A motivic approach to Shimura's zeta functions and attached p-adic L-functions via admissible measures

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Abstract

A motivic approach is presented to Shimura's zeta functions $\mathcal{Z}(s,\mathbf{f})$ [47] attched to holomorphic automorphic forms $\mathbf{f}$ on unitary groups $U_K(n,n)$ over an imaginary quadratic field $K=\mathbb{Q}(\sqrt{-D_K})$. A motivically normalized L-function $\mathcal{D}(s,\mathbf{f})$ attached to $\mathcal{Z}(s,\mathbf{f})$ is defined in accordance with Deligne's conjectures [14]. An explicit description of Shimura's Γ-factors is used.

The attached p-adic L-functions of $\mathcal{D}(s,\mathbf{f})$ satisfies conjecture of Coates-Perrin-Riou [11] and it is constructed via *admissible measures of Amice-Vélu*, see also [31]. The p-ordinary case was treated in [17] via algebraic geometry (method of Katz).

The main result is stated in terms of the Hodge polygon $P_H(t):[0,d]\to\mathbb{R}$ and the Newton polygon $P_N(t)=P_{N,p}(t):[0,d]\to\mathbb{R}$ of the zeta function $\mathcal{D}(s,\mathbf{f})$ of degree $d=4n$. Main theorem gives a p-adic analytic interpolation of the L values in the form of certain integrals with respect to Mazur-type measures, saisfying Coates-Perrin-Riou conjectures.

Both Rankin-Selberg and doubling methods are used.

1 Euler products of Shimura

Explicit Euler products were constructed by G. Shimura from the Hecke eigenvalues of automorphic forms on a classical group (see [47], [45], [46]), where their analytic continuation to the whole complex plane was proved when the group is a unitary group over a CM field and the eigenform is holomorphic. Also, an analytic continuation of an Eisenstein series was proved on another unitary group, containing the group just mentioned defined with such an eigenform.

The idea of a motivic interpretation of Shimura's zeta functions comes from their explicite Gamma factors, see [47] compared with the Deligne-Serre's Gamma factors of Hodge structures on the Betti and de Rham realisations of a motive [43], [14]. A proceedure of bringing Gamma factors to a canonical form is described by H.Cohen [13]. When applied to the classical Eisenstein series E_k, it gives the Gamma factor $\Gamma(s)$ of a classical cusp form. An evidence for the motivic nature of the Gamma factors could be deduces from the attached p-adic L-functions using their Hodge and Newton polygons $P_H(t)$ and $P_{N,p}(t)$ (at p).

1.1 Hermitian modular group $\Gamma_{n,K}$ and the standard zeta function $\mathcal{Z}(s, \mathbf{f})$ (definitions)

The followng function $\mathcal{Z}(s, \mathbf{f})$ is a special c!ase of Euler products constructed by G. Shimura. Let $\theta = \theta_K$ be the quadratic character attached to $K = \mathbb{Q}(\sqrt{-D_K})$, $n' = \left[\frac{n}{2}\right]$.

$$\Gamma_{n,K} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_K) | M\eta_n M^* = \eta_n \right\},\ \eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

$$\mathcal{Z}(s, \mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

(defined via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a} \subset \mathcal{O}_K$)

$$= \prod_{\mathfrak{q}} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1} (\text{an Euler product over primes } \mathfrak{q} \subset \mathcal{O}_K,$$

with $\deg \mathcal{Z}_{\mathfrak{q}}(X) = 2n$, the Satake parameters $t_{i,\mathfrak{q}}, i = 1, \cdots, n$),

$$\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f}) \ \text{(Motivically normalized standard zeta function}$$

with a functional equation $s \mapsto \ell - s$; $\mathrm{rk} = 4n$, and motivic weight $\ell - 1$).

Main result: Assuming $\ell > 2n$, a p-adic interpolation is constructed of all critical values $\mathcal{D}(s, \mathbf{f}, \chi)$ normalized by $\times\Gamma_{\mathcal{D}}(s)/\Omega_{\mathbf{f}}$, in the critical strip $n \leq s \leq \ell - n$ for all χ mod p^r in both *bounded or unbounded case* , i.e. when the product $\alpha_{\mathbf{f}} = \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)}$ is *not a p-adic unit.*

1.2 Example: Ikeda's lifting $f \rightsquigarrow \mathbf{f} = Lift(f)$

Its L-function gives a crucial motivation for both complex and p-adic theory of L-functions on unitary groups, and extends to a general (not necessarily lifted) case. Recall that in [19]

$$S_{2k+1}(\Gamma_0(D), \theta) \ni f \rightsquigarrow \mathbf{f} = Lift(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}),\ \text{if } n = 2n' \text{ is even } (E)$$

$$S_{2k}(\mathrm{SL}(\mathbb{Z})) \ni f \rightsquigarrow \mathbf{f} = Lift(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}),\ \text{if } n = 2n' + 1 \text{ is odd } (O)$$

the standard L-function of $\mathbf{f} = Lift^{(n)}(f)$ is a nice product: $\mathcal{Z}(s, \mathbf{f}) =$

$$\prod_{i=1}^{n} L(s + k + n' - i + (1/2), f) L(s + k + n' - i + (1/2), f, \theta) \ \ [19]$$

$$= \prod_{i=0}^{n-1} L(s + \ell/2 - i - (1/2), f) L(s + \ell/2 - i - (1/2), f, \theta).$$

Notice $k + n' = \ell/2$, then the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1 - s$ becomes $\Gamma_{\mathcal{Z}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s + \ell/2 - i - (1/2))^2$.

