ASYMPTOTIC BEHAVIOR OF AUTOMORPHIC SPECTRA AND THE TRACE FORMULA, II

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Abstract. In this article we discuss some recent developments concerning the asymptotic behavior of the discrete spectrum of the right regular representation in $L^2(\Gamma\backslash G)$ for a lattice Γ in a semisimple Lie group G.

1. Introduction

This article is a continuation of [34]. Some of the problems raised in [34] have been solved by now and I will report on some of the new developments.

Let G be a connected linear semisimple Lie group of non-compact type with a fixed choice of a Haar measure. Let $\Pi(G)$ denote the set of all equivalence classes of irreducible unitary representations of G, equipped with the Fell topology. We fix a Haar measure on G. Let $\Gamma \subset G$ be a lattice in G, i.e., a discrete subgroup such that $\mathrm{vol}(\Gamma\backslash G) < \infty$. Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$. Let $L^2_{\mathrm{disc}}(\Gamma\backslash G)$ be the span of all irreducible subrepresentations of R_Γ and denote by $R_{\Gamma,\mathrm{disc}}$ the restriction of R_Γ to $L^2_{\mathrm{disc}}(\Gamma\backslash G)$. Then $R_{\Gamma,\mathrm{disc}}$ decomposes discretely as

$$R_{\Gamma,\mathrm{disc}} \cong \widehat{\bigoplus}_{\pi\in\Pi(G)} m_\Gamma(\pi)\pi, \tag{1.1}$$

where

$$m_\Gamma(\pi) = \dim \mathrm{Hom}_G(\pi, R_\Gamma) = \dim \mathrm{Hom}_G(\pi, R_{\Gamma,\mathrm{disc}})$$

is the multiplicity with which π occurs in R_Γ. The multiplicities are known to be finite under a weak reduction-theoretic assumption on (G,Γ), which is satisfied if G has no compact factors or if Γ is arithmetic. The study of the multiplicities $m_\Gamma(\pi)$ is one of the main concerns in the theory of automorphic forms. Apart from special cases like discrete series representations, one cannot hope in general to describe the multiplicity function on $\Pi(G)$ explicitly. A more feasible and interesting problem is the study of the asymptotic behavior of the multiplicities with respect to the growth of various parameters such as the level of congruence subgroups or the infinitesimal character of π. This is closely related to the study of families of automorphic forms (see [42]).

We pick three representative problems which we will discuss in some detail.

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The first problem in this context is the Weyl law. It is related to the basic question of the existence of cusp forms. Let K be a maximal compact subgroup of G. Fix an irreducible representation σ of K. Let $\Pi(G;\sigma)$ be the subspace of all $\pi \in \Pi(G)$ such that $[\pi|_K\colon \sigma] > 0$. Especially, if σ is the trivial representation, then $\Pi(G;\sigma)$ is the spherical dual $\Pi^{\mathrm{sph}}(G)$. Given $\pi \in \Pi(G)$, denote by $\lambda_\pi = \pi(\Omega)$ the Casimir eigenvalue of π. For $\lambda \geq 0$ let the counting function be defined by

$$N_\Gamma^\sigma(\lambda) = \sum_{\substack{\pi\in\Pi(G;\sigma)\\ |\lambda_\pi|\leq\lambda}} m_\Gamma(\pi). \tag{1.2}$$

Then the problem is to determine the behavior of the counting function as $\lambda \to \infty$.

Another basic problem is the limit multiplicity problem, which is the study of the asymptotic behavior of the multiplicities if $\mathrm{vol}(\Gamma\backslash G) \to \infty$. For $G = \mathrm{GL}_n(\mathbb{R})$ this corresponds to the study of harmonic families of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, $\mathbb{A}$ being the ring of adeles (see [42]). More precisely, for a given lattice Γ define the discrete spectral measure μ_Γ on $\Pi(G)$, associated to Γ, by

$$\mu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi\in\Pi(G)} m_\Gamma(\pi)\delta_\pi, \tag{1.3}$$

where δ_π is the Dirac measure at π. Then the limit multiplicity problem is concerned with the study of the asymptotic behavior of μ_Γ as $\mathrm{vol}(\Gamma\backslash G) \to \infty$. For appropriate sequences of lattices (Γ_n) one expects that the measures μ_{Γ_n} converge to the Plancherel measure μ_{pl} on $\Pi(G)$.

One can also consider more sophisticated functions of the spectrum. An important example is the Ray-Singer analytic torsion $T_X(\rho)$ of a compact Riemannian manifold X and a finite dimensional representation ρ of its fundamental group $\pi_1(X)$ [40]. The analytic torsion $T_X(\rho)$ is defined as a weighted product of regularized determinants of the Laplace operators $\Delta_p(\rho)$ on p-forms on X with values in the flat vector bundle associated to ρ. In the present context X is a compact locally symmetric space $\Gamma\backslash G/K$, where K is a maximal compact subgroup of G and Γ is a uniform, torsion free lattice in G. Of particular interest are representations of Γ which arise as the restriction of a representation of G. Let (Γ_n) be a tower of normal subgroups of Γ. Put $X_n = \Gamma_n\backslash G/K$, $n \in \mathbb{N}$. Then $X_n \to X$ is a sequence of finite normal coverings of X. For appropriate representations, called strongly acyclic, Bergeron and Venkatesch [4] studied the asymptotic behavior of $\log T_{X_n}(\rho)$ as $n \to \infty$. One of their main results is

$$\lim_{n\to\infty} \frac{\log T_{X_n}(\rho)}{\mathrm{vol}(X_n)} = \log T_X^{(2)}(\rho), \tag{1.4}$$

where $T_X^{(2)}(\rho)$ is the L^2-torsion [20], [22]. Using the equality of analytic torsion and Reidemeister torsion [7], [27], (1.4) implies results about the growth of the torsion subgroup in the integer homology of arithmetic groups. Let $\mathbf{G}$ be a semisimple algebraic group over $\mathbb{Q}$, $G = \mathbf{G}(\mathbb{R})$ and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ a co-compact, arithmetic subgroup. As shown in [4], there are strongly acyclic representations ρ of G on a finite dimensional vector space V such that V

