

A SURVEY ON THE GLOBAL GAN-GROSS-PRASAD CONJECTURE FOR  
FOURIER-JACOBI CASE

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ABSTRACT. The global Gan-Gross-Prasad (GGP) conjecture predicts that the non-vanishing of certain periods is equivalent to the non-vanishing of certain central value of some  $L$ -function. There are two types of GGP conjectures : Bessel case, Fourier-Jacobi case. In 2015, Hang Xue proved the Fourier-Jacobi GGP conjecture for skew-hermitian case on the same rank group. But his result is under certain local restriction to apply relative trace formula. We suggest a way to prove one direction of the general Fourier-Jacobi case for skew-hermitian unitary group without such local restrictions. This survey article is based on the ongoing joint work with Hiraku Atobe.

1. Fourier-Jacobi period

Let  $E/F$  be a quadratic extension of number fields with adele rings  $\mathbb{A}_E$  and  $\mathbb{A}_F$  respectively. We denote the nontrivial automorphism of  $E$  fixing  $F$  by  $x \rightarrow \bar{x}$ . Let  $\omega$  be the non-trivial quadratic character associated to  $F^\times \backslash \mathbb{A}_F^\times$  by the global class field theory and fix a character  $\mu : E^\times \backslash \mathbb{A}_E^\times$  such that  $\mu|_{\mathbb{A}_F^\times} = \omega$ . Sometimes, we view  $\mu$  as a character of  $GL_n(\mathbb{A}_E)$  and in that case, it does mean  $\mu \circ \det$ . We also fix a nontrivial character  $\psi$  of  $E \backslash \mathbb{A}_E$ . If  $v$  is a place of  $F$ , we write  $E_v = E \otimes F_v$ . Let  $W_m \subset W_n$  be  $m$  and  $n$ -dimensional skew-Hermitian spaces over  $E$  such that  $W_n = X \oplus W_m \oplus X^*$  where  $X \oplus X^*$  is the direct sum of  $r$  hyperbolic planes and the restriction of hermitian form of  $W_n$  to  $W_m$  is non-degenerate.

Let  $G_n, G_m$  be the isometry group of  $W_n, W_m$  respectively and regard  $G_m$  as a subgroup of  $G_n$  which acts trivially on the orthogonal complement of  $W_m$  in  $W_n$ . We fix a complete flag of  $X$  and let  $\overline{P_{n,r}}$  the parabolic subgroup of  $G_n$  which stabilize this flag, with the unipotent radical  $N_{n,r}$ . Then the group  $G_m$  acts on  $N_{n,r}$  through conjugation. Put  $H = N_{n,r} \rtimes G_m$ . There is an  $H(F)$ -invariant automorphic Weil representation  $\nu_{\psi^{-1}, \mu^{-1}, W_m}$  of  $H(\mathbb{A}_F)$  realized on Schwartz space  $\mathcal{S}$ . For each  $f \in \mathcal{S}$ , we can define a certain theta series  $\Theta_{\psi^{-1}, \mu^{-1}}(h, f)$  defined on  $H(\mathbb{A}_F)$ .

Let  $\pi_1, \pi_2$  be two irreducible cuspidal automorphic representation of  $G_n(\mathbb{A}_F)$  and  $G_m(\mathbb{A}_F)$  respectively. We regard  $H$  as a subgroup of  $G_n$  through the map  $(n, g) \rightarrow ng$ . For  $\varphi_1 \in \pi_1, \varphi_2 \in \pi_2, f \in \nu_{\psi^{-1}, \mu^{-1}, W_m}$ , we define its Fourier-Jacobi period to be the integral as

$$\mathcal{FJ}_{\psi, \mu}(\varphi_1, \varphi_2, f) := \int_{[N_{n,r} \rtimes G_m]} \varphi_1(ng)\varphi_2(g)\Theta_{\psi^{-1}, \mu^{-1}}((n, g), f) dndg,$$

where  $[N_{n,r} \rtimes G_m] = N_{n,r}(F) \rtimes G_m(F) \backslash N_{n,r}(\mathbb{A}_F) \rtimes G_m(\mathbb{A}_F)$ .