1.3 A motivic normalization for the scalar-valued automorphic forms

For the general case see in [9].

The Gamma factor $\Gamma_{\mathcal{Z}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$. suggests the following nice motivic normalization $\mathcal{D}(s) = \mathcal{Z}(s-(\ell/2)+(1/2)) = \prod\limits_{i=0}^{n-1} L(s-i,f)L(s-i,f,\theta)$ with the *Gamma factor* $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{Z}}(s-(\ell/2)+(1/2)) = \prod\limits_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s-i)^2$, and the L-function $\mathcal{D}(s)$ satisfies the symmetry $s \mapsto \ell - s$ of motivic weight $\ell - 1$ with the *slopes* $2\cdot 0, 2\cdot 1, \ldots 2\cdot(n-1)$,
$2\cdot(\ell-n), \cdots, 2\cdot(\ell-1)$, so that *Deligne's critical values* (as in [14]) are at $s = n, \ldots, s = \ell - n$.

Moreover the existence of p-adic L-funcions in Ikeda's case $\mathbf{f} = Lift(\Delta)$ of degree 3 directly follows in this case from the abnove product (even in the non-ordinary case, e.g. for $f = \Delta$, $p = 7$, $\tau(7) = -16744 = -2^3\cdot 7\cdot 13\cdot 23$, for any K, but $n = 2n'+1$ must be odd) with $\ell = 2k+2n' = 12+2n'$.

1.4 General zeta functions: critical values and coefficients

More general zeta functions are Euler products of degree d

$$\mathcal{D}(s,\chi) = \sum_{n=1}^{\infty} \chi(n)a_n n^{-s} = \prod_p \frac{1}{\mathcal{D}_p(\chi(p)p^{-s})},\ \Lambda_{\mathcal{D}}(s,\chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s,\chi),$$

where $\deg \mathcal{D}_p(X) = d$ for all but finitely many p, and $\mathcal{D}_p(0) = 1$.

In many cases algebraicity of the zeta values was proven as

$$\frac{\mathcal{D}^*(s_0,\chi)}{\Omega^{\pm}_{\mathcal{D}}} \in \mathbb{Q}(\{\chi(n), a_n\}_n),\ \text{where } \mathcal{D}^*(s,\chi) \text{ is normalized by } \Gamma_{\mathcal{D}},$$

at critical points $s_0 \in \mathbb{Z}_{crit}$ as linear combinations of *coefficients* a_n dividing out *periods* $\Omega^{\pm}_{\mathcal{D}}$, where $\mathcal{D}^*(s_0,\chi) = \Lambda_{\mathcal{D}}(s_0,\chi)$ if $h^{\ell,\ell} = 0$.

In p-adic analysis, the Tate field is used $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$, the completion of an algebraic closure $\bar{\mathbb{Q}}_p$, in place of $\mathbb{C}$. Let us fix embeddings $\begin{cases} i_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \\ i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}, \end{cases}$ and continue analytically the normalized zeta values $\mathcal{D}^{alg}(s,\chi) = A(s,\chi).\frac{\mathcal{D}^*(s,\chi)}{\Omega^{\pm}_{\mathcal{D}}}$ to $s \in \mathbb{Z}_p$, $\chi \bmod p^r$, where $A(s,\chi)$ is a polynomial elementary factor of [11], p.29.

1.5 The Hodge and Newton polygons of $\mathcal{D}(s)$

are used in order to state our Main result.

The Hodge polygon $P_H(t) : [0,d] \to \mathbb{R}$ of the function $\mathcal{D}(s)$ and the Newton polygon $P_{N,p}(t) : [0,d] \to \mathbb{R}$ at p are piecewise linear functions:

The Hodge polygon of weight w has the slopes j of $length_j = h^{j,w-j}$ given by Serre's Gamma factors of the functional equation of the form $s \mapsto w+1-s$, relating $\Lambda_{\mathcal{D}}(s,\chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s,\chi)$ and $\Lambda_{\mathcal{D}^\rho}(w+1-s,\bar{\chi})$, where ρ is the complex conjugation of a_n, and $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^\rho}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s) = \prod_{j \le \frac{w}{2}} \Gamma_{j,w-j}(s)$, where

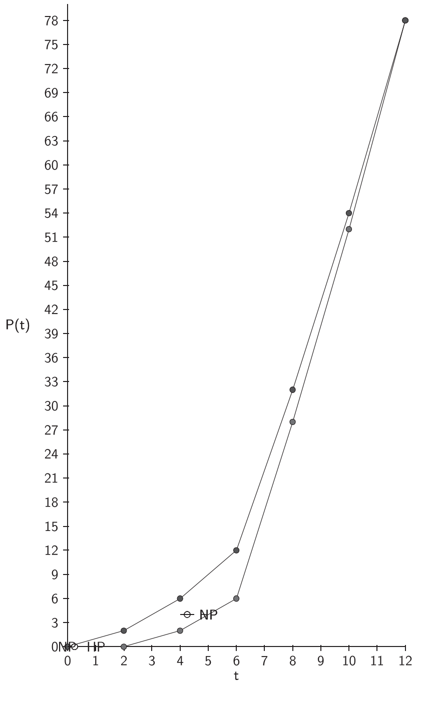
$$\Gamma_{j,w-j}(s) = \begin{cases} \Gamma_{\mathbb{C}}(s-j)^{h^{j,w-j}}, & \text{if } j < w, \\ \Gamma_{\mathbb{R}}(s-j)^{h^{j,j}_+}\Gamma_{\mathbb{R}}(s-j+1)^{h^{j,j}_-}, & \text{if } 2j = w,\ \text{where} \end{cases}$$

$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s)$, $h^{j,j} = h_{+}^{j,j} + h_{-}^{j,j}$, $\sum_j h^{j,w-j} = d$, see [13] for the various examples with Gamma factors.

