ℓ-modular Deligne representations of the Weil group, local Langlands correspondence and local constants

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Abstract

This is an expositary note on our papers [6] and [7] joint with R. Kurinczuk, on which I gave a talk at the RIMS conference "Automorphic forms, automorphic representations and related topics" in January 2019. Let F be a non-archimedean local field of residual characteristic p and W_F its Weil group. For ℓ a prime number different from p, we classify equivalence classes of W_F-semi-simple Deligne ℓ-modular representations in terms of isomorphism classes of irreducible ℓ-modular representations of W_F. After extending to the ℓ-modular setting the constructions of local constants by Jacquet, Piatetski-Shapiro and Shalika on the reductive group side, and by Artin and Deligne on the Weil group side, we define a variant of the ℓ-modular local Langlands correspondence of Vignéras which satisfies a preservation of local constants property for pairs of generic representations.

1 Introduction

Let F be a non-archimedean local field of residual cardinality q and ℓ a prime number which does not divide q. We consider smooth representations of locally profinite groups, and call them ℓ-adic when they act on $\overline{\mathbb{Q}_\ell}$-vector spaces, and ℓ-modular when they act on $\overline{\mathbb{F}_\ell}$-vector spaces.

We denote by W_F the Weil group of F. The local Langlands correspondence LLC ([2], [4]) for $\mathrm{GL}_n(F)$ is a bijection between the set of isomorphism classes of ℓ-adic irreducible representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of ℓ-adic n-dimensional W_F-semi-simple Deligne representations, characterized by the fact that the Rankin–Selberg local constants of a pair of irreducible ℓ-adic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$ and the Artin–Deligne local constants of the corresponding tensor product of Deligne representations of W_F are equal.

After having developed the theory of ℓ-modular representations of reductive p-adic groups (see [11]), Vignéras proved the ℓ-modular local Langlands correspondence in [12] and characterized it by compatibility with LLC via congruences modulo ℓ in a non naive manner. The V-correspondence as we shall call it is a bijection between the set of isomorphism classes of ℓ-modular irreducible representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of ℓ-modular n-dimensional W_F-semi-simple Deligne representations with nilpotent Deligne operator.

In [6], we extend the theory of Rankin–Selberg local constants of Jacquet, Piatetski-Shapiro and Shalika ([5]) to pairs ℓ-modular generic representations, but the local constants defined there do not match the Artin–Deligne constants via the V-correspondence. This non matching issue also occurs when one considers the ℓ-modular Godement-Jacquet local constants defined by Mínguez in [8].

In [7], we classify the equivalence classes of ℓ-modular indecomposable W_F-semi-simple Deligne representations in terms of isomorphism classes of irreducible representations of W_F. We then extend the definitions of Artin–Deligne factors to this setting, and define a modified version of the ℓ-modular local Langlands correspondence of Vignéras which we call the C-correspondence, which makes the Rankin–Selberg factors of pairs of generic representations on one side match the Artin–Deligne factors of the corresponding representations on the other.

In what follows, we describe the main results of [6] and [7]. We end the paper by some explicit computations of C-parameters for $\mathrm{GL}_2(F)$ and show that under our modified correspondence, the Godement-Jacquet local constants and the Artin-Deligne standard constants of their C-parameter coincide for $\mathrm{GL}_2(F)$.

Acknowledgements. I thank the organizers of the conference Prof. Satoshi Wakatsuki and Prof. Shunsuke Yamana, and I also thank Prof. Taku Ishii for giving me the opportunity to give a talk at the RIMS. This note has benefited from useful comments of R. Kurinczuk and I thank him too.

2 Rankin-Selberg ℓ-modular local constants

2.1 Definition and basic properties

We refer the reader to [6] for this section, which is the adaptation of [5] to the ℓ-modular setting. We denote by R one of the fields $\overline{\mathbb{F}_\ell}$ or $\overline{\mathbb{Q}_\ell}$, and consider a nontrivial (smooth) R-valued additive character ψ of F. Once and for all we fix a square root $q^{1/2}$ of q in R which we use to normalize parabolic induction, but require that its choice in $\overline{\mathbb{Q}_\ell}$ and $\overline{\mathbb{F}_\ell}$ is compatible with reduction modulo ℓ. We denote by val the $\mathbb{Z}$-valued valuation on F. Let π be an R-representation of Whittaker type of $\mathrm{GL}_n(F)$ (see [6, Proposition 2.17]). We denote by $\widetilde{\pi}$ the representation $g \mapsto \pi({}^t g^{-1})$ of $\mathrm{GL}_n(F)$, which is again of Whittaker type and isomorphic to the (smooth) contragredient $\pi^\vee$ of π when π is irreducible. Setting w_n for the usual long Weyl element of $\mathrm{GL}_n(F)$, for $W \in W(\pi,\psi)$ we set $\widetilde{W} : g \mapsto W(w_n {}^t g^{-1})$ so that the map $W \mapsto \widetilde{W}$ is a vector-space isomorphism between $W(\pi,\psi)$ and $W(\widetilde{\pi},\psi^{-1})$. We denote by $\mathrm{N}_n(F)$ the subgroup of upper triangular matrices with ones on the diagonal in $\mathrm{GL}_n(F)$ and by η_n the element $(0,\dots,0,1) \in F^n$. For $k \in \mathbb{Z}$, setting

