Nonvanishing properties of Fourier coefficients for Siegel modular forms

by
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1 Introduction and main result

Let $\mathbb{H}_n$ be the Siegel upper half space of degree n with the usual action of the group $Sp(n,\mathbb{R})$, given by $(M,Z) \longmapsto M < Z >:= (AZ+B)(CZ+D)^{-1}$. For a polynomial representation $\rho : GL(n,\mathbb{C}) \longrightarrow Aut(V)$ on a finite-dimensional vector space $V = V_\rho$ we define an action of $Sp(n,\mathbb{R})$ on V-valued functions on $\mathbb{H}_n$ by

$$(f,M) \longmapsto (f \mid_\rho M)(Z) = \rho(CZ+D)^{-1} f(M < Z >).$$

We choose the smallest nonnegative integer k such that $\rho = \det^k \otimes \rho_0$ with ρ_0 is still polynomial and we call this $k = k(\rho)$ the weight of ρ; if ρ itself is scalar-valued, we often write k instead of $\det^k$. We denote by M^n_ρ the space of Siegel modular forms for ρ w.r.t.$\Gamma := Sp(n,\mathbb{Z})$, i.e. the set of all holomorphic functions $f : \mathbb{H}_n \longrightarrow V$ satisfying $f \mid_\rho M = f$ for all $M \in \Gamma$; in the case $n = 1$ the usual condition in cusps must be added. The subspace of cusp forms will be denoted by S^n_ρ.
It is often convenient to realize $V = V_\rho$ as $\mathbb{C}^m$. The Fourier expansion of f is then of type

$$f(Z) = \sum_T a_f(T) e^{2\pi i trace(TZ)},$$

where the Fourier coefficients $a_f(T)$ are in $\mathbb{C}^m$ and T runs over the set $\Lambda^n_\geq$ of all symmetric half-integral matrices of size n, which are positive-semidefinite.

The problem: It is quite natural to ask whether one can assure the nonvanishing of Fourier coefficients for specific simple types of $T \in \Lambda^n_>$.
For a long time the focus was on the class of *primitive* Fourier coefficients (first for Hecke eigenforms, then without such assumption) culminating in the work of Yamana [7]; we also refer to [7] for more details on the history of this topic. Then Saha [5] opened a new chapter by showing that in degree two there is always a nonvanishing Fourier coefficient such that det(2T) is odd and squarefree. Saha's work had important applications in his (and his coauthors') theorem on transfer from Siegel cusp forms of degree 2 to

automorphic forms on $GL(4)$, see [3].
To formulate our main result, it is convenient to use for $T \in \Lambda^n_>$ the "absolute discriminant" $d(T)$, defined by

$$d(T) = \begin{cases} \det(2T) & n \text{ even} \\ \frac{1}{2} det(2T) & n \text{ odd} \end{cases}$$

Theorem: *Let $f = \sum_T a_f(T) e^{2\pi i tr(TZ)}$ be a nonzero modular form in M^n_ρ and assume that $k(\rho) \geq 2 + \frac{n-1}{2}$. Then there exist infinitely many $T \in \Lambda^n_>$ such that $a_f(T)$ is nonzero and $d(T)$ is odd squarefree.*

Remarks:

- Even for degree one this is a nontrivial statement (of the analytic number theory of modular forms)
- Certainly some condition on the weight is necessary : If S is an even uninmodular positive definite quadratic quadratic form of size n, then the degree n modular form

  $$\theta_{S,\nu}(Z) = \sum_{X \in \mathbb{Z}^{(n,n)}} det(X)^\nu e^{\pi i tr(S[X]Z)} \quad (\nu = 0, 1)$$

  gives an example violating the theorem.
- Even if we are only interested in scalar-valued modular forms, our proof (induction on n) makes it necessary to go through vector-valued forms.
- We mention that A.Pollack [4] has obtained the functional equation for the spinor L-function attached to a degree 3 Hecke eigenform f provided that f has a nonzero Fourier coefficient $a_f(T)$ where the ternary quadratic form T comes from a *maximal* order in a definite quaternion algebra over $\mathbb{Q}$. [1])
- There is also a quantative version of our main result:

  $$\sharp\{d \leq X \mid d \quad \text{odd and squarefree}, \exists\, T \in \Lambda^n : d(T) = d, a_f(T) \neq 0\}$$

