}_2(\mathbb{R})_\delta)}{|D(\Phi_\delta)|^{1/2}}\, \hat{J}^{\delta}_{2,\mathbb{R}}(\hat{h}), \qquad \hat{J}^{\mathrm{ss}}_{2,\mathbb{C}}(\hat{h}) = \sum_{\delta'} \frac{\mathrm{vol}(\mathrm{GL}_2(\mathbb{Z})_{\delta'}\backslash\mathrm{GL}_2(\mathbb{R})_{\delta'})}{|D(\Phi_{\delta'})|^{1/2}}\, \hat{J}^{\delta'}_{2,\mathbb{C}}(\hat{h}),$$

where $\{\delta\}$ (resp. $\{\delta'\}$) runs over all the $\mathrm{GL}_2(\mathbb{Z})$-conjugacy classes in $\mathrm{GL}_2(\mathbb{Z})$ whose characteristic polynomial Φ_δ (resp. $\Phi_{\delta'}$) is $\mathbb{Q}$-irreducible and no complex roots (resp. has two complex roots).

♠ The terms $\hat{J}^{\delta}_{2,\mathbb{R}}(\hat{h})$ and $\hat{J}^{\delta'}_{2,\mathbb{C}}(\hat{h})$ come from the invariant weighted orbital integrals $I_{\mathsf{M}_1}(\gamma, f)$ with $\gamma = \mathrm{diag}(\pm 1, \delta) \in \mathsf{M}_1(\mathbb{Q})$ such that

$$\mathsf{M}_1(\mathbb{Q})_\gamma = \mathsf{G}(\mathbb{Q})_\gamma \cong \mathbb{Q}^\times \times (\mathbb{Q}[X]/\Phi_\delta[X])^\times.$$

♠ The explicit formulas of $J^{\delta}_{2,\mathbb{R}}(\hat{h})$ and $J^{\delta'}_{2,\mathbb{C}}(\hat{h})$ involve the special function

$$b(s,z)=\sum_{n=1}^{\infty}\frac{z^n}{n+s}=z\int_0^1\frac{x^s}{1-zx}\,\mathrm{d}x \quad (|z|<1,\ \mathrm{Re}(s)>-1),$$

with the integral being convergent for any $z\in\mathbb{C}-[1,+\infty)$ ([Ho5], [Ho6], [Ho4]). We set

$$b(s):=b(s,-1)=-\psi\left(\frac{s}{2}+1\right)+\psi(s+1)-\log 2.$$

The geometric side II (singular terms)

The terms $\hat{D}_j(\hat{h})$ $(j=1,\dots,10)$ arise from the invariant weighted orbital integrals $I_{\mathsf{M}}(\gamma)$ with $\mathsf{M}\in\{\mathsf{M}_0,\mathsf{M}_1,\mathsf{G}\}$ and $\gamma\in\mathsf{M}(\mathbb{Q})$ such that $\mathsf{M}(\mathbb{Q})_\gamma$ is not a torus. The general form of $\hat{D}_j(\hat{h})$ is

$$\hat{D}_j(\hat{h})=\Phi_j^{\mathsf{G}}\,\hat{h}(0,0)+\tfrac{1}{2\pi}\int_{\mathbb{R}}\Phi_j^{\mathsf{M}_1}(t)\,\hat{h}(t,0)\,\mathrm{d}t+\left(\tfrac{1}{2\pi}\right)^2\iint_{\mathbb{R}^2}\Phi_j^{\mathsf{M}_0}(t_1,t_2)\,\hat{h}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2,$$

where $\Phi_j^{\mathsf{M}_0}(t_1,t_2)$ is a smooth function on $\mathbb{R}^2$ and $\Phi_j^{\mathsf{M}_1}(t)$ is a smooth function on $\mathbb{R}$ with the estimates $|\Phi_j^{\mathsf{M}_0}(t_1,t_2)|\ll(1+|t_1|+|t_2|)^3$ and $|\Phi_j^{\mathsf{M}_1}(t)|\ll(1+|t|)^2$, and Φ_j^{G} is a constant.