contains a Γ-invariant lattice M. Let $\mathcal{M}$ be the local system of free $\mathbb{Z}$-modules over X, attached to M. Then the cohomology $H_*(X,\mathcal{M})$ of X with coefficients in $\mathcal{M}$ is a finite abelian group. Denote by $|H_*(X,\mathcal{M})|$ its order. Assume that $d=\dim(X)$ is odd. Then by [4] one has

$$\lim_{n\to\infty}\sum_{p=1}^{d}(-1)^{p+\frac{d-1}{2}}\frac{\log|H_p(X_n,\mathcal{M})|}{[\Gamma\colon\Gamma_n]}=c_{M,G}\operatorname{vol}(X),$$

where $c_{M,G}$ is a constant that depends only on G and M. Moreover, if $\delta(G):=\operatorname{rank}G-\operatorname{rank}K=1$, then $c_{M,G}>0$. It is conjectured that the limit

$$\lim_{n\to\infty}\frac{\log|H_j(X_n,\mathcal{M})|}{[\Gamma\colon\Gamma_n]} \tag{1.5}$$

always exists and is equal to zero, unless $\delta(G)=1$ and $j=(d-1)/2$. In the latter case it is equal to $c_{M,G}$ times $\operatorname{vol}(X)$. The conjecture is known to be true for $G=\mathrm{SL}_2(\mathbb{C})$.

An important problem is to extend these results to the non-compact case and we will discuss the current status of the problem. In the present article we will focus on the Weyl law and the analytic torsion.

2. The Weyl law

The Weyl law is concerned with the study of the asymptotic behavior of the counting function (1.2) as $\lambda\to\infty$. This is the first problem which needs to be solved in order to be able to pursue a deeper study of the cuspidal automorphic spectrum. For example, the study of statistical properties of the automorphic spectrum requires first of all to know that the spectrum is infinite and has the right asymptotic properties. This, in particular, concerns the study of families of automorphic forms (see [42]).

The investigation of the asymptotic behavior of the counting function (1.2) is closely related to the study of the counting function of the eigenvalues of the Laplace operator on a compact Riemannian manifold. We refer to [34, Sect. 3] for details.

The connection with the estimation of the counting function (1.2) is established as follows. Let $\widetilde{X}=G/K$. It can be equipped with a G-invariant metric which is unique up to scaling. Let $X=\Gamma\backslash\widetilde{X}$. Assume that Γ is torsion free. Then X is a complete Riemannian manifold of finite volume. Let $\sigma\in\widehat{K}$ and let $\widetilde{E}_\sigma\to\widetilde{X}$ be the homogeneous vector bundle associated to σ, which is equipped with the invariant Hermitian metric induced by σ. Let $E_\sigma=\Gamma\backslash\widetilde{E}_\sigma$ be the corresponding locally homogeneous vector bundle over X. Let ∇^σ be the connection in E_σ induced by the canonical connection in $\widetilde{E}_\sigma$. Let $\Delta_\sigma=(\nabla^\sigma)^*\nabla^\sigma$ be the Bochner-Laplace operator, acting in $C^\infty(X,E_\sigma)$. It is an elliptic, second order, formally self-adjoint differential operator of Laplace type, i.e., its principal symbol is given by $\|\xi\|_x^2\operatorname{Id}_{E_{\sigma,x}}$. Let $\Omega\in\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the Casimir element and $R_\Gamma(\Omega)$ the Casimir operator acting in $C^\infty(\Gamma\backslash G)$.

Let $C^\infty(X, E_\sigma)$ be the space of smooth sections of E_σ. Recall that there is a canonical isomorphism

$$C^\infty(X, E_\sigma) \cong \left(C^\infty(\Gamma\backslash G) \otimes V_\sigma\right)^K. \tag{2.1}$$

With respect to this isomorphism, the Bochner-Laplace operator is related to the Casimir operator $R_\Gamma(\Omega)$ by

$$\Delta_\sigma = -R_\Gamma(\Omega) + \lambda_\sigma \, \mathrm{Id}, \tag{2.2}$$

where λ_σ is the Casimir eigenvalue of σ. Assume that X is compact. Then Δ_σ has a pure discrete spectrum consisting of a sequence of eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$ of finite multiplicities. Let

$$N_\Gamma(\lambda; \sigma) = \#\{j : \lambda_j \le \lambda\}$$

be the counting function of the eigenvalues, where eigenvalues are counted with their multiplicity. Using (2.1) and (2.2), it follows that the counting function (1.2) has the same asymptotic behavior as $N_\Gamma(\lambda; \sigma)$. Applying the heat equation method [5], [16], we obtain the following Weyl law

$$N_\Gamma(\lambda, \sigma) = \frac{\dim(\sigma)\,\mathrm{vol}(\Gamma\backslash G/K)}{(4\pi)^{d/2}\mathbf{\Gamma}(d/2+1)}\lambda^{d/2} + o(\lambda^{d/2}), \quad \lambda \to \infty, \tag{2.3}$$

where $d = \dim(X)$.

Remark 2.1. *We emphasize that the heat equation method does not lead to any nontrivial estimation of the remainder term. Instead one has to use the wave equation* [17]. *For a locally symmetric manifold this means to use the Selberg trace formula. So far estimations of the remainder term are only known if σ is the trivial representation, i.e., for the case of the Laplace operator on functions.*

Remark 2.2. *For a locally symmetric space $X = \Gamma\backslash\widetilde{X}$ there is not only the Laplace operator, but the whole algebra of invariant differential operators $\mathcal{D}(\widetilde{X})$ on $\widetilde{X}$, which one needs to consider. This leads to corresponding asymptotic formulas which contain more information about the distribution of the discrete spectrum then just the Weyl law.*

If Γ is not co-compact, then Δ_σ has a nonempty continuous spectrum which consists of a half-line $[c, \infty)$ for some $c \ge 0$. Then the heat equation method breaks down, because the heat operator $e^{-t\Delta_\sigma}$ is not trace class anymore. One of the basic tools to study the cuspidal automorphic spectrum in the finite volume case is the trace formula.

