On the Fourier transforms of weighted orbital integrals on the real symplectic group of rank three

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We report on the first step of a method for computing the Fourier transforms of weighted orbital integrals, which appear in the Arthur-Selberg trace formula. For more details on that formula and its role in the theory of automorphic forms, see [1]. The full computation of those Fourier transforms has been carried out for groups of rank up to two, and the first step, which consists in the solution of a certain holonomic system of differential equations, has been performed for groups of rank up to three. The case of the symplectic group was the subject of the diploma thesis by Christian Dietz [2]. We give an introduction and an overview of the results. The statement has been somewhat streamlined in order to allow for generalisation to higher rank.

1 Weighted orbital integrals

Weighted orbital integrals are the terms on the geometric side of the Arthur-Selberg trace formula. They are distributions on the set of adelic points of a reductive linear algebraic group, which we assume to be connected. Splitting formulas express them in terms of weighted orbital integrals on the points of that group with coordinates in the local completions of $\mathbb{Q}$. Our method of studying them by means of differential equations works only for the field $\mathbb{R}$, whence we restrict to that case and denote the set of real points of our group by G. The local weighted orbital integral over the orbit of an element γ, which we assume to be semisimple for simpicity, is then defined as

$$J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{G/G_\gamma} f(x\gamma x^{-1}) v_M(x)\, d\dot{x},$$

where M is a Levi subgroup of G (i.e., a Levi component of parabolic subgroup P of G), f is a Schwarz function on G,

$$D(\gamma) = \det_{\mathfrak{g}/\mathfrak{g}_\gamma}(\mathrm{Ad}(\gamma) - \mathrm{id})$$

is the Weyl discriminant, $d\dot{x}$ is an invariant measure on G/G_γ and the weight factor v_M is a specific function on G/M whose definition we will not recall here. In order for the integrand to be left-invariant under the centraliser G_γ, we have to assume that the latter be contained in M.

The set G_{reg} of strongly regular points is characterised by the property that G_γ be a Cartan subgroup. After fixing a Cartan subgroup T of M and restricting γ to $T \cap G_{\mathrm{reg}}$, the measure $d\dot{x}$ on G/T can be chosen independent of γ.

2 Differential equations

Let $\mathfrak{Z}_G$ be $\mathbb{C}$-algebra of bi-invariant differential operators on G. We say that an admissible representation π of G has the infinitesimal character $\chi : \mathfrak{Z}_G \to \mathbb{C}$ if $\pi(z) = \chi(z) \cdot \mathrm{id}$ for all $z \in \mathfrak{Z}_G$. The Harish-Chandra homomorphism $\mathfrak{Z}_G \to \mathfrak{Z}_M$, $z \mapsto z_M$ is characterised by the condition that whenever π is parabolically induced from a representation of M with infinitesimal character ω, we have $\chi(z) = \omega(z_M)$. In fact, this homomorphism is determined by the complexified Lie algebras of G and M. Thus we may replace M by T, in which case we get the Harish-Chandra isomomorphism of $\mathfrak{Z}_G$ onto the set of elements of $\mathfrak{Z}_T$ invariant under the Weyl group W_T.

Arthur has shown that for all Levi subgroups M containing T there exist maps

$$\partial_M : (T \cap G_{\mathrm{reg}}) \times \mathfrak{Z}_G \to \mathfrak{Z}_T$$

such that $\partial_G(z) = z_T$ and

$$J_M(\gamma, zf) = \sum_{L \supset M} \partial_M^L(\gamma, z_L) J_L(\gamma, f),$$

where the sum is taken over all Levi subgroups L of G containing M and ∂_M^L is the analogue of ∂_M in which the ambient group G is replaced by L.