[1]After the RIMS conference we showed that for odd degree n one can get nonzero Fourier coefficients $a_F(T)$ with $d(T)$ being prime; in particular, Pollack's result is now valid unconditionally

$$>> \begin{cases} X^{1-\epsilon} & n \quad odd, f \text{ cuspidal} \\ X^{\frac{5}{8}-\epsilon} & n \quad even, f \text{ cuspidal} \\ X & \text{noncuspidal case} \end{cases}$$

Some Generalizations (not completed, but under consideration)

- It should be possible give a general uniform treatment of our main result for all classical tube domains in a style similar to [7]. Saha's result was already generalized to Hermitian forms of degree 2 [1].
- There should be also a "mod p version" of our result. Note however that one will need additional condititions to avoid cases like $f \equiv 1 \bmod p$ or $f \equiv g(p \cdot z) \bmod p$ for some modular form g.
- One can try to investigate the simultaneous nonvanishing of such Fourier coefficients for two modular forms.
- For a modular form with level N one should get nonvanishing Fourier coefficicients $a_f(T)$ with $d(T) = d_1 \cdot d_2$ where d_1 is odd, squarefree and coprime to N and $d_2 \mid N^\infty$.

2 The proof

For the proof of our theorem, we use the Fourier-Jacobi expansion and induction over the degree n; finally we have to prove a nonvanishing result for Fourier coefficients of certain elliptic modular forms.

2.1 Preparing the induction

This will be done in 3 steps, all of them mainly algebraic in nature.

2.1.1 Step 1:

For a nonzero modular form $f \in M^n_\rho$ we consider its Fourier- Jacobi expansion

$$f(\mathcal{Z}) = \sum_{T \in \Lambda^{n-1}_{\geq}} \phi_T(\tau, \mathfrak{z}) e^{2\pi i tr(TZ)}$$

where we decompose $\mathcal{Z} \in \mathbb{H}_n$ as

$$\mathcal{Z} = \begin{pmatrix} \tau & \mathfrak{z} \\ \mathfrak{z}^t & Z \end{pmatrix} \qquad (\tau \in \mathbb{H}_1,\ Z \in \mathbb{H}_{n-1}).$$

Claim: *There is $T \in \Lambda_{>}^{n-1}$ such that $\phi_T \neq 0$ with $d(T)$ odd and squarefree.*
To prove this statement, one considers a Taylor expansion of f w.r.t. $\mathfrak{z} = (z_2, \ldots, z_n)$ around $\mathfrak{z} = 0$:

$$f(\mathcal{Z} = \sum_{\lambda \in \mathbb{N}_0^{n-1}} f_\lambda \cdot z_2^{\lambda_2} \ldots z_n^{\lambda_n}$$

For

$$\nu_o := Min\{\nu := \sum \lambda_i \mid f_\lambda \neq 0\}$$

we define a $V \otimes \mathbb{C}[X_2, \ldots, X_n]_{\nu_0}$-valued function on $\mathbb{H}_1 \times \mathbb{H}_{n-1}$ by

$$\chi_{\nu_0}(\tau, Z) := \sum_{\lambda_2 + \cdots + \lambda_n = \nu_0} f_\lambda X_2^{\lambda_2} \ldots X_n^{\lambda_n}.$$

As a function of Z, this is a nonzero modular form for the representation

$$\rho' := \rho_{|GL(n-1,\mathbb{C})} \otimes Sym^{\nu_0}.$$

Observing that $k(\rho') \geq k(\rho)$, we get by induction that χ_{ν_0} has a nonzero T-Fourier coefficient with $d(T)$ odd and squarefree; this implies the claim of step 1.

2.1.2 Step 2:

Let ϕ_T be as above; then (after choosing coordiantes of $V \simeq \mathbb{C}^m$ appropriately) there is a nonzero component $\phi_T^{(r)}$ of

$$\phi_T = (\phi_T^{(1)}, \ldots, \phi_T^{(m)})^t,$$

which is an ordinary scalar valued Jacobi form of index T.
Indeed, an inspection of the transformation laws for Jacobi forms shows that the last nonzero component of ϕ_T has the requested property.

2.1.3 Step 3:

This is concerned with a property of Jacobi forms and should be of independent interest. It will then be applied to $\phi_T^{(r)}$ from step 2.

Lemma: *Let $M \in \Lambda_>^n$ have odd squarefree absolute discriminant $d(M)$. Then there exists $\mu \in \mathbb{Z}^n$ such that the rational number $\frac{1}{4}M^{-1}[\mu]$ has exact denominator d if n is even and $4d$ if n is odd.*
In the sequel, we call such μ as in the lemma *M-primitive*.