(I) The terms from $I_{\mathsf{G}}(1_3)=f(1_3)$:
This is settled by the well-known inversion formula of spherical Fourier transform (Plancherel formula).

$$\hat{D}_1(\hat{h})=\mathrm{vol}(\Gamma\backslash\mathrm{SL}_3(\mathbb{R}))\left(\tfrac{1}{\pi}\right)^3\iint_{\mathbb{R}^2}t_1t_2(t_1+t_2)\tanh\left(\tfrac{t_1\pi}{2}\right)\tanh\left(\tfrac{t_2\pi}{2}\right)\tanh\left(\tfrac{(t_1+t_2)\pi}{2}\right)\hat{h}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2.$$

with $\mathrm{vol}(\Gamma\backslash\mathrm{SL}_3(\mathbb{R}))=\frac{1}{8}\hat{\zeta}(2)\hat{\zeta}(3)$.

(II) Term from $I_{\mathsf{M}_1}(1_3)$:

$$\begin{aligned}12\hat{D}_2(\hat{h})=&\left(\tfrac{1}{2\pi}\right)^2\iint_{\mathbb{R}^2}(t_1+t_2)\tanh\left(\tfrac{t_2\pi}{2}\right)\{\psi(1)-\psi(-it_1+1)\}\hat{h}(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2\\&+\left(\tfrac{1}{2\pi}\right)^2\iint_{\mathbb{R}^2}t_2\tanh\left(\tfrac{t_2\pi}{2}\right)\left\{\psi(it_1+1)-\psi\left(\tfrac{it_1}{2}+1\right)-\log 2\right\}\hat{h}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2.\end{aligned}$$

(III) Term from $I_{\mathsf{M}_1}(\mathrm{diag}(1,-1,-1))$:

$$\begin{aligned}&12\hat{D}_3(\hat{h})\\&=\left(\tfrac{1}{2\pi}\right)^2\iint_{\mathbb{R}^2}t_2\left\{\tanh\left(\tfrac{(t_1+t_2)\pi}{2}\right)+\tanh\left(\tfrac{t_2\pi}{2}\right)\right\}\left\{\psi(it_1+1)-\psi\left(\tfrac{it_1}{2}+1\right)-\log 2\right\}\hat{h}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2\\&+\left(\tfrac{1}{2\pi}\right)^2\iint_{\mathbb{R}^2}\left\{t_2\tanh\left(\tfrac{t_2\pi}{2}\right)-(t_1+t_2)\tanh\left(\tfrac{(t_1+t_2)\pi}{2}\right)\right\}\tfrac{1}{it_1}\hat{h}(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2\\&+\tfrac{6}{\pi}\int_{\mathbb{R}}(-t)\tanh\left(\tfrac{\pi t}{2}\right)\hat{h}(t,0)\,\mathrm{d}t.\end{aligned}$$

(IV) Term from $I_{\mathsf{M}_0}(1_3)$:

$$\hat{D}_4(\hat{h}) = \tfrac{8}{3}\hat{h}(0,0) + \tfrac{6}{\pi}\int_{\mathbb{R}} \left\{ 2\psi(1) - 2\log 2 - \psi\left(1+\tfrac{it}{2}\right) - \psi\left(1+\tfrac{-it}{2}\right) \right\} \hat{h}(t,0)\,\mathrm{d}t$$
$$+ \tfrac{1}{3}\left(\tfrac{1}{2\pi}\right)^2 \iint_{\mathbb{R}^2} \Phi_4^{\mathsf{M}_0}(t_1,t_2)\,\hat{h}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2,$$

where $\Phi_4^{\mathsf{M}_0}(t_1,t_2)$ is given by the formula below in terms of functions $b(s) = -\psi\left(\frac{s}{2}+1\right) + \psi(s+1) - \log 2$ and

$$\Upsilon_-(s_1,s_2) := \int_0^1 \int_0^1 \frac{(x^{s_2} - x^{s_1+1})y^{s_1+s_2}}{(1-x)(1-xy)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x, \quad (s_1,s_2)\in i\mathbb{R}^2$$