3 Fourier transforms

The Fourier transform of a Schwarz function f on G is the function $\hat{f}$ on the tempered dual $\Pi_{\mathrm{temp}}(G)$ of G defined by

$$\hat{f}(\pi) = \mathrm{tr}\, \pi(f), \quad \text{where} \quad \pi(f) = \int_G f(g)\pi(g)\, dg.$$

The Fourier transform $\hat{I}$ of a tempered distribution I on G invariant under inner automorphisms is defined by

$$I(f) = \hat{I}(\hat{f}).$$

If it exists, one says that I is supported on characters. In order to take care of the reducibility of certain induced representations, Arthur has introduced a set $T(G)$ of isomorphism classes of virtual representations whose distributional characters span the same vector space as those of the tempered representations. All elements of $T(G)$ are parabolically induced virtual representations τ^G, where $\tau \in T(M)$ is elliptic for a Levi subgroup M. The Fourier transform of I is a regular distribution if there exists a function Φ on $T(G)$ such that

$$I(f) = \sum_{[M]} \int_{W_M \backslash T_{\mathrm{ell}}(M)} \Phi(\tau^G) \hat{f}(\tau^G)\, d\tau,$$

where the sum runs over all conjugacy classes of Levi subgroups and $d\tau$ is the image of a measure on $T_{\mathrm{ell}}(M)$ invariant under the action of the group of unramified characters of M.

4 Invariant distributions

The weighted orbital integrals $J_M(\gamma, f)$ are non-invariant tempered distributions evaluated on a test function f. There is an invariant version of the trace formula in which the geometric terms are invariant distributions obtained from the weighted orbital integrals and so-called weighted tempered characters. They also satisfy a splitting formula, and their archimedean components $I_M(\gamma, f)$ satisfy the same differential equations. The knowledge of their Fourier transforms is useful for applications of the trace formula.

Arthur has shown that, although those Fourier transforms are not regular distributions, a similar formula

$$I_M(\gamma, f) = \int_{W_M \backslash T_{\mathrm{disc}}(M)} \Phi_M(\gamma, \tau^G) \hat{f}(\tau^G)\, d\tau$$

is valid in which the set $T_{\mathrm{ell}}(M)$ of elliptic virtual representations is replaced by a larger set $T_{\mathrm{disc}}(M)$ which is still discrete modulo the action of the group of unramified characters of M.

If we fix τ with infinitesimal character χ and a Cartan subgroup T, the tuple $(\Phi_M)_{M \supset T}$ of functions of γ, restricted to $T \cap G_{\mathrm{reg}}$, satisfies the system of differential equations

$$\chi(z) \cdot \Phi_M(\gamma, \tau) = \sum_{L \supset M} \partial_M^L(\gamma, z_L) \Phi_L(\gamma, \tau).$$

Thus, the first step in computing the Fourier transforms is the determination of the finite-dimensional space of solutions of this holonomic system for a given infinitesimal character χ.

Such a character is determined by the orbit of a linear form λ on the complexified Lie algebra $\mathfrak{t}_{\mathbb{C}}$ under the action of the complex Weyl group W by $\chi(z) = z_T(\lambda)$, where the invariant differential operator z_T on the torus $T_{\mathbb{C}}$ can be interpreted as an element of the symmetric algebra of its Lie algebra or as a polynomial on the dual space. The equation for $M = G$ is particularly simple, as $\partial_G(z) = z_T$, and if λ is regular, it has the solutions $\Phi_G(\gamma) = \gamma^\lambda$, which for non-integral λ exist only on the universal cover of $T_{\mathbb{C}}$.

5 Standard solutions

Let P be minimal among the parabolic subgroups of G containing T, Σ_P the set of roots of $\mathfrak{t}_{\mathbb{C}}$ in the unipotent radical of $\mathfrak{p}_{\mathbb{C}}$, and Λ_P the semigroup generated by Σ_P. We cover the complexified torus $T_{\mathbb{C}}$ up to a set of measure zero by the chambers

$$T_{P,\mathbb{C}} = \{\gamma \in T_{\mathbb{C}} \,:\, |\gamma^\alpha| > 1 \;\forall\, \alpha \in \Sigma_P\}$$

for various P and restrict our differential equations to each of them. The reason for doing so is the fact that the map $(\gamma, \mu) \mapsto \gamma^{-\mu}$ embeds $T_{P,\mathbb{C}}$ into the affine toric variety $\mathrm{Hom}_{\mathrm{rings}}(\Lambda_P, \mathbb{C})$, on which we get a system of differential equations with a regular singularity at zero.