$$\mathrm{GL}_n(F)^{(k)} = \{g \in \mathrm{GL}_n(F),\ \mathrm{val}(\det(g)) = k\},$$

a Whittaker function $W \in W(\pi,\psi)$ has compact support modulo $\mathrm{N}_n(F)$ when restricted to $\mathrm{GL}_n(F)^{(k)}$. For π and π' representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$ respectively, $W \in W(\pi,\psi)$ and $W' \in W(\pi',\psi^{-1})$ and $\Phi \in \mathcal{C}_c^\infty(F^n)$ a Schwartz-Bruhat function, we set

$$c_k(W,W') = \int_{\mathrm{N}_m(F)\backslash \mathrm{GL}_m(F)^{(k)}} W(\mathrm{diag}(g,\mathrm{I}_{n-m}))W'(g)dg$$

if $m < n$ and

$$c_k(W,W',\Phi) = \int_{\mathrm{N}_n(F)\backslash \mathrm{GL}_n(F)^{(k)}} W(g)W'(g)\phi(\eta_n g)dg$$

if $m = n$. These coefficients vanish for $k << 0$ and we define the *Rankin-Selberg integral*

$$I(X,W,W') = \sum_{k\in\mathbb{Z}} c_k(W,W')q^{k(n-m)/2}X^k$$

when $m < n$ and

$$I(X,W,W',\Phi) = \sum_{k\in\mathbb{Z}} c_k(W,W',\phi)X^k$$

when $n = m$. These Laurent series turn out to be elements of $R(X)$, and in fact span $R[X^{\pm 1}]$-fractional ideals of $R(X)$ when W and W' vary (and Φ as well in the $n = m$ case), generated by a unique Euler factor. We denote by

$$L(X, \pi, \pi')$$

this Euler factor and call it the L-factor attached to the pair (π, π'), and declare it to be symmetric in π and π' as a convention, thus defining such a factor when $n < m$ as well. The Rankin-Selberg integrals satisfy functional equations (see [6, Section 3.2]) involving $L(X, \pi, \pi')$ and a unit of $R[X^{\pm 1}]$ called the *epsilon factor* of the pair (π, π') and denoted by

$$\epsilon(X, \pi, \pi', \psi).$$

The last factor in $R(X)$ of interest for us is the *gamma factor* defined as

$$\gamma(X, \pi, \pi', \psi) = \epsilon(X, \pi, \pi', \psi) \frac{L(q^{-1}X^{-1}, \widetilde{\pi}, \widetilde{\pi'})}{L(X, \pi, \pi')}.$$

Let $r_\ell : \overline{\mathbb{Z}_\ell} \to \overline{\mathbb{F}_\ell}$ be the reduction modulo ℓ map. Let $\overline{\psi}$ be a nontrivial ℓ-modular character of F and denote by ψ an ℓ-adic character of F such that $r_\ell(\psi) = \overline{\psi}$. We show in [7, Theorem 3.13] that if $\overline{\pi}$ and $\overline{\pi'}$ are $\overline{\mathbb{F}_\ell}$-representations of Whittaker type of $\mathrm{GL}_n(F)$, and if π and π' are ℓ-adic Whittaker lifts (see [7, Definition 2.21]) of $\overline{\pi}$ and $\overline{\pi'}$ respectively, then

$$\gamma(X, \overline{\pi}, \overline{\pi'}, \overline{\psi}) = r_\ell(\gamma(X, \pi, \pi', \psi))$$

and

$$L(X, \overline{\pi}, \overline{\pi'}) \backslash r_\ell(L(X, \pi, \pi')).$$

Note that the division of L-factors above can be strict, for example when $q \equiv 1[\ell]$, the L-factor on the left is always equal to 1. When the representations are generic we obtain much more precise results concerning the reduction modulo ℓ of L-factors of pairs.

2.2 Local constants of generic representations and reduction modulo ℓ

We denote by ν the character from $\mathrm{GL}_n(F)$ to $R^\times$, which is the normailzed absolute value of F composed with the determinant. If ρ is a cuspidal representation of $\mathrm{GL}_n(F)$ with coefficients in R, we denote its cuspidal line by

$$\mathbb{Z}_\rho = \{\nu^k \rho, \ k \in \mathbb{Z}\}.$$

In [9], the authors introduce the notion of banal representation of $\mathrm{GL}_n(F)$: an irreducible representation is banal if its cuspidal support contains no cuspidal line (this is always the case when $R = \overline{\mathbb{Q}_\ell}$, i.e. all irreducible ℓ-adic representations of $\mathrm{GL}_n(F)$ are banal). Now in [6, Section 2.6], we show that a generic representation π of $\mathrm{GL}_n(F)$ can be written uniquely as a product $\pi_b \times \pi_{tnb}$ (in the sense of normalized parabolic induction), where π_b is a banal representation of $\mathrm{GL}_m(F)$ with m as large as possible for this property (and π_{tnb} consequently non banal, in fact we say totally non banal). One of the main results of [6] (Theorem 4.19) asserts the following.

Theorem 2.1. Let π and π' be two generic R-representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$ respectively, then $L(X, \pi, \pi') = L(X, \pi_b, \pi'_b)$.

As we already saw that γ-factors of pairs are compatible with reduction modulo ℓ, one has a full understanding of reduction modulo ℓ of local factors thanks to the next statement, which is [6, 4.18].