The Newton polygon at p is the convex hull of points $(i, \mathrm{ord}_p(a_i))$ $(i = 0, \ldots, d)$; its slopes λ are the p-adic valuations $\mathrm{ord}_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X) \in \bar{\mathbb{Q}}[X] \subset \mathbb{C}_p[X]$: $length_\lambda = \sharp\{i \mid \mathrm{ord}_p(\alpha_i) = \lambda\}$. According to [6], Th. 8.36, $P_{Newton,p}(t) \geq P_{Hodge}(t)$ on $[0, d]$.

Hodge/Newton polygons for $\mathbf{f} = Lift(\Delta)$, $n = 3$, $U(3,3)$

Let us draw $P_{Hodge}(t)$ (slopes 0,1,2,11,12,13), and $P_{Newton,p}(t)$ (slopes 1,2,3,10,11,12), symmetry for slopes: $j \mapsto 13 - j$, for $p = 7$, $\boldsymbol{f} = Lift(\Delta)$, $k = 12$, $n' = 1$, $\ell = 14 = k + 2n'$, $d = 4n = 12$, $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathbb{C}}(s)^2\Gamma_{\mathbb{C}}(s-1)^2\Gamma_{\mathbb{C}}(s-2)^2$, symmetry $s \mapsto 14 - s$. $P_{Newton,p}(6) = 12$, $P_{Hodge}(6) = 6$,



$h = 6$ ("a motivic Hasse invariant")

1.6 p-adic analytic interpolation of $\mathcal{D}(s, \mathbf{f}, \chi)$

The result expresses the zeta values as integrals with respect to p-adic Mazur-type measures. These measures are constructed from the Fourier coefficients of Hermitian modular forms, and from eigenvalues of Hecke operators on the unitary group.

Pre-ordinary case: $P_H(t) = P_{N,p}(t)$ **at** $t = \frac{d}{2}$ The integrality of measures is proven by T.Bouganis [7], representing $\mathcal{D}^*(s,\chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s,\chi)$ as a Rankin-Selberg type integral at critical points $s = m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce certain bounded measures $\mu_{\mathcal{D}}$ from integral representations and Petersson product, [10]. For the case of p inert in K, see [7].

Admissible case: $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2}) > 0$ The zeta distributions are unbounded, but their sequence produce h-admissible (growing) measures of Amice-Vélu-type, allowing to integrate any continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) = \mathcal{Y}_p$. A general result is used on the existence of h-admissible (growing) measures from binomial congruences for the coefficients of Hermitian modular forms. Their p-adic *Mellin transforms* $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y(x)d\mu_{\mathcal{D}}(x)$, $\mathcal{L}_{\mathcal{D}} : \mathcal{Y}_p \to \mathbb{C}_p$ give p-adic analytic interpolation of growth $\log_p^h(\cdot)$ of the L-values: the values $\mathcal{L}_{\mathcal{D}}(\chi x_p^m)$ are integrals given by $i_p\left(\frac{\mathcal{D}^*(m,\mathbf{f},\chi)}{\Omega_{\mathbf{f}}}\right) \in \mathbb{C}_p$.

2 The explicit form of the standard zeta function

For all integral ideals $\mathfrak{a} \subset \mathcal{O}$ let $T(\mathfrak{a})$ denotes the Hecke operator associated to it as in [47], page 162, using the action of double cosets $\Gamma\xi\Gamma$ with $\xi = \mathrm{diag}(\hat{D}, D)$, $(\det(D)) = (\alpha)$, $\hat{D} = (D^*)^{-1}$, $\alpha \in \mathfrak{a}$.

Consider a non-zero Hermitian modular form $\mathbf{f} \in \mathcal{M}_\ell(\Gamma)$, for a (congruence) subgroup $\Gamma \subset \Gamma_{n,K}$, and assume $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a}) \in \mathbb{C}$ for all integral ideals $\mathfrak{a} \subset \mathcal{O}$. Then

$$\mathcal{Z}(s,\mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s-i+1,\theta^{i-1})\right)\sum_{\mathfrak{a}} \lambda(\mathfrak{a})N(\mathfrak{a})^{-s},$$

the sum is over all integral ideals of $\mathcal{O}_K$.

This series has an Euler product representation $\mathcal{Z}(s,\mathbf{f}) = \prod_{\mathfrak{q}}(\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$, where the product is over all prime ideals of $\mathcal{O}_K$, $\mathcal{Z}_{\mathfrak{q}}(X)$ is the numerator of the series $\sum_{r\geq 0}\lambda(\mathfrak{q}^r)X^r \in \mathbb{C}(X)$, computed by Shimura as follows.