as

$$\begin{aligned}
\Phi_4^{\mathsf{M}_0}(t_1,t_2) =& \Upsilon_-(-it_1, -it_1 - it_2) \\
&+ (\psi(1) - \psi(-it_1 - it_2 + 1))(\psi(1) - \psi(-it_2+1)) \\
&+ 2(\psi(1) - \psi(-it_1+1))(\psi(1) - \psi(-it_2+1)) \\
&+ (\psi(1) - \psi(-it_1+1))\,(b(it_1+it_2) + b(it_2)) \\
&+ (\psi(1) - \psi(-it_1 - it_2 + 1))\,(b(it_1) + b(it_2)) \\
&+ (\psi(1) - \psi(-it_2+1))\ (b(it_1) + b(it_1+it_2)) \\
&+ b(it_1)b(it_1+it_2) + b(it_1+it_2)b(it_2) + b(it_2)b(it_1) \\
&+ \tfrac{1}{2}\,\tfrac{1}{it_1+it_2}\Big\{ -\psi\left(\tfrac{it_1}{2}+1\right) + \psi\left(\tfrac{-it_2}{2}+1\right) + \psi\left(\tfrac{-it_1}{2}+1\right) - \psi\left(\tfrac{it_2}{2}+1\right) \\
&+ 2(\psi(it_1+1) - \psi(-it_2+1)) + 2(-\psi(-it_1+1) + \psi(it_2+1)) \Big\}
\end{aligned}$$

(V) Term from $I_{\mathsf{M}_0}(\mathrm{diag}(1,-1,-1))$:

$$\hat{D}_5(\hat{h}) = 8\hat{h}(0,0)$$
$$+ \tfrac{6}{\pi}\int_{\mathbb{R}} \left\{ 2\psi(1) - 2\log 2 - \psi\left(1+\tfrac{it}{2}\right) - \psi\left(1+\tfrac{-it}{2}\right) + 4b(it) + 4b(-it) \right\} \hat{h}(t,0)\,\mathrm{d}t$$
$$+ \left(\tfrac{1}{2\pi}\right)^2 \iint_{\mathbb{R}^2} \Phi_5^{\mathsf{M}_0}(t_1,t_2)\,\hat{h}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2,$$

where $\Phi_5^{\mathsf{M}_0}(t_1,t_2)$ is given by the formula below in terms of functions $b(s)$ and

$$\Upsilon_+(s_1,s_2) := \int_0^1 \int_0^1 \frac{(x^{s_2} - x^{s_1+1})y^{s_1+s_2}}{(1+x)(1+xy)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x, \quad (s_1,s_2)\in i\mathbb{R}^2$$

$$\begin{aligned}\Phi_5^{\mathsf{M}_0}(t_1,t_2) = {} & \Upsilon_+(-it_1,-it_1-it_2)\\ & + (\psi(1)-\psi(it_2+1))(b(-it_1)+b(it_1)+b(-it_1-it_2)+b(it_1+it_2))\\ & + b(it_1)(b(-it_1-it_2)+b(-it_2)) + b(it_1+it_2)(b(-it_1)+b(-it_2))\\ & + b(-it_1)b(-it_1-it_2)+b(-it_1-it_2)b(-it_2)+b(-it_2)b(-it_1)\\ & + \tfrac{1}{it_1}\Big\{\psi(it_2+1)+\psi(-it_2+1)+b(it_2)+b(-it_2)\\ & - \psi(it_1+it_2+1)-\psi(-it_1-it_2+1)-b(it_1+it_2)-b(-it_1-it_2)\Big\}.\end{aligned}$$

(VI) Term from $I_{\mathsf{M}_0}\left(\left[\begin{smallmatrix}1&0&0\\0&1&1\\0&0&1\end{smallmatrix}\right]\right)$:

$$\gamma_0^{-1}\,\hat{D}_6(\hat{h}) = \tfrac{6}{\pi}\int_{\mathbb{R}}\hat{h}(t,0)\,\mathrm{d}t + \left(\tfrac{1}{2\pi}\right)^2 \iint_{\mathbb{R}^2}\left\{\psi(1)-\log 2-\psi\left(\tfrac{it_2}{2}+1\right)\right\}\hat{h}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2.$$