The theory of such systems of differential equations implies that, for regular $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, there is a unique solution $(\Phi_M)_{M \supset T}$ on the universal cover of $T_{P,\mathbb{C}}$ such that

- $\Phi_G(\gamma) = \gamma^\lambda$,

- $\Phi_M(\gamma) \to 0$ as $\gamma \underset{P}{\to} \infty$ for $M \neq G$,

where $\gamma \underset{P}{\to} \infty$ means that all $|\gamma^\alpha| \to \infty$ for all $\alpha \in \Sigma_P$. If we replace G by a Levi subgroup $L \supset T$, the standard solutions $(\Phi_M)_{L \supset M \supset T}$ of the corresponding system can be completed to solutions of the original system by setting $\Phi_M = 0$ for $M \not\subset L$. If we let L run through the Levi subgroups containing T and λ through a regular W-orbit, these tuples span the space of solutions on $T_{P,\mathbb{C}}$. We will not be discuss here the subsequent tasks of glueing the solutions across wall chambers with the help of jump relations and determining the Fourier transforms among them as functions of τ using limit formulas.

6 Series expansions

Due to the theory of holonomic systems with regular singularities, the solutions on $T_{P,\mathbb{C}}$ with regular exponent λ can be expanded in series

$$\Phi_M(\gamma) = \sum_{\mu \in \Lambda_M \cap \Lambda_P} a_M(\mu) \gamma^{\lambda - \mu},$$

where Λ_M is the group generated by the set Σ_M of weights of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}/\mathfrak{m}_\mathbb{C}$.

General explicit formulas for the differential operators are available only if z is the Casimir element corresponding to an invariant symmetric bilinear form $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$. In that case,

$$\partial_G(z)\gamma^\lambda = \langle \lambda, \lambda \rangle \cdot \gamma^\lambda, \qquad \partial_M(z) = -\sum_{\alpha \in \Sigma_M} \frac{|\langle \eta_M^G, \alpha \rangle|}{(\gamma^\alpha - 1)(\gamma^{-\alpha} - 1)}$$

for M maximal in G, while $\partial_M(z) = 0$ otherwise. Here η_M^G is the volume form on $\mathfrak{a}_M^G \cong \mathfrak{a}_M/\mathfrak{a}_G$ used in the definition of the weight factor v_M, where A_M is the largest split torus in the centre of M. The operators $\partial_M(z)$ for $M \neq G$ are of degree zero and also have series expansions, as

$$\frac{1}{(\gamma^\alpha - 1)(\gamma^{-\alpha} - 1)} = -\sum_{m=1}^{\infty} m \gamma^{-m\alpha}.$$

7 The recursion formula

Plugging the series expansions of differential operators and of solutions into the differential equation for the Casimir element and infinitesimal character $W\lambda$ and equating coefficients, we obtain a recursion formula for the coefficients:

$$\langle \mu - 2\lambda, \mu \rangle a_M(\mu) = 2 \sum_{\substack{L \supset M \\ \dim \mathfrak{a}_M^L = 1}} \sum_{\alpha \in \Sigma_M^L \cap \Sigma_P} |\langle \eta_M^L, \alpha \rangle| \sum_{m=1}^{\infty} m a_L(\mu - m\alpha).$$

In case of the standard solution, it allows in principle to determine all components Φ_M descending from $M = G$.