Theorem 2.2. Let $\overline{\pi}$ and $\overline{\pi'}$ be two banal generic $\overline{\mathbb{F}_\ell}$-representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$ respectively, and π and π' be generic ℓ-adic lifts of $\overline{\pi}$ and $\overline{\pi'}$ respectively, then

$$r_\ell(L(X, \pi, \pi')) = L(X, \overline{\pi}, \overline{\pi'}).$$

From these two statements, we obtain the inductivity relation of local constants of ℓ-modular generic representations, as well as an explicit expression of the cuspidal L-factors of pairs in terms of type theory. We also show that the ℓ-modular L-factors of pairs of generic representations are the gcd of the reduction modulo ℓ of the L-factors of their lifts in a certain sense ([6, Theorem 4.22]).

3 Deligne representations of the Weil group

3.1 Definition and first properties

Let $\nu : \mathrm{W}_F \to R^\times$ be the unique character trivial on the inertia subgroup of W_F and sending a geometric Frobenius element to q^{-1}. It is alright to use this notation for ν as it corresponds to the normalized absolute value of $F^\times$ via local class field theory.

Definition 3.1. A *Deligne R-representation* of W_F is a pair (Φ, U) where Φ is a finite dimensional R-representation of W_F, and $U \in \mathrm{Hom}_{\mathrm{W}_F}(\nu\Phi, \Phi)$. Moreover, we say that (Φ, U) is a *W_F-semi-simple Deligne R-representation* if Φ is semi-simple as a representation of W_F.

We say that two Deligne representations (of W_F) (Φ, U) and (Φ', U') are isomorphic if there is a W_F-isomorphism A from V_Φ to V'_Φ such that $A \circ U = U' \circ A$. When (Φ, U) is a Deligne representation, we typically write $U = N + S$ for its Jordan decomposition. It is immediate to check that (Φ, N) and (Φ, S) are also Deligne representations. We say that (Φ, U) is nilpotent when $U = N$, which is always the case when $R = \overline{\mathbb{Q}_\ell}$. The basic operations on Deligne representations are the following.

Definition 3.2. Let $(\Phi, U), (\Phi', U')$ be Deligne R-representations and let $U = S + N$.

(1) The *direct sum* of (Φ, U) and (Φ', U') is defined by

$$(\Phi, U) \oplus (\Phi', U') = (\Phi \oplus \Phi', U \oplus U').$$

(2) The *dual* of (Φ, U) is defined by

$$(\Phi, U)^\vee = (\Phi^\vee, S^\vee - N^\vee).$$

(3) The *tensor product* of of (Φ, U) and (Φ', U') is defined by

$$(\Phi, U) \otimes (\Phi', U') = (\Phi \otimes \Phi', U \otimes \mathrm{Id} \oplus \mathrm{Id} \otimes U').$$

The first two operations preserve W_F-semi-simplicity, but the third one does not in general (see [7, Example 3.11]) though it does when Φ or Φ' is a direct sum of characters, or when $R = \overline{\mathbb{Q}_\ell}$. As we are really interested in W_F-semi-simple Deligne representations, we shall define in the next section a "semi-simple" tensor product of W_F-semi-simple Deligne representations. First we introduce a useful set of notations.

Notation 3.3.

- $\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$ the set of isomorphism classes of W_F-semi-simple Deligne R-representations.
- $\mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R)$ the isomorphism classes of indecomposable Deligne representations in $\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$.
- $\mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, R)$ the isomorphism classes of irreducible Deligne R-representations in $\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$.
- $\mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, R)$ the isomorphism classes of Deligne representations $(\Phi, U) \in \mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$ with $U = N$. In particular $\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{Q}_\ell}) = \mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{Q}_\ell})$.

The following relaxation of the ismomorphism relation on $\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$ turns out to be at the same time fundamental and natural when studying ℓ-modular W_F-semi-simple Deligne representations.

Definition 3.4. The definition is in two steps:

(1) Deligne R-representations $(\Phi, U), (\Phi', U') \in \mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R)$ are *equivalent*, denoted

$$(\Phi, U) \sim (\Phi', U'),$$

if there exists $\lambda \in R^\times$ such that

$$(\Phi', U') = (\Phi, \lambda U).$$

(2) In the general case, $(\Phi, U), (\Phi', U') \in \mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$ are *equivalent*, still denoted

$$(\Phi, U) \sim (\Phi', U'),$$

if one can decompose $(\Phi', U') = \bigoplus_{i=1}^r (\Phi_i, U_i)$ and $(\Phi, U) = \bigoplus_{i=1}^r (\Phi_i, U_i)$ such that $(\Phi_i, U_i) \sim (\Phi_i, U_i)$ in $\mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R)$.

Definition 3.4 defines an equivalence relation because the decomposition of a W_F-semi-simple Deligne R-representation into indecomposable Deligne R-representations is unique (see [7, Remark 4.9]). In fact it coincides with the isomorphism relation on $\mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, R)$ by [7, Proposition 4.11]. The sets $\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$, $\mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, R)$, $\mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R)$, and $\mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, R)$ are unions of $\sim$-classes.

Notation 3.5.