2.1 Euler factors of the standard zeta function, [47], p. 171

The Euler factors $\mathcal{Z}_{\mathfrak{q}}(X)$ in the Hermitian modular case at the prime ideal $\mathfrak{q}$ of $\mathcal{O}_K$ are

$$\text{(i) } \mathcal{Z}_{\mathfrak{q}}(X) = \prod_{i=1}^{n} \left((1 - N(\mathfrak{q})^{n-1} t_{\mathfrak{q},i} X)(1 - N(\mathfrak{q})^{n} t_{\mathfrak{q},i}^{-1} X)\right)^{-1},$$

if $\mathfrak{q}^{\rho} = \mathfrak{q}$ and $\mathfrak{q} \not| \mathfrak{c}$, (*the inert case outside level* $\mathfrak{c}$),

$$\text{(ii) } \mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^{2n} \left((1 - N(\mathfrak{q}_1)^{2n} t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1} X_1)(1 - N(\mathfrak{q}_2)^{-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i} X_2)\right)^{-1},$$

if $\mathfrak{q}_1 \neq \mathfrak{q}_2, \mathfrak{q}_1^{\rho} = \mathfrak{q}_2$ and $\mathfrak{q}_i \not| \mathfrak{c}$ for $i = 1, 2$ *(the split case outside level)* ,

$$\text{(iii) } \mathcal{Z}_{\mathfrak{q}}(X) = \prod_{i=1}^{n} \left(1 - N(\mathfrak{q})^{n-1} t_{q,i} X\right)^{-1}, \text{ if } \mathfrak{q}^{\rho} = \mathfrak{q} \text{ and } \mathfrak{q}|\mathfrak{c} \textit{ (inert level divisors)},$$

$$\text{(iv) } \mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^{n} \left((1 - N(\mathfrak{q}_1)^{n-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1} X_1)(1 - N(\mathfrak{q}_2)^{n-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i} X_2)\right)^{-1},$$

if $\mathfrak{q}_1 \neq \mathfrak{q}_2, \mathfrak{q}_i|\mathfrak{c}$ for $i = 1, 2$ *(split level divisors).*

where the $t_{?,i}$ above for $? = \mathfrak{q}, \mathfrak{q}_1\mathfrak{q}_2$, are the *Satake parameters* of the eigenform $\mathbf{f}$.

2.2 The standard motivic-normalized zeta $\mathcal{D}(s, \mathbf{f}, \chi)$

The standard zeta function of $\mathbf{f}$ is defined by means of the p-parameters as the following Euler product:

$$\mathcal{D}(s, \mathbf{f}, \chi) = \prod_{p} \prod_{i=1}^{2n} \left\{ \left(1 - \frac{\chi(p)\alpha_i(p)}{p^s}\right) \left(1 - \frac{\chi(p)\alpha_{4n-i}(p)}{p^s}\right) \right\}^{-1},$$

where χ is an arbitrary Dirichlet character. Motivically, this should be $L(s, \mathrm{Res}_{K/\mathbb{Q}}(M_{\boldsymbol{f}}) \otimes \chi)$ with $\mathrm{rk}_K(M_{\boldsymbol{f}}) = 2n$, $\mathrm{rk}_{\mathbb{Q}}(\mathrm{Res}_{K/\mathbb{Q}}(M_{\boldsymbol{f}})) = 4n$. The p–parameters $\alpha_1(p), \ldots, \alpha_{4n}(p)$ of $\mathcal{D}(s, \mathbf{f}, \chi)$ for p not dividing the level C of the form $\mathbf{f}$ are related to the the $4n$ characteristic numbers

$$\alpha_1(p), \ \cdots, \ \alpha_{2n}(p), \ \alpha_{2n+1}(p), \ \cdots, \ \alpha_{4n}(p)$$

of the product of all $\mathfrak{q}$-factors $\mathcal{Z}_{\mathfrak{q}}(N\mathfrak{q}^{(\ell-1)/2)}X)^{-1}$ for all $\mathfrak{q}|p$, which is a polynomial of degree $4n$ of the variable $X = p^{-s}$ (for almost all p) with coefficients in a number field $T = T(\mathbf{f})$.

The relation between the two normalizations $\mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f}) = \mathcal{D}(s, \mathbf{f})$ also was discussed in [9] and [22] for more general L-functions.

2.3 Description of the Main theorem

Let $\Omega_{\mathbf{f}}$ be a period attached to an Hermitian cusp eigenform $\mathbf{f}$, $\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ the standard zeta function, and
$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i}\right) p^{-n(n+1)}, \quad h = \mathrm{ord}_p(\alpha_{\mathbf{f},p})$.
The number $\alpha_{\mathbf{f}}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ (the Hermitian Fourier expansion) on some $\mathbf{f_0}$, and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$, $d = 4n$, $\dfrac{d}{2} = 2n$.

Definition 1. Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^* : \mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then

- a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a *(bounded) measure* M-valued measure on $\mathbb{Z}_p^*$.

- $\mu : \mathcal{C}^h \to M$ is called an h *admissible measure* M-valued measure on $\mathbb{Z}_p^*$ measure if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

for $j = 0, 1, ..., h-1$, and et $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ be the space of definition of *p-adic Mellin transform*

Theorem 2 ([1], [31])**.** *For an h-admissible measure μ, the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \to \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).*

Theorem 3 (Main Theorem)**.** *Let $\mathbf{f}$ be a Hermitian cusp eigenform of degree $n \geq 2$ and of weight $\ell > 2n$. There exist distributions $\mu_{\mathcal{D},s}$ for $s = n, \cdots, \ell - n$ with the properties:*

i) for all pairs (s, χ) such that $s \in \mathbb{Z}$ with $n \leq s \leq \ell - n$,

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s,\chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}}$$

(under the inclusion i_p), with elementary factors $A_p(s,\chi) = \prod_{\mathfrak{q}|p} A_{\mathfrak{q}}(s,\chi)$ including a finite Euler product, Satake parameters $t_{\mathfrak{q},i}$, gaussian sums, the conductor of χ; the integral is a finite sum.

(ii) if $\mathrm{ord}_p \left((\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i}) p^{-n(n+1)} \right) = 0$ then the above distributions $\mu_{\mathcal{D},s}$ are bounded measures, we set $\mu_{\mathcal{D}} = \mu_{\mathcal{D},s^*}$ and the integral is defined for all continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =: \mathcal{Y}_p$.