(VIII) Term from $I_{\mathsf{M}_1}(\mathrm{diag}(1,-1,-1))$:

$$\begin{aligned}\hat{D}_8(\hat{h}) = {} & \left(\gamma_0 - \tfrac{1}{2}\log 2\right)\left[\tfrac{12}{\pi}\int_{\mathbb{R}}\hat{h}(t,0)\,\mathrm{d}t\right.\\ & \left. + \tfrac{1}{\pi^2}\iint_{\mathbb{R}^2}\left\{\psi(it_2+1)-\psi\left(\tfrac{it_2}{2}+1\right)-\log 2\right\}\hat{h}(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2\right].\end{aligned}$$

(IX) Term from $I_{\mathsf{G}}\left(\left[\begin{smallmatrix}1&0&0\\0&1&1\\0&0&1\end{smallmatrix}\right]\right)$ **and** $I_{\mathsf{G}}(\mathrm{diag}(1,-1,-1))$:

$$\hat{D}_9(\hat{h}) = \left\{\left(2\tfrac{\zeta'(2)}{\zeta(2)}+3\log 2\right)\tfrac{\pi}{3}-\tfrac{1}{2}(\log 2)^2\right\}\frac{1}{8\pi}\left(\tfrac{1}{2\pi}\right)^2\iint_{\mathbb{R}^2}\hat{h}(t_1,t_2)\,t_2\,\tanh\left(\frac{\pi t_2}{2}\right)\mathrm{d}t_1\mathrm{d}t_2.$$

(X) Term from $I_{\mathsf{G}}\left(\left[\begin{smallmatrix}1&1&0\\0&1&1\\0&0&1\end{smallmatrix}\right]\right)$ **and** $I_{\mathsf{G}}\left(\mathrm{diag}(1,-1,-1)\left[\begin{smallmatrix}1&0&0\\0&1&1\\0&0&1\end{smallmatrix}\right]\right)$:

$$\hat{D}_{10}(\hat{h}) = \left\{\tfrac{2}{3}(\gamma_0^2-\gamma_1)+\tfrac{1}{4}(\gamma_0+\log 2)\log 2\right\}\left(\tfrac{1}{2\pi}\right)^2\iint_{\mathbb{R}^2}\hat{h}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2.$$

The geometric side III (peripheral term)

This term arises from the splitting formula, which expresses the semi-local invariant weighted orbital integral

$$I_{\mathsf{M}_0(\mathbb{Q}_2\times\mathbb{R})}(\mathrm{diag}(1,-1,-1), f\otimes 1_{\mathrm{GL}_3(\mathbb{Z}_2)})$$

as a sum of similar integrals for $\mathsf{M}_0(\mathbb{R})$ and $\mathsf{M}_0(\mathbb{Q}_2)$.

(VII) Term from $I_{\mathsf{M}_0}^{\mathsf{M}_1}(\mathrm{diag}(1,-1,-1),f)$:

$$\begin{aligned}(\log 2)^{-1}\,\hat{D}_7(\hat{h}) = {} & \left(\tfrac{1}{2\pi}\right)^2\iint_{\mathbb{R}^2}\left\{2\psi(1)-\psi(it_2+1)-\psi(-it_2+1)\right\}\hat{h}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2\\ & + \tfrac{3}{\pi}\int_{\mathbb{R}}\hat{h}(t,0)\,\mathrm{d}t.\end{aligned}$$

2.3. **Extension of the class of test functions.** In Theorem 2.1, the test function $\hat{h} : \mathbb{C}^2 \to \mathbb{C}$ should be related to a function

$$f \in C_{\mathrm{c}}^{\infty}(\mathrm{SL}_3(\mathbb{R}) /\!\!/ \mathrm{SO}(3))$$

as $\widehat{h_f} = h$, which means that $\hat{h}$ is in the Paley-Wiener class. For example, the "heat kernel"

$$(s_1, s_2) \mapsto \exp\left(\frac{-2}{3}T(s_1^2 + s_1 s_2 + s_2^2)\right) \tag{2.1}$$

is not in the Paley-Wiener class. For application, it is better to broaden the class of test functions. Toward that direction, we have the following result.