E. g., if M is maximal,

$$\Phi_M(\gamma) = \sum_{\alpha\in\Sigma_M\cap\Sigma_P}\sum_{m=1}^{\infty}\frac{|\eta_M(\check{\alpha})|}{m-\lambda(\check{\alpha})}\gamma^{\lambda-m\alpha}.$$

Now let $\dim\mathfrak{a}_M = 2$ and L_1, L_2 intermediate Levi subgroups:

$$\begin{tikzcd} & G \arrow[dl, dash] \arrow[dr, dash] & \\ L_1 \arrow[dr, dash] & & L_2 \arrow[dl, dash] \\ & M & \end{tikzcd}$$

We have contributions to $\partial_M^{L_i}(\gamma,z)\Phi_{L_i}(\gamma)$ of the form

$$|\langle\eta_M^{L_i},\alpha_i\rangle| m_i\gamma^{-m_i\alpha_i}\frac{|\eta_{L_i}(\check{\beta}_i)|}{n_i-\lambda(\check{\beta}_i)}\gamma^{\lambda-n_i\beta_i}.$$

For $m_1\alpha_1 + n_1\beta_1 = \mu = m_2\alpha_2 + n_2\beta_2$ they can be brought on a common denominator. Under certain assumptions on $A = \{\alpha_1,\alpha_2\}$ and $B = \{\beta_1,\beta_2\}$, the numerator becomes $\langle\mu - 2\lambda,\mu\rangle$, hence cancels against the same factor on the left-hand side of the recursion formula. These assumptions motivate the following notion.

8 Root cones

Definition 1. *Let Σ be a subset of a root system. A root cone in Σ is a quadruple $\mathcal{K} = (A, B, c, \succ)$, where A, B are subsets of Σ spanning the same real vector space V (hence $\check{A}$, $\check{B}$ span the same vector space $\check{V}$), $c : V\times\check{V}\to\mathbb{R}$ is a perfect pairing and $\succ$ is a relation between A and B satisfying the following conditions.*

(i) The sets A and B are the sets of maximal proper faces resp. vertices of an abstract polytope (see [4]) of rank $\dim V$ with incidence relation $\succ$.

(ii) Given $\alpha\in A$, $\beta\in B$ with $\alpha\succ\beta$, we have

$$c(\alpha,\check{\beta}) = 0$$

and, for all $\alpha'\in A$, $\beta'\in B$,

$$\alpha'\succ\beta,\ \alpha\succ\beta' \ \Rightarrow\ c(\alpha',\check{\beta}')\geq 0$$

with strict inequality for some α', β'.

(iii) For all $\alpha\in A$, $\beta\in B$ we have $c(\alpha,\check{\beta})\in\mathbb{Z}$.

(iv) For all β, $\beta'\in B$ we have

$$2\langle\beta,\beta'\rangle = c(\beta,\check{\beta}')\langle\beta',\beta'\rangle + c(\beta',\check{\beta})\langle\beta,\beta\rangle.$$

We say that the root cone $\mathcal{K}$ is convex if

$$C_A = \{\lambda \in V \mid c(\lambda, \check{\beta}) > 0 \quad \forall \beta \in B\},$$
$$C_{\check{B}} = \{X \in \check{V} \mid c(\alpha, X) > 0 \quad \forall \alpha \in A\}$$

are polyhedral cones with edges $\mathbb{R}_+\alpha$ for $\alpha \in A$ (resp. $\mathbb{R}_+\check{\beta}$ for $\beta \in B$).

It follows from (iv) that $c(\beta, \check{\beta}) = 1$ for all $\beta \in B$. In the convex case, the first condition in (ii) is an equivalence and $\succ$ is encoded in c.

Via c, the elements of A define hyperplanes in V. If we choose an orientation on V, a point X not lying on any of these hyperplanes induces an orientation on each face of the polytope, hence the latter becomes a cycle in V. It can be viewed as a cone over a cycle in a hyperplane $W \subset V$ through X, which represents a multiple of the fundamental class in the homology group of $W \setminus \{X\}$ with respect to the orientation induced by $0 \in V$. In this way, we get a multiplicity $m(X)$ defined for generic $X \in V$. In the convex case, m is just the characteristic function of $C_{\check{B}}$.