2. Automorphic forms

For a connected reductive algebraic group  $G$  over  $F$ , we fix a minimal  $F$ -parabolic subgroup  $P_0$  of  $G$  with a Levi decomposition  $P_0 = M_0U_0$  and a maximal compact subgroup  $K = \prod_v K_v$

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of  $G(\mathbb{A}_F)$  which satisfies

$$G(\mathbb{A}_F) = P_0(\mathbb{A}_F)K, \quad P(\mathbb{A}_F) \cap K = (M(\mathbb{A}_F) \cap K)(U(\mathbb{A}_F) \cap K)$$

and  $M(\mathbb{A}_F) \cap K$  is still maximal compact in  $M(\mathbb{A}_F)$  for every standard parabolic subgroup  $P = UM$  of  $G$  where  $M_0 \subset M$ . (see [10, I.1.4]) Note that the Levi factor  $M_0$  is the centralizer of a maximal split torus  $T_0$ . Throughout the rest the paper,  $P$  always denote a standard subgroup of  $G$  unless mentioned.

Let  $\mathcal{A}_P(G)$  be the space of automorphic forms on  $U(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$ . i.e., smooth,  $K$ -finite and  $\mathfrak{z}$ -finite functions on  $U(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$  of moderate growth, where  $\mathfrak{z}$  is the center of the universal enveloping algebra of the complexified Lie algebra of the product of the archimedean localization of  $G(\mathbb{A}_F)$ . When  $P = G$ , we simply write  $\mathcal{A}(G)$  for  $\mathcal{A}_G(G)$ . For a cuspidal automorphic representation  $\rho$  of  $M(\mathbb{A}_F)$ , we write  $\mathcal{A}_P^\rho(G)$  for the subspace of functions  $\phi \in \mathcal{A}_P(G)$  such that for all  $k \in K$ , the function  $m \rightarrow |\delta_P(m)|^{-1} \cdot \phi(mk)$  belongs to the space of  $\rho$ . (Here,  $\rho_P$  is the modulus function of  $P(\mathbb{A}_F)$ .) (see [10, I.2.17])

We extend the definition of automorphic forms from reductive groups to special non-reductive groups. Let  $N$  be a unipotent group over  $F$  which admits a  $G$ -action and denote this action by  $\sigma : G \rightarrow \text{Aut}(N)$ . Using  $\sigma$ , we can consider the semi-direct product  $N \rtimes G$  and define automorphic forms on  $N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F)$  as follows.

For a function  $\varphi : N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F) \rightarrow \mathbb{C}$  and arbitrary  $n \in N(\mathbb{A}_F)$ , denote  $\varphi_n : G(\mathbb{A}_F) \rightarrow \mathbb{C}$  by  $\varphi_n(g) := \varphi(n, g)$ . We say that  $\varphi$  is an automorphic form on  $N(F) \rtimes G(F) \backslash N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F)$  if

- $\varphi((\delta, \gamma) \cdot (n, g)) = \varphi(n, g)$  for  $(\delta, \gamma) \in N(F) \rtimes G(F)$
- $\varphi$  is smooth
- $\varphi_n$  is right  $K$ -finite for a maximal compact subgroup  $K$  of  $G(\mathbb{A}_F)$  for any  $n \in N(\mathbb{A}_F)$
- $\varphi_n$  is  $\mathfrak{z}$ -finite function for any  $n \in N(\mathbb{A}_F)$
- $\varphi_n$  is of moderate growth for any  $n \in N(\mathbb{A}_F)$