Theorem 2.2. *For $\kappa > 2$, set $\mathscr{D}_\kappa = \mathbb{R}^2 + i\,\mathrm{Convex}(\{w(1+\kappa, 1+\kappa) | w \in \mathfrak{S}_3\})$. Let ϕ be a holomorphic function defined in a neighborhood of $\mathscr{D}_\kappa$ with the following properties:*

(i) *$\phi(w\Lambda) = \phi(\Lambda)$ for all $w \in \mathfrak{S}_3$.*
(ii) *There exists $\mu > 5$ such that*

$$|\phi(\Lambda)| \ll (1 + \|\mathrm{Re}(\Lambda)\|)^{-\mu} \quad \textit{for } \Lambda \in \mathscr{D}_\kappa.$$

Then, the series-integrals $\hat{I}_{\mathrm{spect}}(\phi)$ and $\hat{I}_{\mathrm{geom}}(\phi)$ are absolutely convergent in a certain sense, and fit in the identity :

$$\hat{I}_{\mathrm{spect}}(\phi) = \hat{I}_{\mathrm{geom}}(\phi).$$

3. Ingredients of the proof

3.1. **Global ingredient.** There are two main ingredients :

(1) Arthur's works on invariant trace formulas.
(2) A new description of the unipotent terms including an explicit determination of weight factors in the unipotent weighted orbital integrals (Hoffmann-Wakatsuki [HW]. *cf.* Flicker [Fli], Matz [Matz1]).

3.2. **Local ingredient.** After a global consideration, the main task boils down to the determination of Fourier transform of the invariant weighted orbital integrals on the real group $\mathrm{GL}(3, \mathbb{R})$. Recall that the orbital integral of $\gamma \in \mathsf{G}(\mathbb{R})$ is defined as

$$J_{\mathsf{G}}(\gamma, f) = \int_{\mathsf{G}(\mathbb{R})_\gamma \backslash \mathsf{G}(\mathbb{R})} f(g^{-1}\gamma g)\, \mathrm{d}\dot{g}, \quad f \in C_{\mathrm{c}}^{\infty}(\mathsf{G}(\mathbb{R})).$$

Obviously, $f \mapsto J_{\mathsf{G}}(\gamma, f)$ is $\mathrm{Ad}(\mathsf{G}(\mathbb{R}))$-invariant. As such, $J_{\mathsf{G}}(\gamma, f)$ is expected to be a "superposition" of characters $I_{\mathsf{G}}(\pi, f)$ of irreducible unitary (tempered) representations π of $\mathsf{G}(\mathbb{R})$:

$$J_{\mathsf{G}}(\gamma, f) = \int_{\Pi(\mathsf{G})} I_{\mathsf{G}}(\pi, f)\, \Phi_{\mathsf{G}}(\gamma, \pi)\, \mathrm{d}\pi.$$

For instance when γ is regular semisimple, $\Phi_{\mathsf{G}}(\gamma, \pi)$ (the Fourier transform) is determined by Harish-Chandra [HC] and R. Herb [Herb2]. Consider a Levi M and a point $\gamma \in \mathsf{M}(\mathbb{R})$.

If $\mathsf{G}(\mathbb{R})_\gamma = \mathsf{M}(\mathbb{R})_\gamma$, then

$$J_{\mathsf{M}}(\gamma, f) = \int_{\mathsf{G}(\mathbb{R})_\gamma \backslash \mathsf{G}(\mathbb{R})} f(g^{-1}\gamma g) \underbrace{v_{\mathsf{M}}(g)}_{\text{weight factor}} \, \mathrm{d}g$$

If $\mathsf{G}(\mathbb{R})_\gamma \neq \mathsf{M}(\mathbb{R})_\gamma$, then

$$J_{\mathsf{M}}(\gamma, f) = \lim_{a \to 1} \sum_{\mathsf{M} \subset L} r_{\mathsf{M}}^L(\gamma, a) \, J_L(\gamma a, f)$$

where $a \in A_{\mathsf{M}}(\mathbb{R})^0$ with $\mathsf{G}(\mathbb{R})_{\gamma a} \subset \mathsf{M}(\mathbb{R})$ (for more detail see [Ar6]). Due to the factor $v_{\mathsf{M}}(g)$, the distribution $J_{\mathsf{M}}(\gamma)$ (and $J_{\mathsf{M}}^L(\gamma)$) is *not* invariant when M is proper. Arthur invented a way to obtain a set of invariant distributions $I_{\mathsf{M}}^L(\gamma)$ from the family $J_{\mathsf{M}}^L(\gamma)$ ($\mathsf{M} \subset L$) by means of the "weighted characters" of Levi subgroups containing M inductively ([Ar5], [Ar7], [Ar8]). Since $I_{\mathsf{M}}(\gamma) = I_{\mathsf{M}}^{\mathsf{G}}(\gamma)$ is now invariant, one can speak of its Fourier transform $\pi \mapsto \Phi_{\mathsf{M}}(\gamma, \pi)$ which fits in the formula:

$$I_{\mathsf{M}}(\gamma, f) = \int_{\Pi(\mathsf{G}(\mathbb{R}))} I_{\mathsf{G}}(\pi, f) \, \Phi_{\mathsf{M}}(\gamma, \pi) \, \mathrm{d}\pi.$$

An explicit determination of $\Phi_{\mathsf{M}}(\gamma, \pi)$ is partially done by Arthur []; indeed, when a real reductive group in question admits discrete series representations (which is not the case for $\mathrm{GL}_3(\mathbb{R})$), the discrete part of $\Phi_{\mathsf{M}}(\gamma, \pi)$ has been completely determined. For $\mathrm{GL}(3, \mathbb{R})$ (and some other groups of real rank 2 as well as for all real rank 1 groups), W. Hoffmann ([Ho4]) solved the problem at least when γ is regular semisimple (i.e., $\mathsf{G}(\mathbb{R})_\gamma^0$ is a torus) by finding a solution to the holonomic system known by Arthur, introducing a bunch of new generalized hypegeometric functions such as $b(s, z)$ (for rank 1) and

$$\tilde{b}(s_1, s_2, z_1, z_2) = \sum_{n_2=1}^{\infty} \sum_{n_1=n_2}^{\infty} \frac{z_1^{n_1} z_2^{n_2}}{(s_1 + n_1)(s_2 + n_2)}.$$

Thus for GL(3) the regular case is settled, and the singular case (i.e., $\mathsf{G}(\mathbb{R})_\gamma^0$ is not a torus) remains unsettled. Both cases are necessary for the description of trace formula. It is shown by Arthur that the limit formula

$$I_{\mathsf{M}}(\gamma, f) = \lim_{a \to 1} \sum_{\mathsf{M} \subset L} r_{\mathsf{M}}^L(\gamma, a) \, I_L(\gamma a, f)$$

holds true. This suggests that once $\Phi_{\mathsf{M}}(\gamma)$ is obtained for $\gamma \in \mathsf{M}(\mathbb{R})$ with $\mathsf{G}(\mathbb{R})_\gamma \subset \mathsf{M}(\mathbb{R})$, then

$$\Phi_{\mathsf{M}}(\gamma, \pi) = \lim_{a \to 1} \sum_{\mathsf{M} \subset L} r_{\mathsf{M}}^L(\gamma, a) \, \Phi_L(\gamma a, \pi)$$

should yields the desired formula for the singular case (when $\mathsf{G}(\mathbb{R})_\gamma \neq \mathsf{M}(\mathbb{R})_\gamma$). Knowing the regular case settled by Hoffmann, we calculate the limit for every singular case that is necessary to write down the trace formula for $\mathrm{SL}_3(\mathbb{Z})$ this time.

4. Applications

4.1. **Error term estimate in Weyl's law.** Recall the sequence $\Lambda_j \in \mathbb{C}^2/\mathfrak{S}_3$ $(j \geqslant 0)$ of the discrete spectral parameters of $\mathrm{SL}_3(\mathbb{Z})$. We are interested in the discrete spectral counting function of $\Gamma = \mathrm{SL}_3(\mathbb{Z})$:

$$N_{\mathrm{disc}}^{\Gamma}(X) = \#\{j|\, \|\Lambda_j\| \leqslant X\}, \quad X > 0,$$

where

$$\|\Lambda\|^2 := \frac{2}{3}(x^2 + xy + y^2), \quad \Lambda = (x, y) \in \mathbb{C}^2.$$