We turn now to the case of a general lattice. We assume that $G = \mathbf{G}(\mathbb{R})$, where $\mathbf{G}$ is a connected semisimple algebraic group over $\mathbb{Q}$. Let $X = \Gamma\backslash\widetilde{X} = \Gamma\backslash G/K$ and $E_\sigma \to X$ be as above. Let $\Delta_\sigma : C^\infty(X, E_\sigma) \to C^\infty(X, E_\sigma)$ be the Bochner-Laplace operator. As operator in $L^2(X, E_\sigma)$ it is essentially self-adjoint. Let $L^2_{\mathrm{disc}}(X, E_\sigma)$ be the subspace of $L^2(X, E_\sigma)$ which is the closure of the span of all L^2-eigensections of Δ_σ. Recall that a cusp form for

Γ is a smooth K-finite function $\phi\colon \Gamma\backslash G \to \mathbb{C}$ which is a joint eigenfunction of the center of the universal enveloping algebra $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ and which satisfies

$$\int_{\Gamma\cap N_P\backslash N_P} \phi(nx)\, dn = 0$$

for all unipotent radicals N_P of proper rational parabolic subgroups P of G, i.e., $P = P(\mathbb{R})$, where P is a rational parabolic subgroup of $\mathbf{G}$. Put

$$L^2_{\mathrm{cus}}(X, E_\sigma) := (L^2_{\mathrm{cus}}(\Gamma\backslash G) \otimes V_\sigma)^K.$$

Then $L^2_{\mathrm{cus}}(X, E_\sigma)$ is contained in $L^2_{\mathrm{disc}}(X, E_\sigma)$. The orthogonal complement $L^2_{\mathrm{res}}(X, E_\sigma)$ of $L^2_{\mathrm{cus}}(X, E_\sigma)$ in $L^2_{\mathrm{disc}}(X, E_\sigma)$ is called the *residual subspace*. By Langland's theory of Eisenstein series it follows that $L^2_{\mathrm{res}}(X, E_\sigma)$ is spanned by iterated residues of cuspidal Eisenstein series. By definition we have an orthogonal decomposition

$$L^2_{\mathrm{disc}}(X, E_\sigma) = L^2_{\mathrm{cus}}(X, E_\sigma) \oplus L^2_{\mathrm{res}}(X, E_\sigma).$$

Let $N^{\mathrm{disc}}_\Gamma(\lambda;\sigma)$, $N^{\mathrm{cus}}_\Gamma(\lambda;\sigma)$, and $N^{\mathrm{res}}_\Gamma(\lambda;\sigma)$ be the counting function of the eigenvalues with eigensections belonging to the corresponding subspace. The following results about the growth of the counting functions hold for any lattice Γ in a real semisimple Lie group. Let $d = \dim X$. Donnelly [8] has proved the following bound for the cuspidal spectrum

$$\limsup_{\lambda\to\infty} \frac{N^{\mathrm{cus}}_\Gamma(\lambda,\sigma)}{\lambda^{d/2}} \le \frac{\dim(\sigma)\,\mathrm{vol}(X)}{(4\pi)^{d/2}\mathbf{\Gamma}\left(\frac{d}{2}+1\right)}. \tag{2.4}$$

For the full discrete spectrum, we have at least an upper bound for the growth of the counting function. The main result of [28] states that

$$N^{\mathrm{disc}}_\Gamma(\lambda,\sigma) \ll (1+\lambda^{2d}). \tag{2.5}$$

This result implies that invariant integral operators are of trace class on the discrete subspace which is the starting point for the trace formula. The proof of (2.5) relies on the description of the residual subspace in terms of iterated residues of Eisenstein series.

Let $N^{\mathrm{cus}}_\Gamma(\lambda)$ be the counting function with respect to the trivial representation σ_0 of K, i.e., the counting function of the cuspidal spectrum of the Laplacian on functions. Then Sarnak [41] conjectured that if $\mathrm{rank}(G/K) > 1$, Weyl's law holds for $N^{\mathrm{cus}}_\Gamma(\lambda)$, which means that equality holds in (2.4). Furthermore, one expects that the growth of the residual spectrum is of lower order than the cuspidal spectrum.

In the meantime Sarnak's conjecture has been verified in quite a number of cases. A. Reznikov proved it for congruence groups in a group G of real rank one, S. Miller [26] proved it for $\mathbf{G} = \mathrm{SL}(3)$ and $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, the author [31] established it for $\mathbf{G} = \mathrm{SL}(n)$ and a congruence group Γ. The most general result is in the spherical case is due to Lindenstrauss and Venkatesh [19] who proved the following theorem.

Theorem 2.3. *Let* $\mathbf{G}$ *be a split adjoint semi-simple group over* $\mathbb{Q}$ *and let* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *be a congruence subgroup. Let* $d = \dim S$*. Then*

$$N_\Gamma^{\text{cus}}(\lambda) \sim \frac{\operatorname{vol}(\Gamma\backslash\widetilde{X})}{(4\pi)^{d/2}\mathbf{\Gamma}\left(\frac{d}{2}+1\right)}\lambda^{d/2}, \quad \lambda \to \infty. \tag{2.6}$$

The method used by Lindenstrauss and Venkatesh is based on the construction of convolution operators with pure cuspidal image. It avoids the delicate estimates of the contributions of the Eisenstein series to the trace formula. This proves existence of many cusp forms for these groups.