- For $(\Phi, U) \in \mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$, we let $[\Phi, U]$ denote its equivalence class.
- $[\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)] = \mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)/\sim$.
- $[\mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, R)] = \mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, R)/\sim$.
- $[\mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R)] = \mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R)/\sim$.
- $[\mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, R)] = \mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, R)/\sim = \mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, R)$.

As already said when $R = \overline{\mathbb{Q}_\ell}$ one has $[\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{Q}_\ell})] = \mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{Q}_\ell})$. Note that the operations $\oplus$ and $(\Phi, U) \mapsto (\Phi, U)^\vee$ on $\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$ descend to $[\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)]$ whereas $\otimes$ does not when $R = \overline{\mathbb{F}_\ell}$ (see [7, Example 4.13]). We shall also take care of this problem in the next section. We however already notice that $[\Phi, U] \otimes [\Phi', U]$ is well defined in $[\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{F}_\ell})]$ as soon as Φ is a direct sum of characters and $U = N$ (see [7, Lemma 4.14]).

3.2 Classification

In this section we describe the set $[\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)]$, note that when $R = \overline{\mathbb{Q}_\ell}$ the result we discuss here have been known for a long time. The problem is in fact immediately reduced to the description of $[\mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R)]$, we first deal with $[\mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, R)]$. Let's give a typical example of an element in $\mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, R)$. We start with $\Psi \in \mathrm{Irr}(\mathrm{W}_F, R)$ the set of isomorphism classes of irreducible W_F representations. We denote by

$$o(\Psi)$$

the cardinality of the "irreducible line"

$$\mathbb{Z}_\Psi = \{\nu^k \Psi, k \in \mathbb{Z}\},$$

which is equal to $+\infty$ when $R = \overline{\mathbb{Q}_\ell}$ and divides $\ell - 1$ (in fact it divides the order of q in $(\mathbb{Z}/\ell\mathbb{Z})^\times$) when $R = \overline{\mathbb{F}_\ell}$.

Example 3.6. Take $\Psi \in \mathrm{Irr}(\mathrm{W}_F, R)$

- Then $\Psi := (\Psi, 0) \in \mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{Q}_\ell}) \cap \mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{Q}_\ell})$ and $[\Psi] = \Psi$.
- If $R = \overline{\mathbb{F}_\ell}$, let I be an isomorphism from $\nu^{o(\Psi)}$ to Ψ. Define a Deligne $\overline{\mathbb{F}_\ell}$-representation

$$C(\Psi, I) = (\Phi(\Psi), C_I)$$
$$\Phi(\Psi) = \bigoplus_{k=0}^{o(\Psi)-1} \nu^k \Psi$$
$$C_I(x_0, \ldots, x_{o(\Psi)-1}) = (I(x_{o(\Psi)-1}), x_0, \ldots, x_{o(\Psi)-2}).$$

  Then $C(\Psi, I) \in \mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{F}_\ell})$ and one checks ([7, Lemmas 4.20 to 4.23]) that its isomorphism class only depends on $\mathbb{Z}_\Psi$, and its equivalence class is independent of I, so we set

$$C(\mathbb{Z}_\Psi) := [C(\Psi, I)] \in [\mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, \overline{\mathbb{F}_\ell})].$$

In fact we prove in [7, Section 4.2] that there are no other type of irreducible Deligne representations.

Theorem 3.7. Take $\Phi \in \mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, R)$. If $R = \overline{\mathbb{Q}_\ell}$ there is a unique $\Psi \in \mathrm{Irr}(\mathrm{W}_F, R)$ such that $\Phi = \Psi$, whereas if $R = \overline{\mathbb{F}_\ell}$ there is either a unique $\Psi \in \mathrm{Irr}(\mathrm{W}_F, R)$ such that $\Phi = \Psi$, or a unique irreducible line $\mathbb{Z}_\Psi$ such that $[\Phi] = C(\mathbb{Z}_\Psi)$ (both cases being disjoint even when $\mathbb{Z}_\Psi = \{\Psi\}$).

Now we give a famous example of an element in $\mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R) \cap \mathrm{Nilp}_{\mathrm{ss}}(\mathrm{D}, R)$.

Example 3.8. For $r \geqslant 1$

$$[0, r-1] = (\Phi(r), N(r))$$
$$\Phi(r) = \bigoplus_{k=0}^{r-1} \nu^k,$$
$$N(r)(x_0, \ldots, x_{r-1}) = (0, x_0, \ldots, x_{r-2}).$$

Note that because $[0, r-1]$ is a direct sum of characters as a W_F-representation and $N(r)$ is nilpotent, the class $[0, r-1] \otimes [\Phi, U]$ is well defined in $[\mathrm{Rep}_{ss}(\mathrm{D}, R)]$ for any (Φ, U) in $\mathrm{Rep}_{ss}(\mathrm{D}, R)$. The classification theorem of $[\mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R)]$ ([7, Section 4.3]) is now easy to state.

Theorem 3.9. Let $[\Phi, U]$ belong to $[\mathrm{Indec}_{\mathrm{ss}}(\mathrm{D}, R)]$, then there is a unique $r \geqslant 1$ and a unique Θ in $[\mathrm{Irr}_{\mathrm{ss}}(\mathrm{D}, R)]$ such that $[\Phi, U] = [0, r-1] \otimes \Theta$.