Their Mellin transforms $\mathcal{L}_{\mu_{\mathcal{D},s}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D},s}$, $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \to \mathbb{C}_p$, give bounded p-adic analytic interpolation of the above L-values to on the $\mathbb{C}_p$-analytic group $\mathcal{Y}_p$; and these distributions are related by: $\displaystyle\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^*-s} \mu_{\mathcal{D},s^*}$, $X = \mathbb{Z}_p^*$, where $s^* = \ell - n$, $s_* = n$.

(iii) in the admissible case assume that $0 < h \leq s^* - s_* + 1 = \ell + 1 - 2n$, where $h = \mathrm{ord}_p \left((\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i}) p^{-n(n+1)} \right) > 0$, Then there exists an h–admissible measure $\mu_{\mathcal{D}}$ whose integrals $\int_{\mathbb{Z}_p^*} \chi x_p^s d\mu_{\mathcal{D}}$ are given by $i_p \left(A_p(s,\chi) \dfrac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}} \right) \in \mathbb{C}_p$ with $A_p(s,\chi)$ as in (i); their Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, belong to the type $o(\log x_p^h)$.
(iv) the functions $\mathcal{L}_{\mathcal{D}}$ are determined by (i)-(iii).

Remarks.
(a) Interpretation of s^*: the smallest of the "big slopes" of P_H
(b) Interpretation of $s_* - 1$: the biggest of the "small slopes" of P_H.

3 Proofs via the doubling method

Based on the *pull-back identity*, it is valid for many classical groups. In the Sp case it gives a *double integral representation* for $\mathcal{D}(\boldsymbol{f}, s, \chi)$ and its critical values at t with $k + t = \ell$, through a certain Eisenstein series $\mathfrak{E}^{2n}_{k,\chi}$ on the Hermitian space $\mathcal{H}_{2n}$ of degree $2n$ using an algebraic linear form $g \mapsto \mathcal{F}(g)$ applied to a function $\mathcal{H}_{t,\chi}(z, w)$ in a certain tensor product of arithmetical nearly holomorphic functions

$$\mathcal{D}(\boldsymbol{f}, t, \chi) = \mathcal{F}(\mathcal{H}_{t,\chi})$$

described in [5] for a holomorphic Siegel modular form $\mathbf{f}$ of weight ℓ for the congruence subgroup $\Gamma_0(N)$ in Sp_n which is a Hecke eigenform.

Special values of the standard L-function used in the proof are of the type $\mathcal{D}^{(M)}(\mathbf{f}, s)$ and attached to $\boldsymbol{f}$ and to twists $\mathcal{D}^{(M)}(\boldsymbol{f}, s, \chi)$ of the L-function by Dirichlet characters χ, where M denotes a common multiple of N and the conductor of χ, and the Euler factors at primes dividing M are removed.

3.1 A formula for $\mathcal{D}(\boldsymbol{f}, t, \chi) = \mathcal{F}(\mathcal{H}_{t,\chi})$

Is the following pull-back double integral representation:

$$\mathcal{F}(g) = \frac{\left\langle \langle \boldsymbol{f} \mid_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, g(*,*)\rangle^w_{\Gamma_0(N^2p)}, \boldsymbol{f} \mid_\ell \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} \right\rangle^z_{\Gamma_0(N^2p)}}{\langle \boldsymbol{f}, \boldsymbol{f}\rangle^2_{\Gamma_0(N^2p)}}$$

Here $g(z, w) = \mathcal{H}_{t,\chi}(-\bar{z}, w)$ is a function in the tensor product of certain spaces of automorphic forms

$$\mathcal{H}_{t,\chi} \in C^\infty M^\ell_n(\Gamma_0(M), \varphi)|_z \otimes_{\mathbb{C}} C^\infty M^\ell_n(\Gamma_0(M), \varphi)|_w,$$

obtained from the above Eisenstein series $\mathfrak{E}^{2n}_{k,\chi}$ applying

1. an arithmetical operator of higher twist,
2. a diagonal differential operator $\overset{\circ}{\mathfrak{D}}{}^\nu_{n,\alpha}$ with $\alpha = k + t$ acting on nearly holomorphic forms and preserving their arithmeticity.

In the Sp case the operator of higher twist, attached to the unipontent radical U of a parabolic P, is described through matrix Gauss sums in [4] .

3.2 Explicit pull-back identity in the Sp case

is based on the *"unfolding trick"*: twisting of the standard zeta function with a Dirichlet character χ equals to a series obtained from a summation of eigenvalues of $\mathbf{f}$, twisted by χ It is represented by the scalar product of a series obtained from a summation of Hecke operators applied to an eigenform for the appropriate Hecke algebra ((2.28) at p.1389 [5]) . Let S be a square free number, $p \mid S$ and