In [3], the name "root cone" was given to what we now call convex root cone, but with a slightly more restrictive condition than (iii) allowing only one nonzero value. For root cones $\mathcal{K} = (A, B, c)$ in that restricted sense, it was proved that $\check{\mathcal{K}} = (\check{B}, \check{A}, \check{c})$ is also a root cone, where $\check{c}(\check{\beta}, \check{\alpha}) = c(\alpha, \beta)$. In [2], it was checked that this duality remains intact for all convex root cones. It is open whether this is true in general, if we set $\check{\beta} \check{\succ} \check{\alpha} \Leftrightarrow \alpha \succ \beta$.

9 Special functions

As in sections 5–7, we fix P and T again. If M is a Levi subgroup of G containing T, a root cone $\mathcal{K} = (A, B, c, \succ)$ in the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ will be called a root cone for (M, P) if A, $B \subset \Sigma_M \cap \Sigma_P$ and the projection $\check{V} \to \mathfrak{a}_M^G$ is bijective. In this case, let $\eta_{\check{V}}$ be the pullback of the volume form η_M^G under that projection. If, moreover, there is no other root cone $(A', B', c, \succ')$ for (M, P) such that $B \subset B'$ and $\succ'$ has the same restriction to $(A \cap A') \times B$ as $\succ$, then we call $\mathcal{K}$ a maximal root cone for (M, P).

Given an element of

$$\Lambda_{\mathcal{K}} = \{\mu \in \Lambda \mid c(\mu, X) \geq 0 \; \forall \, X \in \operatorname{ess\,supp} m\},$$

we define

$$a_{\mathcal{K}}(\mu, \lambda) = \int_V e^{\lambda(X) - c(\mu, X)} m(X) |d\eta_{\check{V}}(X)|.$$

The integral converges if λ is negative on $\operatorname{ess\,supp} m$ and extends meromorphically to $\mathfrak{t}_{\mathbb{C}}^*$ with at most simple poles along the hyperplanes $\lambda(\check{\beta}) \in \mathbb{Z}_{\geq 0}$ for $\beta \in B$. In the special case that $\mathcal{K}$ is simplicial with $B = \{\beta_1, \ldots, \beta_l\}$, we have $\Lambda_{\mathcal{K}} = \Lambda \cap \bar{C}_A$ and

$$a_{\mathcal{K}}(\mu, \lambda) = \frac{|\eta_{\check{V}}(\check{\beta}_1, \ldots, \check{\beta}_l)|}{\prod_{\beta \in B} (c(\mu, \check{\beta}) - \lambda(\check{\beta}))}.$$

In the case $l = 2$, one recognises the common denominator mentioned at the end of section 7.

If, in addition, we are given a set $\Psi \subset \Lambda_{\mathcal{K}}$ of weights, we define a special function of $\gamma \in T_{P,\mathbb{C}}$ with parameter λ by

$$\phi_{\mathcal{K},\Psi}(\gamma,\lambda) = \sum_{\mu \in \Psi} a_{\mathcal{K}}(\mu,\lambda)\gamma^{-\mu}.$$

This series is absolutely convergent. In the special case that Ψ is the submonoid Λ_Π of Λ generated by a linearly independent subset $\Pi \subset \Lambda$, there is a closed formula

$$\phi_{\mathcal{K},\Lambda_\Pi}(\gamma,\lambda) = \int_V \frac{e^{\lambda(X)}}{\prod_{\delta\in\Pi}\left(1-\gamma^{-\delta}e^{-c(\delta,X)}\right)} m(X)|d\eta_{\check{V}}(X)|,$$

Theorem 1 (see [3])**.** *Suppose that G is one of the following:*

(i) a group of real rank one,

(ii) a split group of real rank two, or

(iii) the group GL(4)*.*

Then the standard solution on $T_{P,\mathbb{C}}$ with exponent λ is given by

$$\phi_M(\gamma) = \gamma^\lambda \sum_{\mathcal{K}} \phi_{\mathcal{K},\Psi}(\gamma,\lambda),$$

where the sum is over all maximal root cones $\mathcal{K} = (A,B,c)$ for (M,P) and each Ψ is determined by the corresponding $\mathcal{K}$.