With this theorem at hand one can define an operation $\otimes_{\mathrm{ss}}$ on $\mathrm{Rep}_{ss}(\mathrm{D}, \overline{\mathbb{F}_\ell})$ and its descent still denoted $\otimes_{\mathrm{ss}}$ to $[\mathrm{Rep}_{ss}(\mathrm{D}, \overline{\mathbb{F}_\ell})]$, we refer the reader to [7, Section 4.4].

3.3 Local factors of Deligne representations

Again we fix a nontrivial R-character ψ of F. We first define the Artin L-factor. For V an R-representation of W_F, we denote by V^{I_F} the space of vectors in V fixed by the elements of the inertia subgroup I_F of W_F. We denote by Frob a geometric Frobenius element in W_F.

Definition 3.10. Let $(\Phi, U) \in \mathrm{Rep}_{\mathrm{ss}}(\mathrm{D}, R)$, we set

$$L(X, (\Phi, U)) = \det((\mathrm{Id} - X\Phi(\mathrm{Frob}))|_{\mathrm{Ker}(U)^{I_F}})^{-1}.$$

Note that when $R = \overline{\mathbb{F}_\ell}$, the Deligne operator U can be bijective and in this case $L(X, (\Phi, U)) = 1$. It is also easy to see that this definition only depends on $[\Phi, U]$. Deligne defined in [1] the ϵ-factors of representations in $\mathrm{Irr}(\mathrm{W}_F, R)$ (and more generally of semi-simple R-representations of W_F). Hence with the definition of L-factors above, the usual formula allows one to define γ-factors for $\mathrm{Irr}(\mathrm{W}_F, R)$. We can now define the following more general γ-factors.

Definition 3.11. Take (Φ, U) in $\mathrm{Rep}_{ss}(\mathrm{D}, R)$, and write $\Phi = \Psi_1 \oplus \cdots \oplus \Psi_r$ a decomposition of Φ into elements of $\mathrm{Irr}(\mathrm{W}_F, R)$, then by definition

$$\gamma(X, (\Phi, U), \psi) = \prod_{i=1}^{r} \gamma(X, \Psi_i, \psi).$$

Again this factor depends only on $[\Phi, U]$. Finally one sets

$$\epsilon(X, [\Phi, U], \psi) = \gamma(X, [\Phi, U], \psi) \frac{L(X, [\Phi, U])}{L(q^{-1}X^{-1}, [(\Phi, U)^\vee])}.$$

It is not immediate from the definition, but one checks that this epsilon factor is indeed a unit of $R[X^{\pm 1}]$ (see [7, Proposition 5.6]). Of course all the definitions above agree with the usual ones when $R = \overline{\mathbb{Q}_\ell}$.

4 An ℓ-modular correspondence which preserves local constants

We begin with some notations.

Notation 4.1.
- $\mathrm{Irr}(\mathrm{GL}_n(F), R)$ the set of isomorphism classes of irreducible R-representations of $\mathrm{GL}_n(F)$.
- $\mathrm{Irr}(G, R) = \coprod_{n \geqslant 0} \mathrm{Irr}(\mathrm{GL}_n(F), R)$.
- $\mathrm{Irr}_{\mathrm{gen}}(\mathrm{GL}_n(F), R)$: the set of isomorphism classes of generic representations in $\mathrm{Irr}(\mathrm{GL}_n(F), R)$.
- $\mathrm{Irr}_{\mathrm{gen}}(G, R) = \coprod_{n \geqslant 0} \mathrm{Irr}_{\mathrm{gen}}(\mathrm{GL}_n(F), R)$.

4.1 The ℓ-modular correspondence of Vignéras

We say that an ℓ-adic irreducible representation of $\mathrm{GL}_n(F)$ is integral if it contains a $\overline{\mathbb{Z}_\ell}$-lattice $\mathcal{L}$ which is $\mathrm{GL}_n(F)$-stable. We denote by an index e the subset of integral representations in the classes above (for instance $\mathrm{Irr}_e(\mathrm{GL}_n(F),\overline{\mathbb{Q}_\ell})$, $\mathrm{Irr}_{\mathrm{gen},e}(G,\overline{\mathbb{Q}_\ell})$). For π as above, the ℓ-modular representation $\mathcal{L}\otimes_{\overline{\mathbb{Z}_\ell}}\overline{\mathbb{F}_\ell}$ has finite length, and Vignéras shows in her work that it contains a unique subquotient $J_\ell(\pi)$ with degenerate Whittaker model of the same type as that of π. Moreover she shows that the map

$$J_\ell : \pi \mapsto J_\ell(\pi)$$

is a surjection from $\mathrm{Irr}_e(G,\overline{\mathbb{Q}_\ell})$ to $\mathrm{Irr}(G,\overline{\mathbb{F}_\ell})$.