Let φ be a Dirichlet character $\bmod M > 1$, χ a Dirichlet character $\bmod N$, $N^2|M$, $\ell = k+\nu \in \mathbb{N}$ with $\chi(-1) = (-1)^k\varphi(-1)$ and $g \in \mathcal{S}_n^\ell(\Gamma_0(M), \bar\varphi)$, then in the Sp case one has the following

$$\left\langle \mathbf{f}, \mathfrak{E}_{2n}^{k,\nu}(\star, -\bar z, M, N, \varphi, \chi, \bar s)\right\rangle_{\Gamma_0(M)}$$
$$= \frac{\Omega_{\ell,\nu}(s)}{\mathfrak{L}(k+2s, \bar\chi\bar\varphi)}\chi(-1)^n(N^n)^{2k+\nu+2s-n-1}M^{\frac{n(n+1)}{2}-\frac{n\ell}{2}}$$
$$\times\, \mathcal{D}^{(M)}\left(\mathbf{f}|_\ell \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}, k+2s-n, \bar\chi\right)\mathbf{f}|_\ell \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} |U\left(\frac{M}{N^2}\right),$$

with the normalizing product $\mathfrak{L}(s,\psi) = L(s,\psi)\prod_{i=1}^{n} L(2s-2i, \psi^2)$, Gamma factor $\Gamma_n(s) = \pi^{\frac{n(n-1)}{2}}\prod_{i=1}^{n}\Gamma(s-\frac{n-1}{2})$ of degree n, Atkin's operator $U(L)$, and Hua's integral [24] $\Omega_{l,\nu}(s) = (-1)^{\frac{n\ell}{2}}2^{1+\frac{n(n+1)}{2}-2ns}\pi^{\frac{n(n+1)}{2}}\dfrac{\Gamma_n(l+s-\frac{n}{2})\Gamma_n(l+s-\frac{n+1}{2})}{\Gamma_n(k+s)\Gamma_n(k+s-\frac{n}{2})}$, providing a meromorphic continuation of the twisted L-functions to the whole complex plane (Corollary 3.3 in [5], p.1399).

3.3 Strategies to prove Main theorem by doubling method (Unitary case)

Let us follow the strategy developed for the symplectic group in [5] in the *ordinary case*, and in [15] in the *admissible case*.

To construct the admissible measure μ satisfying extensions of (i) and (ii) for Unitary group we follow four steps: following the technique of Amice-Vélu [1], μ is given by a certain sequence of distrbutions $\{\mu_j\}_{j\geq 0}$ on the ground p-adic space X

1. Construct a certain sequence of *modular (automorphic) distributions* $\{\mathcal{H}_j(z,w)\}_{j\geq 0}$ valued in a tower of arithmetic (nearly holomorphic) automorphic forms of levels $Cp^v (v \geq 1)$.
2. Apply a suitable algebraic linear form (represented by a double Petersson scalar product in (z,w)).
3. Check the admissibility properties (h-admissible measures). This is equivalent to proving congruences for the Fourier coefficients in $\mathcal{T}_1$, $\mathcal{T}_2 \in \Lambda$ (symmetric or hermitian matrices).
4. Prove that certain integrals of Dirichlet characters χ and arithmetical chracters χx^j, $j = s^* - t$ coincide with the algebraic normalized special values of the standard zeta function $\mathcal{D}^{alg}(\boldsymbol{f}, t, \chi)$ twisted with χ.

4 Existence of h-admissible measures

of Amice-Vélu-type gives an unbounded p-adic analytic interpolation of the L-values of growth $\log_p^h(\cdot)$, using the Mellin transform of the constructed measures. This condition says that the product $\prod_{i=1}^{n} t_{\mathfrak{p},i}$ is nonzero and divisible by a certain power of p in $\mathcal{O}$:

$$\mathrm{ord}_p\left(\prod_{\mathfrak{q}|p}\left(\prod_{i=1}^{n} t_{\mathfrak{q},i}\right)p^{-n(n+1)}\right) = h.$$

We use an *easy condition of admissibility* of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p^*$ with values in the semigroup algebra $\mathcal{O}[[q]] = \mathcal{O}[[q^H]]_{H\in\Lambda(\mathcal{O})^+}$ as in Theorem 4.8 of [10]. It suffices to check *congruences* of the type (with $\varkappa = 4$)

$$U^{\varkappa v}\Big(\sum_{j'=0}^{j}\binom{j}{j'}(-a_p^0)^{j-j'}\Phi_{j'}(a+(p^v)\Big) \in Cp^{vj}\mathcal{O}[[q]]$$

for all $j = 0,1,\ldots,\varkappa h-1$. Here $s = s^* - j'$, $\Phi_{j'}(a+(p^v))$ is a certain automorphic-valued distribution in nearly holomorphic arithmetical functions $\mathcal{H}_{t,\chi}(z,w)$ We use a general sufficient condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p^*$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [10].

4.1 Proof of the Main Theorem (iii): admissible case

Using a certain double Petersson product $\mathcal{F}(\mathcal{H}_{t,\chi})$ for $\mathcal{D}^{alg}(t,\mathbf{f},\chi)$ of $\mathcal{H}_{t,\chi}$ times an elementary factor and a power of a fixed Satake parameter written as $\gamma(L)$, with $L = p^v$. Using an eigenfunction $\mathbf{f}_0$ of Atkin's operator $U(p)$ of eigenvalue $\alpha_{\mathbf{f}}$ on $\mathbf{f}_0$ and a certain double Petersson product for $g(z,w) = \mathcal{H}_{t,\chi}(-\bar{z},w)$ one has $\mathcal{F}(\mathcal{H}_{t,\chi}) =$

$$= \frac{\left\langle \langle \mathbf{f}_0 \,|_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, g(*,*)\rangle^w_{\Gamma_0(N^2p)}, \mathbf{f}_0 \,|_\ell \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}\right\rangle^z_{\Gamma_0(N^2p)}}{\langle \mathbf{f}_0, \mathbf{f}_0\rangle^2_{\Gamma_0(N^2p)}}$$