For $M \in \Lambda_>^n$ we consider a nonzero Jacobi form ϕ_M on $\mathbb{H} \times \mathbb{C}^{(1,n)}$ of weight k and index M. The theta expansion is then

$$\phi_M(\tau, \mathfrak{z}) = \sum_{\mu} h_\mu(\tau) \cdot \Theta_M[\mu](\tau, \mathfrak{z}),$$

where μ runs over $\mathbb{Z}^n/2M \cdot \mathbb{Z}^n$ and (with $\tilde{\mu} := (2M)^{-1} \cdot \mu$)

$$\Theta_M[\mu](\tau, \mathfrak{z}) = \sum_{R \in \mathbb{Z}^n} e^{2\pi i M[R+\tilde{\mu}]\tau + 2\mathfrak{z} \cdot M \cdot (R+\tilde{\mu})}.$$

Proposition: *Let ϕ_M be as above with $d(M)$ odd and squarefree; then $h_\mu \neq 0$ for some M-primitive μ.*

Remark: The proposition above can be understood as a nonvanishing statement for some coordinate function in the vector-valued modular form given by the collection of the h_μ. If the associated projective representation π_M of $SL(2, \mathbb{Z})$ would be irreducible, then we would get such nonvanishing for free (indeed for all the h_μ). Already for $n = 1$ however, the irreducibility is not assured unless the index m is a prime number, see [6].

The proof of the proposition is the technically most difficult point of the induction argument. It turns out that it suffices to show that for all fixed $\nu_0 \in \mathbb{Z}^n$, which are not M-primitive, the following submatrix of $\pi_M(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$ is of maximal rank:

$$\left(e^{2\pi i \frac{1}{2}\nu^t M^{-1}\mu}\right)_{\mu,\nu}$$

Here μ runs over all M-primitive elements in $\mathbb{Z}^\ltimes/2M \cdot \mathbb{Z}^n$ and ν runs over all elements in $\mathbb{Z}^n/2M\mathbb{Z}^n$ with the extra property that $\frac{1}{4}M^{-1}[\nu] - \frac{1}{4}M^{-1}[\nu_0] \in \mathbb{Z}$. Checking this property is somewhat laborious.

2.2 The analytic part

For a nonzero modular form of degree n with Fourier expansion $f = \sum_{\mathcal{T}} a_f(\mathcal{T}) e^{2\pi i tr(\mathcal{T}Z)}$ we may choose $M \in \Lambda^{n-1}$ such that $d(M)$ is odd and squarefree and ϕ_M is nonzero (by induction). For such M we study the Fourier coefficents $a_f(\mathcal{T})$ with

$$\mathcal{T} = \begin{pmatrix} l & \frac{1}{2}\mu \\ \frac{1}{2}\mu^t & M \end{pmatrix}.$$

We observe that $\det(\mathcal{T}) = (l - \frac{4}{M}^{-1}[\mu]) \cdot det(M)$. For absolute discriminants this means in the case of odd n (writing d for $d(M)$):

$$d(\mathcal{T}) = \underbrace{(l - \frac{1}{4}M^{-1}[\mu])}_{\frac{\alpha}{d}} \cdot d = \alpha$$

and, by our previous considerations, $\alpha \in \mathbb{N}$ is always coprime to d, if μ is chosen to be M-primitive. Furthermore, as in step 2, we may choose a component $\phi_M^{(r)}$ of the $V = \mathbb{C}^m$-valued Fourier-Jacobi coefficient ϕ_M, such that the modular form

$$h_\mu(\tau) = \sum a_f^{(r)}(\begin{pmatrix} l & \frac{1}{2}\mu \\ \frac{1}{2}\mu^t & M \end{pmatrix}) e^{2\pi i(l - \frac{1}{4}M^{-1}[\mu])\tau}$$

is nonzero, of level d and its Fourier expansion is of the form

$$h_\mu(\tau) = \sum_{(\alpha,d)=1} b(\alpha) e^{2\pi i \frac{\alpha}{d}\tau}$$

We are done, if we can show that there exist infinitly many odd and squarefree α with Fourier $b(\alpha) \neq 0$. This can be achieved analytically by using Rankin-Selberg type arguments. The case of n even works similarly, but h_μ will be a modular form of half-integral weight.
We finally add that the case of noncusp forms needs additional care here.

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