We say that a semi-simple ℓ-adic representation of W_F is integral if it contains a W_F-stable lattice. For $\Phi \in \mathrm{Irr}_e(\mathrm{W}_F,\overline{\mathbb{Q}_\ell})$ (the subset of integral representation in $\mathrm{Irr}(\mathrm{W}_F,\overline{\mathbb{Q}_\ell})$) and $\mathcal{M}$ a W_F-stable lattice in the space of Φ, we denote by $r_\ell(\Phi)$ the semi-simplification of the finite length representation $\mathcal{M}\otimes_{\overline{\mathbb{Z}_\ell}}\overline{\mathbb{F}_\ell}$, which is a W_F-module independent of the choice of $\mathcal{M}$. If (Φ,N) belongs to $\mathrm{Rep}_{ss}(\mathrm{D},\overline{\mathbb{Q}_\ell}) = \mathrm{Nilp}_{ss}(\mathrm{D},\overline{\mathbb{Q}_\ell})$, we write it

$$(\Phi,N) = \sum_{k\in K, i\geqslant 1} [0,i_k-1]\otimes\Psi_k$$

for K a finite set and $\Psi_k \in \mathrm{Irr}(W_F)$, and we say that (Φ,N) is integral if all Ψ_k are. We again put an index e to denote the subset of integral representations in a given class (for instance $\mathrm{Rep}_{ss,e}(\mathrm{D},\overline{\mathbb{Q}_\ell})$, $\mathrm{Irr}_{ss,e}(\mathrm{D},\overline{\mathbb{Q}_\ell})$). We then set

$$r_\ell(\Phi,N) = \sum_{k\in K, i\geqslant 1} [0,i_k-1]\otimes r_\ell(\Psi_k).$$

The reduction modulo ℓ map from $\mathrm{Nilp}_{ss,e}(\mathrm{D},\overline{\mathbb{Q}_\ell})$ to $\mathrm{Nilp}_{ss}(\mathrm{D},\overline{\mathbb{F}_\ell})$ is surjective ([12]).

We denote by LLC (local Langlands correspondence) the well-known bijection from $\mathrm{Rep}_{ss}(\mathrm{D},\overline{\mathbb{Q}_\ell})$ to $\mathrm{Irr}(G,\overline{\mathbb{Q}_\ell})$ established in [2] and [4]. We recall that denoting by $W(\pi,\psi)$ the Whittaker model of the standard module lying over $\pi\in\mathrm{Irr}(G,\overline{\mathbb{Q}_\ell})$ for ψ a non-trivial ℓ-adic character of F, and setting $L(X,\pi,\pi') = L(X,W(\pi,\psi),W(\pi',\psi))$, $\gamma(X,\pi,\pi',\psi) = \gamma(X,W(\pi,\psi),W(\pi',\psi),\psi)$ and $\epsilon(X,\pi,\pi',\psi) = \epsilon(X,W(\pi,\psi),W(\pi',\psi),\psi)$, the bijection LLC satisfies and is characterized ([3]) by the properties, for (Φ,N) and (Φ',N') in $\mathrm{Rep}_{ss}(\mathrm{D},\overline{\mathbb{Q}_\ell})$:

$$L(X,(\Phi,N)\otimes(\Phi',N')) = L(X,\mathrm{LLC}(\Phi,N),\mathrm{LLC}(\Phi',N'))$$

and

$$\epsilon(X,(\Phi,N)\otimes(\Phi',N'),\psi) = \epsilon(X,\mathrm{LLC}(\Phi,N),\mathrm{LLC}(\Phi',N'),\psi)$$

(which in particular imply the same equality for γ-factors).

We denote by $\pi\mapsto\pi_R^*$ the Aubert-Zelevinsky involution on $\mathrm{Irr}(G,R)$ (see [12] for the subtleties of the case $R=\overline{\mathbb{F}_\ell}$). In [12], Vignéras shows that LLC restricts to a bijection from $\mathrm{Rep}_{ss,e}(\mathrm{D},\overline{\mathbb{Q}_\ell})$ to $\mathrm{Irr}_e(G,\overline{\mathbb{Q}_\ell})$, and proves the existence of a bijection $\mathrm{V}:\mathrm{Irr}(G,\overline{\mathbb{F}_\ell})\to\mathrm{Nilp}_{ss}(\mathrm{D},\overline{\mathbb{F}_\ell})$ characterized by the relation for all $(\Phi,N)\in\mathrm{Rep}_{ss,e}(\mathrm{D},\overline{\mathbb{Q}_\ell})$:

$$r_\ell((\Phi,N)) = \mathrm{V}(J_\ell(\mathrm{LLC}(\Phi,N)^*_{\overline{\mathbb{Q}_\ell}})^*_{\overline{\mathbb{F}_\ell}}).$$

This bijection commutes with taking duals, twisting by characters and takes central character to determinant, however it does not satisfy the preservation of local constants as LLC^{-1} does (see [7, Example 6.18]).

4.2 A modified correspondence which preserves local constants

In [7, Section 6], thanks to our classification of $[\mathrm{Rep}_{ss,e}(\mathrm{D},\overline{\mathbb{F}_\ell})]$, we define an injection

$$\mathrm{CV} : \mathrm{Nilp}_{ss}(\mathrm{D},\overline{\mathbb{F}_\ell}) \to [\mathrm{Rep}_{ss,e}(\mathrm{D},\overline{\mathbb{F}_\ell})].$$

It is by definition compatible with direct sums, hence it is sufficient to describe it on elements of $\mathrm{Nilp}_{ss}(\mathrm{D},\overline{\mathbb{F}_\ell})$ supported on irreducible lines. So take $\Psi \in \mathrm{Irr}(\mathrm{W}_F,\overline{\mathbb{F}_\ell})$ and $(\Phi, N) \in \mathrm{Nilp}_{ss}(\mathrm{D},\overline{\mathbb{F}_\ell})$ supported on $\mathbb{Z}_\Psi$:

$$(\Phi, N) = \bigoplus_{i\geqslant 0} \bigoplus_{k=0}^{o(\Psi)-1} a_{i,k}[0, i-1] \otimes \nu^k\Psi.$$

We say that (Φ, N) is *acyclic* if for each fixed i, there is $0 \leqslant k \leqslant o(\Psi) - 1$ such that $a_{i,k} = 0$. We say that (Φ, N) is *cyclic* if for each fixed i, the coefficient $a_{i,k}$ is independent of k.