Moreover, applying Atkin's operator $U(p^v)$ gives

$$\mathcal{D}^{alg}(t,\mathbf{f},\chi) = \alpha_{\mathbf{f}}^{-v}\times \frac{\left\langle \langle \mathbf{f}_0 \,|_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, U(p^v)\big(g(*,*)\rangle\big)^w_{\Gamma_0(N^2p)}, \mathbf{f}_0 \,|_\ell \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}\right\rangle^z_{\Gamma_0(N^2p)}}{\langle \mathbf{f}_0, \mathbf{f}_0\rangle^2_{\Gamma_0(N^2p)}}$$

4.1.1 Modification of the proof in the admissible case

instead of Kummer congruences, to *estimate p-adically the integrals of test functions*: $L = p^v$: $\int_{a+(L)}(x-a)^j d\mathcal{D}^{alg} := \sum_{j'=0}^{j}\binom{j}{j'}(-a)^{j-j'}\int_{a+(L)} x^{j'}d\mathcal{D}^{alg}$, using the *orthogonality of characters* and the sequence of zeta distributions $\int_{a+(L)} x^j d\mathcal{D}^{alg} = \frac{1}{\sharp(\mathcal{O}/L\mathcal{O})^\times}\sum_{\chi \bmod L}\chi^{-1}(a)\int_X \chi(x)x^j d\mathcal{D}^{alg}$, $\int_X \chi d\mathcal{D}^{alg}_{s^*-j} = \mathcal{D}^{alg}(s^*-j,f,\chi) =: \int_X \chi(x)d\mathcal{D}^{alg}_j$.

4.2 Defining automorphic distributions $\{\mathcal{H}_j\}$ through $\mathcal{H}_{L,\chi}^{(t)}$

For $\ell = k + \nu, k = n + t, t \geq 1$, we use the same sequence of functions $\mathcal{H}_{L,\chi}^{(t)}$ as in [5], defined as follows:
If $\chi \neq 1$, then

$$\mathcal{H}_{L,\chi}^{(t)}(z,w) = \mathfrak{L}(k+2s,\varphi\chi)\mathfrak{E}_{2n}^{k,\nu}(w,z,R^2N^2\frac{S}{R_0},RN,\varphi,\chi,s)$$
$$|_\ell^z\, U(L^2)\, |_\ell^w\, U(L^2)\, |_\ell^z \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} |_\ell^w \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}.$$

If $\chi = 1$, then

$$\mathcal{H}_{L,\chi}^{(t)}(z,w) = \mathfrak{L}(k+2s,\varphi\chi')\sum_{i=0}^{n}(-1)^i p^{\frac{i(i-1)}{2}} p^{-in}$$
$$\times\, \mathfrak{E}_{2n}^{k,\nu}(w,z,(R'p)^2N^2\frac{S}{R'_0},R'N,\varphi,\chi',s,i)$$
$$|_\ell^z\, U(L^2)\, |_\ell^w\, U(L^2)\, |_\ell^z \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} |_\ell^w \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}.$$

Defining the functions $\mathcal{H}_{(a,L)}(z,w)$ through the Fourier expansions
The function $\mathcal{H}_{(a,L)}(z,w)$ has the Fourier expansion of the form:

$$\mathcal{H}_{(a,L)}(z,w) = \sum_{\mathcal{T}_1,\mathcal{T}_4\in\Lambda_n^+} \alpha_{a,L}(\mathcal{T}_1,\mathcal{T}_4)\cdot\exp\left(\frac{2\pi i}{N^2p}\mathrm{tr}(\mathcal{T}_1 z + \mathcal{T}_4 w)\right),$$

$$\sum_{\mathcal{T}(\mathcal{T}_2),G,b} \mathfrak{P}_{n,k}^{\nu}(T)\cdot G_n(2T_2,N,\chi)\cdot(\varphi\chi)^2(\det G)\cdot\det(2T[G^{-1}])^{k-\frac{2n+1}{2}}$$
$$\cdot\,(\varphi\chi)(b)\cdot b^{-k}\cdot d(b,T[G^{-1}])\cdot\frac{p}{L}\sum_{\chi}\chi(aN^n)c_\chi^{t-1}(\overline{\varphi\chi})_0(c_\chi)\cdot G(\bar\chi)$$
$$\cdot\,\bar\chi(\det(2T_2))\cdot\chi(\det(G^2))\cdot b)\cdot(1-\overline{(\varphi\chi^0\chi)}_0(p)p^{t-1})\cdot L(t,\epsilon_{T[G^{-1}]}\varphi\chi),$$

where the quadratic character ϵ_T defined as $\epsilon_T(*) := \left(\frac{(-1)^n\det(2T)}{*}\right)$ and $D(T) = \{G\in M_n(\mathbb{Z}^*)|T[G^{-1}]\in\Lambda_n\}, b|\det(2T)$.

5 Defining an algebraic linear form $g \mapsto \mathcal{F}(g)$

Let $f \in S_n^\ell(\Gamma_0(N),\bar\varphi)$ and a function $g(z,w)$ which as a modular form of z and w belongs to $M_n^\ell(\Gamma^0(N^2p),\varphi)$, we consider the following $\mathbb{C}$-valued function:

$$\mathcal{F}(g) = \frac{\left\langle \langle f\,|_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, g(*,*)\rangle^w_{\Gamma_0(N^2p)}, f\,|_\ell \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}\right\rangle^z_{\Gamma_0(N^2p)}}{\langle f,f\rangle^2_{\Gamma_0(N^2p)}}.$$

The pull-back identity gives us our critical values as values of $\mathcal{F}(g)$:

$$\mathcal{F}(\mathcal{H}^{(t)}_{L,\chi}) = \langle f, f\rangle^{-1} \cdot \Omega_{\ell,\nu}(0) \cdot (N^2p)^{\frac{n(n+1)}{2}-\frac{n\ell}{2}}\chi(-1)^n(-1)^{ln} \\ \cdot (Nc_\chi)^{n(\ell+t-1)}\alpha_0(pL^4c_\chi^{-2}) \cdot E_p(t,\chi) \cdot \bar\chi(\frac{p}{(p,c_\chi)})^n \cdot D^{(Np)}(f,t,\bar\chi)$$

for any character χ whose conductor is a power of p, where $E_p(s,\psi) = \prod_{j=1}^{n} \frac{(1-\psi(p)\alpha_j(p)^{-1}p^{s-1})}{(1-\bar\psi(p)\alpha_j(p)p^{-s})}$ is a modified p-Euler factor in accordance with the motivic formula in [11].