Now we consider an arbitrary K-type. To formulate our result, we need to pass to the adelic framework. Let $\mathbf{G}$ be a reductive algebraic group over $\mathbb{Q}$. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$. Denote by $A_{\mathbf{G}}$ the split component of the center of $\mathbf{G}$ and let $A_{\mathbf{G}}(\mathbb{R})^0$ be the component of 1 in $A_{\mathbf{G}}(\mathbb{R})$. Let ξ_0 be the trivial character of $A_{\mathbf{G}}(\mathbb{R})^0$ and denote by $\Pi(\mathbf{G}(\mathbb{A}), \xi_0)$ the set of equivalence classes of irreducible unitary representations of $\mathbf{G}(\mathbb{A})$ whose central character is trivial on $A_{\mathbf{G}}(\mathbb{R})^0$. Let $L^2_{\text{cus}}(\mathbf{G}(\mathbb{Q})A_{\mathbf{G}}(\mathbb{R})^0\backslash\mathbf{G}(\mathbb{A}))$ be the subspace of cusp forms in $L^2(\mathbf{G}(\mathbb{Q})A_{\mathbf{G}}(\mathbb{R})^0\backslash\mathbf{G}(\mathbb{A}))$. Denote by $\Pi_{\text{cus}}(\mathbf{G}(\mathbb{A}), \xi_0)$ the subspace of all π in $\Pi(\mathbf{G}(\mathbb{A}), \xi_0)$ which are equivalent to a subrepresentation of the regular representation in $L^2_{\text{cus}}(\mathbf{G}(\mathbb{Q})A_{\mathbf{G}}(\mathbb{R})^0\backslash\mathbf{G}(\mathbb{A}))$. For $\pi \in \Pi_{\text{cus}}(G(\mathbb{A}), \xi_0)$ let $m(\pi)$ denote the multiplicity with which π occurs in the space of cusp forms $L^2_{\text{cus}}(\mathbf{G}(\mathbb{Q})A_{\mathbf{G}}(\mathbb{R})^0\backslash\mathbf{G}(\mathbb{A}))$. Let A_f be the ring of finite adèles. Any irreducible unitary representation π of $\mathbf{G}(\mathbb{A})$ can be written as $\pi = \pi_\infty \otimes \pi_f$, where π_∞ and π_f are irreducible unitary representations of $\mathbf{G}(\mathbb{R})$ and $\mathbf{G}(\mathbb{A}_f)$, respectively. Let $\mathcal{H}_{\pi_\infty}$ and $\mathcal{H}_{\pi_f}$ denote the Hilbert space of the representation π_∞ and π_f, respectively. Let K_f be an open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$. Denote by $\mathcal{H}_{\pi_f}^{K_f}$ the subspace of K_f-invariant vectors in $\mathcal{H}_{\pi_f}$. Let $\mathbf{G}(\mathbb{R})^1$ be the subgroup of all $g \in \mathbf{G}(\mathbb{R})$ with $|\det(g)| = 1$. Given $\pi \in \Pi(\mathbf{G}(\mathbb{A}), \xi_0)$, denote by λ_π the Casimir eigenvalue of the restriction of π_∞ to $\mathbf{G}(\mathbb{R})^1$. For $\lambda \geq 0$ let $\Pi_{\text{cus}}(\mathbf{G}(\mathbb{A}), \xi_0)_\lambda$ be the space of all $\pi \in \Pi_{\text{cus}}(\mathbf{G}(\mathbb{A}), \xi_0)$ which satisfy $|\lambda_\pi| \leq \lambda$. Then we have the following theorem, which is work in progress and which is joint work with J. Matz.

Theorem 2.4. *Let* $\mathbf{G}$ *be one of the following types of groups: An inner form of* $\operatorname{GL}(n)$ *or* $\operatorname{SL}(n)$*, a quasi-split classical group, or the exceptional group* G_2*. Let* $K_\infty \subset \mathbf{G}(\mathbb{R})^1$ *be a maximal compact subgroup and let* $K_f \subset \mathbf{G}(\mathbb{A}_f)$ *be a congruence subgroup. Let* $d = \dim \mathbf{G}(\mathbb{R})^1/K_\infty$*. For every* $\sigma \in \Pi(K_\infty)$ *we have*

$$\begin{aligned}\sum_{\pi\in\Pi_{\text{cus}}(\mathbf{G}(\mathbb{A}),\xi_0)_\lambda} & m(\pi)\dim\left(\mathcal{H}_{\pi_f}^{K_f}\right)\dim\left(\mathcal{H}_{\pi_\infty}\otimes V_\sigma\right)^{K_\infty} \\ & \sim \dim(\sigma)\frac{\operatorname{vol}(\mathbf{G}(\mathbb{Q})A_{\mathbf{G}}(\mathbb{R})^0\backslash\mathbf{G}(\mathbb{A})/K_f)}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^{d/2}.\end{aligned} \tag{2.7}$$

as $\lambda \to \infty$*. Furthermore,*

$$\sum_{\pi\in\Pi_{\text{res}}(\mathbf{G}(\mathbb{A}),\xi_0)_\lambda} m(\pi)\dim\left(\mathcal{H}_{\pi_f}^{K_f}\right)\dim\left(\mathcal{H}_{\pi_\infty}\otimes V_\sigma\right)^{K_\infty} \ll \lambda^{d/2-1}. \tag{2.8}$$

If $\mathbf{G}$ is semisimple and simply connected, it satisfies strong approximation. Then (2.7) and (2.8) imply the following corollary.

Corollary 2.5. *Let* $\mathbf{G}$ *be as above and assume that* $\mathbf{G}$ *is semisimple and simply connected. Then for every* $\sigma \in \Pi(K_\infty)$ *we have*

$$N_\Gamma^{\mathrm{cus}}(\lambda,\sigma) \sim \frac{\dim(\sigma)\operatorname{vol}(\Gamma\backslash\widetilde{X})}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^{d/2} \tag{2.9}$$

as $\lambda \to \infty$. *Moreover, the residual spectrum satisfies*

$$N_\Gamma^{\mathrm{res}}(\lambda,\sigma) \ll \lambda^{d/2-1}. \tag{2.10}$$

Sketch of the proof. For all details we refer to our forthcoming paper. By Karamata's theorem [5, Theorem 2.42] it follows that in order to prove (2.7), it suffices to show that there exists an asymptotic expansion of the form

$$\sum_{\pi\in\Pi_{\mathrm{cus}}(\mathbf{G}(\mathbb{A}),\xi_0)} m(\pi)e^{t\lambda_{\pi_\infty}}\dim\big(\mathcal{H}_{\pi_f}^{K_f}\big)\dim\big(\mathcal{H}_{\pi_\infty}\otimes V_\sigma\big)^{K_\infty} \sim \frac{\dim(\sigma)\operatorname{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1/K_f)}{(4\pi)^{d/2}}t^{-d/2} \tag{2.11}$$

as $t \to +0$. We first establish the corresponding statement (2.7) where the sum runs over all of $\Pi_{\mathrm{disc}}(\mathbf{G}(\mathbb{A}),\xi_0)$ in place of $\Pi_{\mathrm{cus}}(\mathbf{G}(\mathbb{A}),\xi_0)$. To this end we apply the Arthur trace formula as follows. We choose a certain family of test functions $\widetilde{\phi}_t^1 \in C_c^\infty(\mathbf{G}(\mathbb{A})^1)$, depending on $t>0$. At the infinite place $\widetilde{\phi}_t^1$ is given by the function $h_t^\sigma \in C^\infty(\mathbf{G}(\mathbb{R})^1)$ which is defined in terms of the heat kernel $H_t^\sigma\colon \mathbf{G}(\mathbb{R})^1 \to \operatorname{End}(V_\sigma)$ of the Laplacian $\widetilde{\Delta}_\sigma$ on $\widetilde{X}$ by $h_t^\sigma(g) := \operatorname{tr} H_t^\sigma(g)$, $g \in \mathbf{G}(\mathbb{R})^1$, multiplied by a certain cutoff function φ_t. At the finite places $\widetilde{\phi}_t^1$ is given by the normalized characteristic function of an open compact subgroup K_f of $\mathbf{G}(\mathbb{A}_f)$. Then by the non-invariant trace formula [1] we have the equality

$$J_{\mathrm{spec}}(\widetilde{\phi}_t^1) = J_{\mathrm{geo}}(\widetilde{\phi}_t^1), \quad t>0.$$