All root cones appearing in the above theorem are convex. For each root cone, Ψ is a subset of $\Lambda_A \cap C_A \cap \bar{C}_B$, where Λ_A is the lattice generated by A. It can be a proper subset only in case (iii).

10 The case of the split symplectic group of rank three

Having extendes the notion of root cones, we can now state the result of [2] in a streamlined way.

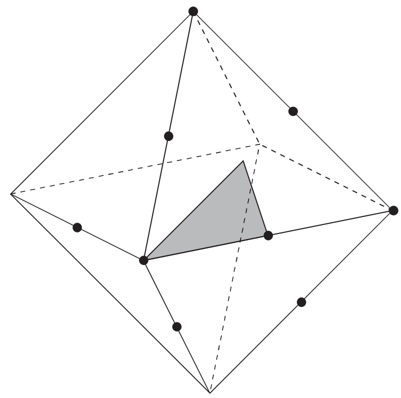
Theorem 2. *The above theorem is also true for the split symplectic group of rank three if we allow nonconvex root cones.*

The component Φ_T is a sum of 198 special functions $\phi_{\mathcal{K},\Psi}$, of which

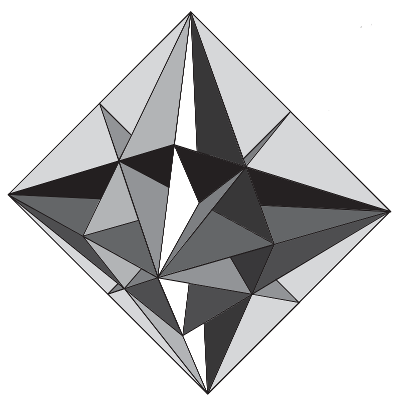
- *154 come from simplicial (three-sided) root cones,*
- *12 come from four-sided root cones,*
- *26 come from five-sided root cones,*
- *6 come from seven-sided root cones.*

The nonconvex root cones are the five- and seven-sided ones.

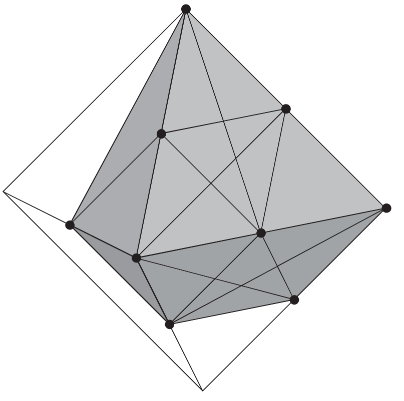
In the rest of this paper, we try to give some feeling for the statement and the proof. In all the subsequent figures, we depict the Lie algebra $\mathfrak{t}$ of the maximal split torus T, which we identify with its dual space using some Weyl-group-invariant pairing. First we depict the convex hull of the roots, which is an octahedron, and indicate some system of positive roots by black dots. The corresponding root spaces make up the unipotent radical of a minimal parabolic subgroup P. The corresponding Weyl chamber intersects the surface of the octahedron in the grey triangle.



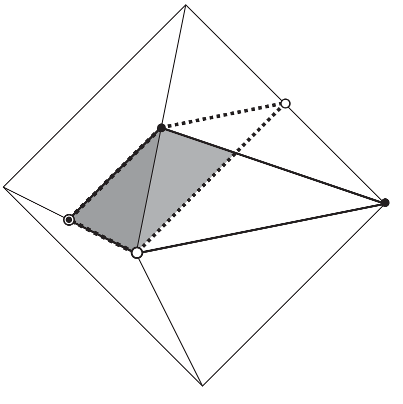
The components ϕ_M of the solution are indexed by the Levi subgroups M of G containing T. The map $M \mapsto \mathfrak{a}_M$ sets up a bijection with the special subspaces of $\mathfrak{t}$, whose intersections with the octahedron are depicted in the next figure.