We denote by  $\mathcal{A}(N \rtimes G)$  the space of automorphic forms on  $N(F) \rtimes G(F) \backslash N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F)$ . Note that if  $N$  is the trivial group  $\mathbf{1}$ , then  $\mathcal{A}(\mathbf{1} \rtimes G)$  equals  $\mathcal{A}(G)$ . For  $\varphi \in \mathcal{A}(N \rtimes G)$ , define  $\phi_P : N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F) \rightarrow \mathbb{C}$  by

$$\phi_P(n, g) := \int_{U_P(\mathbb{A}_F)} \varphi(n, ug) du \text{ for } (n, g) \in N(\mathbb{A}_F) \rtimes G(\mathbb{A}_F)$$

and define  $\varphi^P : G(\mathbb{A}_F) \rightarrow \mathbb{C}$  as

$$\varphi^P(g) := \int_{N(\mathbb{A}_F)} \varphi_P(n, g) dn.$$

**Proposition 2.1.** *For  $\varphi \in \mathcal{A}(N \rtimes G)$ ,  $\varphi^P \in \mathcal{A}_P(G)$  for any standard parabolic subgroup  $P$  of  $G$ .*

*Remark 2.2.* For  $\phi \in \mathcal{A}(G)$ , if we regard  $\phi \in \mathcal{A}(\mathbf{1} \rtimes G)$ , then  $\phi_P = \phi^P$ . Thus  $\phi \rightarrow \phi_P$  sends  $\mathcal{A}(G)$  to  $\mathcal{A}_P(G)$ .

### 3. Mixed truncation

To explain mixed truncation, we first recall some notation regarding Arthur truncation. For more explanation on the notation here, see [1, Sec. 1].

For a connected reductive algebraic group  $G$  over  $F$ , we fix a minimal  $F$ -parabolic subgroup  $P_0$  of  $G$  with a Levi decomposition  $P_0 = U_0M_0$ . Write  $X(G)$  for the  $F$ -rational characters of  $G$ . Let  $\mathfrak{a}_0^*$  be the  $\mathbb{R}$ -vector space spanned by lattice  $X(T_0)$  and  $\mathfrak{a}_0 = \text{Hom}(X(T_0) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R})$  its dual space. The canonical pairing on  $\mathfrak{a}_0^* \times \mathfrak{a}_0$  is denoted by  $\langle \cdot, \cdot \rangle$ . Let  $\Delta_0$  and  $\Delta_0^\vee$  be the sets of simple roots and simple coroots in  $\mathfrak{a}_0^*$  and  $\mathfrak{a}_0$  respectively. Write  $\hat{\Delta}_0^\vee$  and  $\hat{\Delta}_0$  for the dual basis of  $\Delta_0$  and  $\Delta_0^\vee$  respectively. (In other words,  $\hat{\Delta}_0^\vee$  and  $\hat{\Delta}_0$  are set of coweight and weight respectively.) For a standard parabolic subgroup  $P = UM$  of  $G$ , write  $T$  for a maximal split torus in the center of  $M$  and  $\mathfrak{a}_P^* = X(M) \otimes_{\mathbb{Z}} \mathbb{R}$  and  $\mathfrak{a}_P$  for its dual space.

For a pair of standard parabolic subgroups  $Q \subset P$  of  $G$ , there is a canonical injection  $\mathfrak{a}_P \rightarrow \mathfrak{a}_Q$  and surjection  $\mathfrak{a}_Q \rightarrow \mathfrak{a}_P$  induced by two inclusion maps  $M_Q \hookrightarrow M_P$  and  $T_P \hookrightarrow T_Q$ . So we have a canonical decomposition

$$\mathfrak{a}_Q = \mathfrak{a}_Q^P \oplus \mathfrak{a}_P, \quad \mathfrak{a}_Q^* = (\mathfrak{a}_Q^P)^* \oplus \mathfrak{a}_P^*$$

In particular, if we take  $Q = P_0$ , we have a decomposition

$$\mathfrak{a}_0 = \mathfrak{a}_0^P \oplus \mathfrak{a}_P, \quad \mathfrak{a}_0^* = (\mathfrak{a}_0^P)^* \oplus \mathfrak{a}_P^*$$

for all standard subgroup  $P$ .