To describe the formula shortly, for any lattice $\Gamma \subset \mathrm{SL}_3(\mathbb{R})$ we set

$$V(\Gamma) := \frac{\mathrm{vol}(\Gamma\backslash\mathfrak{H}_5)}{\Gamma(7/2)(4\pi)^{5/2}},$$

where vol is taken with respect to the metric $\sqrt{8/3}\,\mathrm{d}\mu(Z)$ on $\mathfrak{H}_5$. An asymptotic formula (if any) of the counting function $N_{\mathrm{disc}}^{\Gamma}(X)$ as X grows to infinity is an obvious analogue of Wey's law, which is established for a spectrum of an elliptic pseudo-differential operator on a compact Riemann manifolds by Hörmander. Since our $\Gamma\backslash\mathfrak{H}_5$ is not compact, Weyl's law for $N_{\mathrm{disc}}^{\Gamma}(X)$ does not follow from a general theorem of Hörmander. For $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, the Wey's law is first established by S. D. Miller [Miller] in the form

$$N_{\mathrm{disc}}^{\mathrm{SL}_3(\mathbb{Z})}(X) \sim V(\Gamma)\, X^5, \quad X \to \infty.$$

By a method of Duistermaat-Kolk-Varadarajan [DKV], an error term estimate for $N_{\mathrm{disc}}^{\Gamma(N)}$ $(N \geqslant 3)$ is obtained by Lapid-Müller (2009):

$$N_{\mathrm{disc}}^{\Gamma(N)}(X) = V(\Gamma(N))\, X^5 + O(X^4(\log X)^3), \quad X \to \infty.$$

Note that Lapid-Müller [LM] actually prove a similar formula for any principal congruence subgroup $\Gamma(N)\,(N \geqslant 3)$ of an arbitrary $\mathrm{SL}_n(\mathbb{R})$. As an application of our explicit trace formula for $\mathrm{SL}_3(\mathbb{Z})$, we have a small improvement:

Theorem 4.1.

$$N_{\mathrm{disc}}^{\mathrm{SL}_3(\mathbb{Z})}(X) = V(\mathrm{SL}_3(\mathbb{Z}))\, X^5 + O(X^4), \quad X \to \infty.$$

4.2. **Small time behavior of the heat trace.** As is well-known, the Poisson summation formula shows the transformation formula for Jacobi's theta function $\theta(t) = \sum_{n\in\mathbb{Z}} \exp(-\pi n^2 t)$ $(t > 0)$ which in turn yields the small-time asymptotic

$$\theta(t) \sim t^{-1/2}, \quad t \to +0.$$

Note that the set of square numbers n^2 coincided with the spectrum of the Laplacian $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ of the flat torus $\mathbb{R}/2\pi\mathbb{Z}$. By our explicit trace formula of $\mathrm{SL}_3(\mathbb{Z})$, we obtain the asymptotic expansion of the heat trace

$$\Theta(T) := \sum_{j=0}^{\infty} \exp(-(\|\Lambda_j\|^2 + 2)\, T), \quad T > 0,$$

which is viewed as a non-commutative analogue of $\theta(t)$. By applying Theorem 2.2 to the heat kernel (2.1) and computing the asymptotic expansion of each term, we obtain the following result. Besides the main term, which is consistent with Wey's law, the second term $c\,T^{-3/2}\log T$ is determined with explicit coefficient c.

Theorem 4.2. *As $T \to +0$, we have the asymptotic expansion of the form*

$$\begin{aligned}\Theta(T) \sim & \Gamma(7/2)\, V(\mathrm{SL}_3(\mathbb{Z}))\, T^{-5/2} \\ & + \tfrac{-1}{3\pi(4\pi)^{5/2}} \operatorname{vol}(\mathsf{M}_1(\mathbb{Z})\backslash \mathsf{M}_1(\mathbb{R}))\, T^{-3/2}\log T \\ & + T^{-3/2}\sum_{n=0}^{\infty} Q_n\, T^{n/2} + T^{-1}(\log T)^2 \sum_{n=0}^{\infty} Q'_n T^{n/2} + T^{-1}\log T \sum_{n=0}^{\infty} Q''_n\, T^{n/2},\end{aligned}$$

where

$$\mathsf{M}_1 = \left\{ \left[\begin{smallmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{smallmatrix}\right] \right\} \cong \mathrm{GL}_1 \times \mathrm{GL}_2.$$

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