Then we study the asymptotic behavior of the spectral and the geometric side as $t \to 0$. To deal with the geometric side, we use the fine $\mathfrak{o}$-expansion [3]

$$J_{\mathrm{geo}}(f) = \sum_{M\in\mathcal{L}}\sum_{\gamma\in(M(\mathbb{Q}_S))_{M,S}} a^M(S,\gamma)J_M(\gamma,f), \tag{2.12}$$

which expresses the distribution $J_{\mathrm{geo}}(f)$ in terms of weighted orbital integrals $J_M(\gamma,f)$. Here M runs over the set of Levi subgroups $\mathcal{L}$ containing the Levi component M_0 of the standard minimal parabolic subgroup P_0, S is a finite set of places of $\mathbb{Q}$, and $(M(\mathbb{Q}_S))_{M,S}$ is a certain set of equivalence classes in $M(\mathbb{Q}_S)$. This reduces our problem to the investigation of weighted orbital integrals. The key result is that

$$\lim_{t\to 0} t^{d/2}J_M(\widetilde{\phi}_t^1,\gamma) = 0,$$

unless $M = \mathbf{G}$ and $\gamma = 1$. This follows from the analysis of the local weighted orbital integrals carried out in [2]. In fact, we show that $J_M(\widetilde{\phi}_t^1, \gamma)$ has a complete asymptotic expansion

$$J_M(\widetilde{\phi}_t^1, \gamma) \sim t^{-d/2+d(\gamma)} \sum_{j=0}^{\infty} \sum_{i=0}^{r_M} b_{ij} t^{j/2} (\log t)^i$$

as $t \to 0$, where $d(\gamma) = \dim \mathcal{O}^\gamma$ and $\mathcal{O}^\gamma \subset \mathbf{G}(\mathbb{R})$ is the unipotent conjugacy class in $\mathbf{G}(\mathbb{R})$ induced from $M(\mathbb{R})$ along $P(\mathbb{R})$. The contributions to (2.12) of the terms where $M = \mathbf{G}$ and $\gamma = 1$ are easy to determine. Using the behavior of the heat kernel $h_t^\sigma(1)$ as $t \to 0$, it follows that

$$J_{\text{geo}}(\widetilde{\phi}_t^1) \sim \frac{\dim(\sigma) \operatorname{vol}(\mathbf{G}(\mathbb{Q})\backslash \mathbf{G}(\mathbb{A})^1/K_f)}{(4\pi)^{d/2}} t^{-d/2} \tag{2.13}$$

as $t \to 0$.

To deal with the spectral side we use the refined expansion of the spectral side of the trace formula [14, Corollary 1]. This allows us to replace $\widetilde{\phi}_t^1$ by a similar function $\phi_t^1 \in \mathcal{C}^1(G(\mathbb{A})^1)$ which is given as the product of the heat kernel h_t at infinity and the normalized characteristic function of K_f. The term in $J_{\text{spec}}(\phi_t^1)$ corresponding to $M = \mathbf{G}$ is $J_{\text{spec},G}(\phi_t^1) = \operatorname{tr} R_{\text{disc}}(\phi_t^1)$, which is equal to the left hand side of (2.11). If M is a proper Levi subgroup of $\mathbf{G}$, then $J_{\text{spec},M}(\phi_t^1)$ is given by a finite sum of integrals [14, Corollary 1], [23, (5.8)] and the main ingredients of the integrals are logarithmic derivatives of intertwining operators

$$M_{Q|P}(\lambda)\colon \mathcal{A}^2(P) \to \mathcal{A}^2(Q),$$

where $P, Q \in \mathcal{P}(M)$ and $\mathcal{A}^2(P)$ and $\mathcal{A}^2(Q)$ are the spaces of automorphic forms for P and Q, respectively (see [34, Sect. 2.2]). To deal with these integrals, we use the standard properties of intertwining operators to reduce the problem to the case of intertwining operators associated to pairs of adjacent parabolic subgroups. Let $\alpha \in \Sigma_M$ be a root and assume that P and Q are adjacent along α. For $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$ let $M_{Q|P}(\pi, s)$, $s \in \mathbb{C}$, be the corresponding rank one intertwining operator. It admits a global normalizing factor $n_\alpha(\pi, s)$ [34, (2.2)]. These factors are meromorphic functions of finite order of $s \in \mathbb{C}$ and satisfy the functional equation $|n_\alpha(\pi, it)| = 1$ for all $t \in \mathbb{R}$. Using [34, (2.2)], the estimation of the spectral side can be reduced to the study of integrals involving logarithmic derivatives of the normalizing factors and of the local intertwining operators.

In the case of $\mathbf{G} = \mathrm{GL}(n)$, the normalizing factors are expressed in terms of Ranking-Selberg L-functions [31]. Using the analytic properties of Rankin-Selberg L-functions, it follows that there exist $C > 0$ and $T > 1$ such that for $\pi = \pi_1 \otimes \pi_2$, $\pi_i \in \Pi_{\text{disc}}(\mathrm{GL}(n_i, \mathbb{A}))$, we have

$$\int_T^{T+1} \left| \frac{n'_\alpha(\pi, i\lambda)}{n_\alpha(\pi, i\lambda)} \right| d\lambda \le C \log(T + \nu(\pi_1 \times \tilde{\pi}_2)), \tag{2.14}$$

where $\nu(\pi_1 \times \tilde{\pi}_2) = N(\pi_1 \times \tilde{\pi}_2)(2 + c(\pi_1 \times \tilde{\pi}_2)$, $N(\pi_1 \times \tilde{\pi}_2)$ is the conductor occurring in the functional equation and $c(\pi_1 \times \tilde{\pi}_2)$ is the analytic conductor defined in [31, (4.21)]. For the proof of (2.14) see [31, Proposition 5.1]. For the groups listed in Theorem 2.4 Tobias

Finis and Erez Lapid [12] used functoriality to transfer the problem of the estimation of the corresponding integrals to the well-understood problem for $\mathrm{GL}(n)$, which implies bounds similar to (2.14). For an arbitrary reductive group $\mathbf{G}$ we have formulated in [15] more general conditions on the normalizing factors, called (TWN) (tempered winding number), which are needed to prove that $\mathbf{G}$ has the limit multiplicity property.