For (Φ, N) as above, we set

$$b_i = \min_k a_{i,k}, \text{ and } c_{i,k} = a_{i,k} - b_i.$$

Then

$$(\Phi, N) = (\Phi, N)_{\mathrm{acyc}} \oplus (\Phi, N)_{\mathrm{cyc}},$$

with

$$(\Phi, N)_{\mathrm{acyc}} = \bigoplus_{i\geqslant 1} \bigoplus_{k=0}^{o(\Psi)-1} c_{i,k}[0, i-1] \otimes \nu^k\Psi$$

$$(\Phi, N)_{\mathrm{cyc}} = \bigoplus_{j\geqslant 1} b_j[0, j-1] \otimes \bigoplus_{k=0}^{o(\Psi)-1} \nu^k\Psi.$$

Then by definition

$$\mathrm{CV}((\Phi, N)_{\mathrm{cyc}}) = \bigoplus_{j\geqslant 1} b_j[0, j-1] \otimes C(\mathbb{Z}_\Psi) \in [\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D},\overline{\mathbb{F}_\ell})],$$

and

$$\mathrm{CV}(\Phi, N) = (\Phi, N)_{\mathrm{acyc}} \oplus \mathrm{CV}((\Phi, N)_{\mathrm{cyc}}) \in [\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D},\overline{\mathbb{F}_\ell})].$$

We then obtain an injection

$$\mathrm{C} := \mathrm{CV} \circ \mathrm{V} : \mathrm{Irr}(G,\overline{\mathbb{F}_\ell}) \to [\mathrm{Rep}_{\mathrm{ss}}(\mathrm{D},\overline{\mathbb{F}_\ell})].$$

The main result of [7] is [7, Theorem 6.15]. It is a consequence of the results of Section 2.2 together with an explicit computation of both sides in the banal cuspidal case.

Theorem 4.2. For $\pi,\ \pi' \in \mathrm{Irr}_{\mathrm{gen}}(G,\overline{\mathbb{F}_\ell})$ and $\psi : F \to \overline{\mathbb{F}_\ell}^\times$ a non trivial character, we have

$$L(X, \mathrm{C}(\pi) \otimes_{\mathrm{ss}} \mathrm{C}(\pi')) = L(X, \pi, \pi'),$$
$$\gamma(X, \mathrm{C}(\pi) \otimes_{\mathrm{ss}} \mathrm{C}(\pi'), \psi) = \gamma(X, \pi, \pi', \psi),$$
$$\epsilon(X, \mathrm{C}(\pi) \otimes_{\mathrm{ss}} \mathrm{C}(\pi'), \psi) = \epsilon(X, \pi, \pi', \psi).$$

4.3 The case of Godement-Jacquet L-factors

In [8] which is the first detailed study of local factors modulo ℓ, Mínguez defines and studies the Godement-Jacquet local constants of elements in $\mathrm{Irr}(G,R)$ (the paper is more generally concerned with inner forms of $\mathrm{GL}_n(F)$). In particular he obtains ([8, Corollaire 4.2]) that the Godement-Jacquet γ-factors are compatible with reduction modulo ℓ whereas L-factors of elements in $\mathrm{Irr}(G,\overline{\mathbb{F}_\ell})$ divide those of their lifts in $\mathrm{Irr}(G,\overline{\mathbb{Q}_\ell})$. He also proves the inductivity relation of L-factors, but only in the banal case ([8, Théorème 5.7]).

One can extract from [8], and especially [8, Section 6], the computation of all local constants of $\mathrm{GL}_2(F)$. We end this survey with a final exercise which is mentioned but not done at the end of [7], and which is to check that the C-correspondence above preserves standard local constants for all irreducible representations of $\mathrm{GL}_2(F)$. First we recall the classification of ℓ-modular irreducible representations of $\mathrm{GL}_2(F)$ due to Vignéras ([10]), and we give for each case the corresponding C-parameter. We refer to [7, Section 6] for the notations that we will use hereunder.

The ℓ-modular irreducible representations of $\mathrm{GL}_2(F)$ are:

1) The supercuspidal representations.
2) The cuspidal non-supercuspidal representations (this happens only if $q\equiv -1[\ell]$), in which case they are of the form $\mathrm{St}_1(\mathbb{Z}_\chi)$ for χ a character of $F^\times$ if $\ell=2$ (in which case $\mathbb{Z}_\chi=\{\chi\}$), or of the form $\mathrm{St}_2(\mathbb{Z}_\chi)$ for χ a character of $F^\times$ if $\ell\neq 2$ (in which case $\mathbb{Z}_\chi=\{\chi,\nu\chi\}$).
3) Irreducible principal series $\chi_1\times\chi_2$ with $\chi_1\chi_2^{-1}\neq\nu^{\pm1}$,
4) The non cuspidal Steinberg representations $\mathrm{St}(2,\chi)$ for χ a character of $F^\times$ (this happens only if $q\not\equiv -1[\ell]$).
5) $\nu^{1/2}\chi\circ\det$ for χ a character of $F^\times$.