5.1 Congruences between the coefficients of the Hermitian modular forms

In order to integrate any locally-analytic function on X, it suffices to check the following binomial congruences for the coefficients of the Hermitian modular form

$$\mathcal{H}_{s^*-j,\chi} = \sum_{\mathcal{T}_1,\mathcal{T}_2} v(\mathcal{T}_1,\mathcal{T}_2,s^*-j,\chi)q_z^{\mathcal{T}_1}q_w^{\mathcal{T}_2} :$$

for $v \gg 0$, and a constant C

$$\frac{1}{\sharp(\mathcal{O}/M\mathcal{O})^\times}\sum_{j'=0}^{j}\binom{j}{j'}(-a)^{j-j'}\sum_{\chi \bmod M}\chi^{-1}(a)v(p^v\mathcal{T}_1,p^v\mathcal{T}_2,s^*-j,\chi)q_z^{\mathcal{T}_1}q_w^{\mathcal{T}_2}i \\ \in Cp^{vj}\mathcal{O}[[q_z,q_w]] \quad \text{(This is a quasimodular form if } j'\neq s^*)$$

The resulting measure $\mu_\mathcal{D}$ allows to integrate all continuous characters in $\mathcal{Y}_p = \mathrm{Hom}_{cont}(X,\mathbb{C}_p^*)$, including Hecke characters, as they are always locally analytic.

Its p-adic Mellin transform $\mathcal{L}_{\mu_\mathcal{D}}$ is an analytic function on $\mathcal{Y}_p$ of the logarithmic growth $\mathcal{O}(\log^h)$, $h = \mathrm{ord}_p(\alpha)$.

5.2 Proof of the main congruences

Thus the double Petersson product in $\ell_\mathbf{f}$ can be expressed through the Fourier coeffcients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coeffcients: $\ell_{\mathcal{T}_i} : h \mapsto b_{\mathcal{T}_i}(i=1,\dots,n)$. It follows that $\ell_\mathbf{f}(h) = \sum_i \gamma_i b_{\mathcal{T}_i}$, where $\gamma_i \in k$. Using the expression for $\ell_f(h_j) = \sum_i \gamma_{i,j}b_{j,\mathcal{T}_i}$, the above congruences reduce to

$$\sum_{i,j}\gamma_{i,j}\beta_j b_{j,\mathcal{T}_i} \equiv 0 \mod p^N.$$

The last congruence is done by an elementary check on the Fourier coefficients $b_{j,\mathcal{T}_i}$.

The abstract Kummer congruences are checked for a family of test elements.

In the admissible case it suffices to check *binomial congruences* for the Fourier coefficients as above in place of Kummer congruences.

5.3 Proving the main congruences as in [15], [10]

Denote $L = p^v$, $\gamma(L) = \langle \boldsymbol{f}_0, \boldsymbol{f}_0 \rangle \alpha_0 (pL^4)^{-1} \Omega_{\ell,\nu}^{-1}(0)(N^2 p)^{\frac{n\ell}{2} - \frac{n(n+2)}{2}} (-1)^n (-1)^{\ell n} N^{n(1-j-\ell)}$,

$$A := \gamma(L) \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v(\mathfrak{T}_1, \mathfrak{T}_2, j, \chi)$$

where $v(L\mathfrak{T}_1, \mathfrak{T}_2, j, \chi)$ are Fourier coefficients of $\mathcal{H}_{L,\chi}^{(j)}$. These summations can be put into the integrals and composed as the derivative of a product:

$$\begin{aligned}
&\int_{a+(L)} (x_p - a_p)^r d\mu \\
&= \int_{x \equiv a \mod L} \sum_{i=0}^{|M|} \mu_i \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{(j+i+1)!}{(j+1)!} x^{j+1} d\mu \\
&= \int_{x \equiv a \mod L} \sum_{i=0}^{|M|} \mu_i x^{-1} \cdot \frac{\partial^i}{\partial x^i} \left(x^{i+1} (x-a)^r \right) d\mu
\end{aligned}$$

We use the method of V. Q. My which can be explained by the following lemma:

Lemma 4 (Lemma 5.2 in [32], page 158)**.** *Suppose that h and q are natural numbers, $h > q$, and $d \equiv -Cd'a \mod m$. Then the number*

$$B_q = \sum_{j=0}^{h} \binom{h}{j} (-a)^{h-j} (-C)^{h-j} d'^{h-j} d^{j-i} j^q \frac{\Gamma(j+1)}{\Gamma(j+1+i)}$$

is divisible by m^{h-i-q}.

Using the orthogonality relations of character χ and the congruence $x \equiv a \mod L$ which gives the congruence $\frac{\partial^i}{\partial x^i} \left(x^{i+1} (x-a)^r \right) \equiv 0 \mod L^{r-i}$ then we have proof for the main congruences. ∎

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