Finally we have to deal with normalized intertwining operators

$$R_{Q|P}(\pi, s) = \otimes_v R_{Q|P}(\pi_v, s).$$

Since the open compact subgroup K_f of $\mathbf{G}(\mathbb{A}_f)$ is fixed, there are only finitely many places v for which we have to consider $R_{Q|P}(\pi_v, s)$. The key result for the estimation of integrals involving the logarithmic derivative of $R_{Q|P}(\pi_v, s)$, which is uniform in π_v, is a generalization of the classical Bernstein inequality [15, Corollary 5.18].

Another problem is that for every Levi subgroup M of $\mathbf{G}$ we have to control the growth of the residual spectrum. In [28] a polynomial bound was obtained. However, this bound is not sufficient for our purpose. We need a bound which is of lower order than the order of the growth of the discrete spectrum predicted by the Weyl law. To achieve this goal for the groups listed in Theorem 2.4 we combine the approach of [28], which is based on the study of rank one intertwining operators, with the estimations of normalizing factors obtained in [12].

Combining these estimations, it follows that for every proper Levi subgroup M of $\mathbf{G}$ we have

$$J_{\mathrm{spec},M}(\phi_t^1) = O(t^{-(d-1)/2}) \tag{2.15}$$

as $t \to +0$. This proves (2.11). As explained above, this implies Theorem 2.4.

The next problem is to estimate the remainder term in the Weyl law. So far an estimation of the remainder term has been only obtained for the spherical spectrum, i.e., the trivial K-type.

For X a congruence quotient of the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ and the cuspidal spectrum of the Laplacian on functions of X, this problem has been studied by E. Lapid and the author in [18]. One of the main results is the following theorem.

Theorem 2.6. *Let* $\widetilde{X} = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ *and* $d = \dim \widetilde{X}$. *Let* $\Gamma(N)$ *be the principal congruence subgroup of* $\mathrm{SL}(n,\mathbb{Z})$ *of level* N. *Then for* $N \geq 3$ *we have*

$$N_{\Gamma(N)}^{\mathrm{cus}}(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash \widetilde{X})}{(4\pi)^{d/2}\mathbf{\Gamma}\left(\frac{d}{2}+1\right)}\lambda^{d/2} + O\left(\lambda^{(d-1)/2}(\log\lambda)^{\max(n,3)}\right), \quad \lambda \to \infty.$$

Actally, we consider not only the cuspidal spectrum of the spherical Laplacian, but the cuspidal spectrum of the whole algebra of invariant differential operators $\mathcal{D}(\widetilde{X})$.

In a recent paper [13], Tobias Finis and Erez Lapid estimated the remainder term of the Weyl law for the spherical cuspidal spectrum of a locally symmetric space X defined by a simply connected, simple Chevalley group $\mathbf{G}$ and a congruence subgroup of $\mathbf{G}(\mathbb{Q})$. The main result of [13] is the following theorem.

Theorem 2.7. *Let* $\mathbf{G}$ *be a simpy connected, simple Chevalley group. Then there exists* $\delta > 0$ *such that for any congruence sugroup* Γ *of* $\mathbf{G}(\mathbb{Z})$ *we have*

$$N_\Gamma^{\mathrm{cus}}(\lambda) = \frac{\mathrm{vol}(X)}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}\lambda^{d/2} + O_\Gamma(\lambda^{d/2-\delta}), \quad \lambda \geq 1,$$

where $X = \Gamma\backslash G(\mathbb{R})/K$ *and* $d = \dim X$.

This is a sharpening of the result of [19]. It is an interesting problem to see, if one can also estimate the remainder term for non-trivial K-types.

3. The analytic torsion

A more sophisticated spectral invariant is the analytic torsion [40]. In the context of locally symmetric spaces it has been used to study torsion in the cohomology of arithmetic groups. Recall its definition.

Let X be a compact Riemannian manifold of dimension n and let $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$ a finite dimensional representation of its fundamental group. Let $E_\rho \to X$ be the flat vector bundle associated with ρ. Choose a Hermitian fiber metric in E_ρ. Let $\Delta_p(\rho)$ be the Laplace operator on E_ρ-valued p-forms with respect to the metrics on X and in E_ρ. It is an elliptic differential operator, which is formally self-adjoint and non-negative. Since X is compact, $\Delta_p(\rho)$ has a pure discrete spectrum consisting of sequence of eigenvalues $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to \infty$ of finite multiplicity. Let

$$\zeta_p(s;\rho) := \sum_{\lambda_j > 0} \lambda_j^{-s} \tag{3.1}$$

be the zeta function of $\Delta_p(\rho)$. The series converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > n/2$ and admits a meromorphic extension to $s \in \mathbb{C}$, which is holomorphic at $s = 0$. Then the Ray-Singer analytic torsion $T_X(\rho) \in \mathbb{R}^+$ is defined by

$$T_X(\rho) := \exp\left(\frac{1}{2}\sum_{p=1}^{n}(-1)^p p \frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}\right). \tag{3.2}$$

It depends on the metrics on X and E_ρ. However, if $\dim(X)$ is odd and ρ acyclic, which means that $H^*(X, E_\rho) = 0$, then $T_X(\rho)$ is independent of the metrics [29]. The analytic torsion has a topological counterpart. This is the Reidemeister torsion $T_X^{\mathrm{top}}(\rho)$ (usually it is denoted by $\tau_X(\rho)$), which is defined in terms of a smooth triangulation of X [40], [27]. It is known that for unimodular representations ρ (meaning that $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(X)$) one has the equality

$$T_X(\rho) = T_X^{\mathrm{top}}(\rho) \tag{3.3}$$

[7], [27]. In the general case of a non-unimodular representation the equality does not hold, but the defect can be described [6].