Now in each case we give the corresponding C-parameter, they are all computed in [7, Example 6.12] except in the last case.

1) If π is banal supercuspidal, then $\mathrm{C}(\pi)=\Psi$ for a unique $\Psi\in\mathrm{Irr}(\mathrm{W}_F,\overline{\mathbb{F}_\ell})\simeq\mathrm{Irr}_{ss}(\mathrm{D},\overline{\mathbb{F}_\ell})\cap\mathrm{Nilp}(\mathrm{D},\overline{\mathbb{F}_\ell})$, whereas if it is non-banal supercuspidal, then $\mathrm{C}(\pi)=C(\mathbb{Z}_\Psi)$ where $\mathbb{Z}_\Psi=\{\Psi\}$ for a unique $\Psi\in\mathrm{Irr}(\mathrm{W}_F,\overline{\mathbb{F}_\ell})$. In both cases Ψ has dimension 2.
2) Here $q\equiv -1[\ell]$. When $\ell=2$ then $\mathrm{C}(\mathrm{St}_1(\mathbb{Z}_\chi))=C(\mathbb{Z}_\chi)\oplus C(\mathbb{Z}_\chi)$ where $\mathbb{Z}_\chi=\{\chi\}$, whereas when $\ell\neq 2$ then $\mathrm{C}(\mathrm{St}_2(\mathbb{Z}_\chi))=C(\mathbb{Z}_\chi)$ where $\mathbb{Z}_\chi=\{\chi,\ \nu\chi\}$.
3) $\mathrm{C}(\chi_1\times\chi_2)=\chi_1\oplus\chi_2$ with $\chi_i\in\mathrm{Irr}(\mathrm{W}_F,\overline{\mathbb{F}_\ell})\simeq\mathrm{Irr}_{ss}(\mathrm{D},\overline{\mathbb{F}_\ell})\cap\mathrm{Nilp}(\mathrm{D},\overline{\mathbb{F}_\ell})$.
4) Here $q\not\equiv -1[\ell]$. $\mathrm{C}(\mathrm{St}(2,\chi))=[0,1]\otimes\chi$ if $q\not\equiv 1[\ell]$, and $\mathrm{C}(\mathrm{St}(2,\chi))=[0,1]\otimes C(\mathbb{Z}_\chi)$ where $\mathbb{Z}_\chi=\{\chi\}$ if $q\equiv 1[\ell]$.
5) $\mathrm{C}(\nu^{1/2}\chi\circ\det)=\chi\oplus\nu\chi$ if $q\not\equiv\pm1[\ell]$, $\mathrm{C}(\nu^{1/2}\chi\circ\det)=[0,1]\otimes\nu^{-1}\chi$ if $q\equiv -1[\ell]$ and $\ell\neq 2$, and $\mathrm{C}(\nu^{1/2}\chi\circ\det)=[0,1]\otimes C(\mathbb{Z}_\chi)$ where $\mathbb{Z}_\chi=\{\chi\}$ if $q\equiv 1[\ell]$.

Because γ-factors are compatible with reduction modulo ℓ and only see the supercuspidal support of a representation, it follows from the properties of the LLC and the definition of C that the γ-factors are preserved by the C-correspondence. Hence it is sufficient to compare L-factors on both sides of the correspondence. One checks from the definitions on the Galois side, and the computations in [8]:

1) If π is supercuspidal, then $L(X, \mathrm{C}(\pi)) = L(X, \pi) = 1$.
2) If π is cuspidal, then $L(X, \mathrm{C}(\pi)) = L(X, \pi) = 1$ again.
3) $L(X, \mathrm{C}(\chi_1 \times \chi_2)) = L(X, \chi_1 \times \chi_2) = L(X, \chi_1)L(X, \chi_2)$, where $L(X, \chi_i) = 1$ if χ_i is ramified or $q \equiv 1[\ell]$ and $L(X, \chi_i) = (1 - \chi_i(\varpi)X)^{-1}$ otherwise (where ϖ is a uniformizer of F).
4) Here $q \not\equiv -1[\ell]$. $L(X, \mathrm{C}(\mathrm{St}(2, \chi))) = L(X, \mathrm{St}(2, \chi))$, and it is equal to 1 if $q \equiv 1[\ell]$ or if χ is ramified, and to $(1 - \chi(w)q^{-1}X)$ if $q \not\equiv 1[\ell]$ otherwise.
5) $L(X, \mathrm{C}(\nu^{1/2}\chi \circ \det)) = L(X, \nu^{1/2}\chi \circ \det)$. It is equal to 1 if χ is ramified or if $q \equiv 1[\ell]$. Otherwise it is equal to $(1 - \chi(\varpi)X)^{-1}$ if $q \equiv -1[\ell]$ and $\ell \neq 2$, and to $(1 - q^{-1}\chi(w)X)^{-1}(1 - \chi(\varpi)X)^{-1}$ if $q \not\equiv \pm 1[\ell]$.

We expect this preservation property to hold for all irreducible representations of $\mathrm{GL}_n(F)$ for all n.

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