First one computes the components ϕ_L for the maximal Levi subgroups L, i. e., those with $\dim \mathfrak{a}_L = 1$, and then the components ϕ_M for the Levi subgroups M with $\dim \mathfrak{a}_M = 2$, in terms of special functions $\phi_{\mathcal{K},\Psi}$ with root cones $\mathcal{K}$ of rank 1 resp. 2. Next one considers the differential equation for ϕ_T. The right-hand side is a sum over the 9 Levi subgroups M of the terms $\partial_T^M \Phi_M$. If we plug in the formulas just obtained, we obtain a sum of 258 series over semigroups generated by triples $\{\alpha, \beta_1, \beta_2\}$. They have to be split up into up to 10 partial sums and recombined into the special functions $\phi_{\mathcal{K},\Psi}$ attached to root cones $\mathcal{K}$ of rank 3. Each of the intermediate partial series runs over the intersection of a lattice with some polyhedral cone in $\mathfrak{t}^*$. The next figure shows the walls of all the cones which appear in this process.

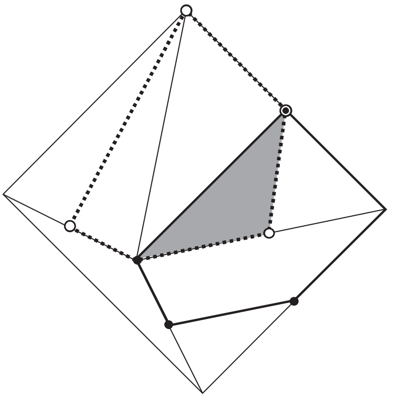


The final result in Theorem 2 is a sum over root cones. The next figure shows one of 154 three-sided root cones. The elements of B are indicated by solid dots, those of A by hollow dots. Via the pairing c, the elements of A determine edges connecting the elements of B, which are drawn as thick lines, whereas the elements of B determine edges connecting the elements of A, which are drawn as dotted lines. The edges and vertices drawn on the octahedron (which is not part of the structure) are just the intersections of faces resp. edges of polyhedral cones with the octahedron. This is the reason for seemingly broken edges.

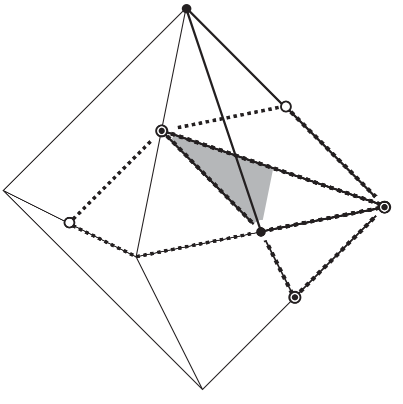


The set Ψ is the intersection of the lattice Λ_A with a cone whose intersection with the surface of the octahedron is shaded in gray. The edges already drawn prevent us from indicating graphically which faces are included in this cone.

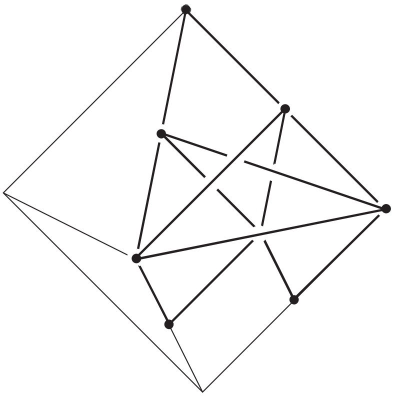
The following figure shows one of 12 four-sided root cones. These are convex as well. In order to characterise the pairing c, it would suffice to indicate the relation $\succ$, e. g., by labelling the edges of one cone and the faces of the other.



Next we show one of 26 five-sided root cones, which are not convex. Here it happens that $c(\alpha, \check{\beta}) = 0$ although $\alpha \not\succ \beta$. To indicate such a spurious incidence in the figure, we have interrupted the edge corresponding to α at the vertex β, and similarly with roles reversed.



Our last figure shows one of 6 seven-sided root cones. For simplicity, we have only drawn the vertices in B and the edges corresponding to the elements of A.



A detailed account of the proof will appear elsewhere.

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