In [4], [21], [36], the equality (3.3) has been applied to study the growth of torsion in the cohomology of co-compact arithmetic groups. A key result used in this context is the approximation of L^2-torsion (see [20], [22] for its definition) for compact locally symmetric spaces. This means the following. Let G be a real connected semisimple Lie group of non-positive type, K a maximal compact subgroup of G and $\widetilde{X} = G/K$ the corresponding symmetric space of non-positive curvature. Let $\Gamma \subset G$ be a co-compact torsion free discrete subgroup. Then Γ acts on $\widetilde{X}$ properly discontinuously and $X = \Gamma\backslash\widetilde{X}$ is a locally symmetric manifold. Let $\{\Gamma_j\}_{j\in\mathbb{N}}$ be a sequence of normal subgroups of finite index of Γ satisfying $\Gamma_{j+1} \subset \Gamma_j$ and $\cap_j\Gamma_j = \{e\}$. Let $X_j = \Gamma_j\backslash\widetilde{X}$. This is normal covering of X of finite index. Let $\tau\colon G \to \mathrm{GL}(V)$ be a finite dimensional representation. Let $\rho_j := \tau|_{\Gamma_j}$ and let $T_{X_j}(\tau)$ denote the analytic torsion of X_j with respect to ρ_j. Let $\theta\colon G \to G$ be the Cartan involution and put $\tau_\theta := \tau \circ \theta$. Then in [4] Bergeron and Venkatesh proved the following theorem concerning the approximation of the L^2-torsion.

Theorem 3.1. *Assume that* $\tau \ncong \tau_\theta$. *Then*

$$\lim_{j\to\infty} \frac{\log T_{X_j}(\tau)}{[\Gamma\colon\Gamma_j]} = \mathrm{vol}(X)t^{(2)}_{\widetilde{X}}(\tau). \tag{3.4}$$

We note that the right hand side is the logarithm of the L^2-torsion of X and τ. This is a key result for the study of the cohomology of arithmetic groups [4].

In view of the potential applications to the cohomology of arithmetic groups, it is very desirable to extend Theorem 3.1 to the non-compact case. The first problem one faces is that the corresponding Laplace operators have a nonempty continuous spectrum and therefore, the heat operators are not trace class and the analytic torsion can not be defined as above. This problem has been studied by Raimbault [39] for hyperbolic 3-manifolds and in [35] for hyperbolic manifolds of any dimension.

So let $G = \mathrm{SO}^0(n,1)$, $K = \mathrm{SO}(n)$ and $\widetilde{X} = G/K$. Equipped with a suitably normalized G-invariant metric, $\widetilde{X}$ becomes isometric to the n-dimensional hyperbolic space $\mathbb{H}^n$. Let $\Gamma \subset G$ be a torsion free lattice. Then $X = \Gamma\backslash\widetilde{X}$ is an oriented n-dimensional hyperbolic manifold of finite volume. As above, let $\tau\colon G \to \mathrm{GL}(V)$ be a finite dimensional complex representation of G. The first step is to define a regularized trace of the heat operators $e^{-t\Delta_p(\tau)}$. To this end one uses an appropriate height function to truncate X at sufficient high level $Y > Y_0$ to get a compact manifold $X(Y) \subset X$ with boundary $\partial X(Y)$, which consists of a disjoint union of $n-1$-dimensional tori. Let $K^{p,\tau}(t,x,y)$ be the kernel of the heat operator $e^{-t\Delta_p(\tau)}$. Using the spectral resolution of $\Delta_p(\tau)$, it follows that there exist $\alpha(t) \in \mathbb{R}$ such that $\int_{X(Y)} \mathrm{tr}\, K^{p,\tau}(t,x,x)\, dx - \alpha(t)\log Y$ has a limit as $Y \to \infty$. Then we define the regularized trace as

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) := \lim_{Y\to\infty}\left(\int_{X(Y)} \mathrm{tr}\, K^{p,\tau}(t,x,x)\, dx - \alpha(t)\log Y\right). \tag{3.5}$$

We note that the regularized trace is not uniquely defined. It depends on the choice of truncation parameters on the manifold X. However, if $X_0 = \Gamma_0\backslash\mathbb{H}^n$ is given and if truncation parameters on X_0 are fixed, then every finite covering X of X_0 is canonically equipped with truncation parameters, namely one simply pulls back the height function on X_0 to a height function on X via the covering map.

Let θ be the Cartan involution of G with respect to $K = \mathrm{SO}(n)$. Let $\tau_\theta = \tau \circ \theta$. If $\tau \ncong \tau_\theta$, it can be shown that $\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ is exponentially decreasing as $t \to \infty$ and admits an asymptotic expansion as $t \to 0$. Therefore, the regularized zeta function $\zeta_{\mathrm{reg},p}(s;\tau)$ of $\Delta_p(\tau)$ can be defined as in the compact case by

$$\zeta_{\mathrm{reg},p}(s;\tau) := \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) t^{s-1}\, dt. \tag{3.6}$$

The integral converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > n/2$ and admits a meromorphic extension to the whole complex plane, which is holomorphic at $s = 0$. So in analogy with the compact case, the regularized analytic torsion $T_X(\tau) \in \mathbb{R}^+$ can be defined by the same formula (3.2).

In even dimension the analytic torsion is rather trivial. Therefore, we assume that $n = 2m+1$. Furthermore, for technical reasons we assume that every lattice $\Gamma \subset G$ satisfies the following condition: For every Γ-cuspidal parabolic subgroup P of G one has

$$\Gamma \cap P = \Gamma \cap N_P, \tag{3.7}$$

where N_P denotes the unipotent radical of P. Let Γ_0 be a fixed lattice in G and let $X_0 = \Gamma_0\backslash\widetilde{X}$. Let Γ_j, $j \in \mathbb{N}$, be a sequence of finite index torsion free subgroups of Γ_0. This sequence is called to be *cusp uniform*, if the tori which arise as cross sections of the cusps of the manifolds $X_J := \Gamma_j\backslash\widetilde{X}$ satisfy some uniformity condition (see [35, Definition 8.2]).

One of the main results of [35] is the following theorem which may be regarded as an analog of Theorem 3.1 for oriented finite volume hyperbolic manifolds.

Theorem 3.2. *Let Γ_0 be a lattice in G and let Γ_i, $i \in \mathbb{N}$, be a sequence of finite-index normal subgroups which is cusp uniform and such that each Γ_i, $i \geq 1$, is torsion-free and satisfies* (3.7). *If $\lim_{i\to\infty}[\Gamma_0 : \Gamma_i] = \infty$ and if each $\gamma_0 \in \Gamma_0 - \{1\}$ only belongs to finitely many Γ_i, then for each τ with $\tau \neq \tau_\theta$ one has*

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{[\Gamma : \Gamma_i]} = t^{(2)}_{\mathbb{H}^n}(\tau)\,\mathrm{vol}(X_0). \tag{3.8}$$

In particular, if under the same assumptions Γ_i is a tower of normal subgroups, i.e. $\Gamma_{i+1} \subset \Gamma_i$ for each i and $\cap_i \Gamma_i = \{1\}$, then (3.8) *holds.*

This theorem has applications to the study of the growth of torsion in the cohomology of congruence subgroups of $\mathrm{SO}^0(n,1)$ [37]. It is based on [38], which establishes a relation between analytic torsion and topological torsion similar to (3.3) with an additional defect term which can be controlled.

The next goal is to extend Theorem 3.2 to higher rank groups. In joint work with J. Matz [25] we have defined the analytic torsion for a locally symmetric space defined by a quasi-split reductive group $\mathbf{G}$ and a congruence subgroup of $\mathbf{G}(\mathbb{Q})$. For simplicity assume that $\mathbf{G}$ is a connected semisimple algebraic group over $\mathbb{Q}$. Assume that $\mathbf{G}$ is not anisotropic. Let K_∞ be a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. Put $\widetilde{X} := \mathbf{G}(\mathbb{R})/K_\infty$. Let $K_f \subset \mathbf{G}(\mathbb{A})$ be an open compact subgroup. Then we consider the adelic quotient

$$X(K_f) := \mathbf{G}(\mathbb{Q})\backslash(\widetilde{X} \times \mathbf{G}(\mathbb{A}_f))/K_f.$$

Recall that $X(K_f)$ is the disjoint union of finitely many locally symmetric spaces $\Gamma_i\backslash\widetilde{X}$, $i = 1, ..., l$. If $\mathbf{G}$ is simply connected, then by strong approximation we have

$$X(K_f) = \Gamma\backslash\widetilde{X},$$

where $\Gamma = (\mathbf{G}(\mathbb{R}) \times K_f) \cap \mathbf{G}(\mathbb{Q})$. We will assume that K_f is need so that $X(K_f)$ is a manifold. Let $\tau\colon \mathbf{G}(\mathbb{R}) \to \mathrm{GL}(V_\tau)$ be a finite dimensional complex representation. Let $E_\tau \to X(K_f)$ be the associated flat vector bundle and $\Delta_p(\tau)$ the Laplacian on p-forms with values in E_τ. To define the regularized trace $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)})$ we proceed as above, using Arthur's truncation. Let J_{geo} be the geometric side of the Arthur trace formula. It turns out that

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)}) = J_{\mathrm{geo}}(\phi_t^{\tau,p})$$

for an appropriate test function $\phi_t^{\tau,p} \in C^\infty(G(\mathbb{A}))$ which at the infinite place is given by the heat kernel for the Laplace operators on p-forms on $\widetilde{X}$ with values in the lifted flat bundle $\widetilde{E}_\tau$. This is the key fact that allows us to determine the asymptotic behavior of the regularized trace as $t \to 0$ and $t \to \infty$. Then we can use the analogous formula (3.6) to define the regularized zeta function and the analytic torsion.

Now we can formulate our main result. Let $n \geq 2$. Put $\widetilde{X}_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. Let $K_n(N) \subset \mathrm{SL}(n, \mathbb{A}_f)$ be the principal congruence subgroup of level $N \geq 3$. Put $X_n(N) := X(K_n(N))$. Note that $X_n(N) = \Gamma(N)\backslash\widetilde{X}_n$, where $\Gamma(N) \subset \mathrm{SL}(n,\mathbb{Z})$ is the principal congruence subgroup of level N. Then J. Matz and I proved in [24] the following theorem

Theorem 3.3. *Let $\tau \in \mathrm{Rep}(\mathrm{SL}(n,\mathbb{R}))$. Assume that $\tau \not\cong \tau_\theta$. Then for $n \geq 2$ we have*

$$\lim_{N\to\infty} \frac{\log T_{X_n(N)}(\tau)}{\mathrm{vol}(X_n(N))} = t^{(2)}_{\widetilde{X}_n}(\tau).$$

Moreover, if $n > 4$, then $t^{(2)}_{\widetilde{X}_n}(\tau) = 0$, and if $n = 3, 4$, then $t^{(2)}_{\widetilde{X}_n}(\tau) > 0$.

Remark 3.4. *The number $t^{(2)}_{\widetilde{X}}(\rho)$ can be defined for every finite dimensional representation (cf.* [4, 4.4]*). Moreover, it can be computed explicitly* [4, §5]*. For example, for the trivial representation τ_0 of $\mathrm{SL}(n,\mathbb{R})$, $n = 3, 4$, one has*

$$t^{(2)}_{\widetilde{X}_3}(\tau_0) = \frac{\pi}{2\,\mathrm{vol}(\widetilde{X}_3^c)}, \quad t^{(2)}_{\widetilde{X}_4}(\tau_0) = \frac{124\pi}{45\,\mathrm{vol}(\widetilde{X}_4^c)}$$

[4, 5.9.3, Example 2]. *Here* $\widetilde{X}_j^c$ *denotes the compact dual of* $\widetilde{X}_j$*, and the metric on* $\widetilde{X}_j^c$ *is the one induced from the metric on* $\widetilde{X}_j$*. For the second equality we used that* $\mathrm{SL}(4,\mathbb{R})$ *is a double covering of* $\mathrm{SO}(3,3)$*, and as explained at the beginning of section 5.8 in* [4]*, the corresponding number for* $\mathrm{SO}(3,3)$ *agrees with that for* $\mathrm{SO}(5,1)$*. Finally,* $t^{(2)}_{\mathbb{H}^5}(\tau_0)$ *is computed in* [4, 5.9.3, Example 1].

The next goal is to extend Theorem 3.3 to other reductive groups and use it to study the growth of torsion in the cohomology of congreunce subgroups similar to the case of hyperbolic manifolds.

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