For every standard subgroup  $P$ , let  $\Delta_P \subset \Delta_0$  be the set of non-trivial restriction of simple roots to  $\mathfrak{a}_P$ . For any pair of standard subgroups  $Q \subset P$ , denote by  $\Delta_Q^P$  the subset of  $\Delta_Q$  appearing in the root decomposition of the Lie algebra of unipotent radical  $U_Q \cap M_P$ . Then for  $H \in \mathfrak{a}_P$ ,  $\langle \alpha, H \rangle = 0$  for all  $\alpha \in \Delta_Q^P$  and so  $\Delta_Q^P \subset (\mathfrak{a}_Q^P)^*$ . Note that  $\Delta_P^P = \Delta_P$ . For any  $\alpha \in \Delta_Q^P$ , there is a  $\tilde{\alpha} \in \Delta_0$  whose restriction to  $\mathfrak{a}_Q^P$  is  $\alpha$ . Write  $\alpha^\vee$  for the projection of  $\tilde{\alpha}^\vee$  to  $\mathfrak{a}_Q^P$ . Define

$$(\Delta_Q^P)^\vee = \{\alpha^\vee \mid \alpha \in \Delta_Q^P\}.$$

Define  $(\hat{\Delta}^\vee)_Q^P \subset (\mathfrak{a}_Q^P)^*$  and  $\hat{\Delta}_Q^P \subset \mathfrak{a}_Q^P$  to be the dual basis of  $\Delta_Q^P$  and  $(\Delta_Q^P)^\vee$  respectively. We simply write  $\hat{\Delta}_P^\vee$  for  $(\hat{\Delta}^\vee)_P^P$  and  $\hat{\Delta}_P$  for  $\hat{\Delta}_P^P$ , respectively.

Let  $\tau_Q^P$  be the characteristic function of the subset

$$\{H \in \mathfrak{a}_0 : \langle \alpha, H \rangle > 0 \text{ for all } \alpha \in \Delta_Q^P\} \subset \mathfrak{a}_0$$

and let  $\hat{\tau}_Q^P$  be the characteristic function of the subset

$$\{H \in \mathfrak{a}_0 : \langle \varpi, H \rangle > 0 \text{ for all } \varpi \in \hat{\Delta}_Q^P\} \subset \mathfrak{a}_0.$$

Note that these two functions depends only on the projection of  $\mathfrak{a}_0$  to  $\mathfrak{a}_Q^P$ . We write  $\tau_P$  and  $\hat{\tau}_P$  for  $\tau_P^P$  and  $\hat{\tau}_P^P$ , respectively.

For each parabolic subgroup  $P = UM$ , we have height map

$$H_P : G(\mathbb{A}_F) \rightarrow \mathfrak{a}_P$$

characterized by the following properties : (see [1, page 917])

- $|\chi|(m) = e^{\langle \chi, H_P(m) \rangle}$  for all  $\chi \in X(M)$  and  $m \in M(\mathbb{A}_F)$
- $H_P(nmk) = H_P(m)$  for all  $n \in U(\mathbb{A}_F), m \in M(\mathbb{A}_F), k \in K$ .

The restriction of  $H_P$  on  $M(\mathbb{A}_F)$  is surjective homomorphism. Denote the kernel of  $H_P|_{M(\mathbb{A}_F)}$  by  $M(\mathbb{A}_F)^1$  and the connected component of 1 in  $T(\mathbb{R})$  by  $T(\mathbb{R})^0$ . Then  $M(\mathbb{A}_F)$  is the direct product of normal subgroup  $M(\mathbb{A}_F)^1$  with  $T(\mathbb{R})^0$  and  $H_P$  gives an isomorphism between  $T(\mathbb{R})^0$  and  $\mathfrak{a}_P$ . Denote the inverse of this map by  $X \rightarrow e^X$ . We simply write  $H(g)$  for  $H_{P_0}(g)$ . Note that  $H_P(g)$  is the projection of  $H(g)$  onto  $\mathfrak{a}_P$ .

Let  $T \in \mathfrak{a}_0$ . For  $\phi, \phi' \in \mathcal{A}(N \rtimes G)$ , we define a mixed truncation by

$$\Lambda_m^T(\phi \otimes \phi')(g) = \sum_P (-1)^{\dim \mathfrak{a}_P^*} \sum_{\gamma \in P(F) \backslash G(F)} \left( \int_{N(F) \backslash N(\mathbb{A}_F)} \phi_P(n, \gamma g) \phi'_P(n, \gamma g) dn \right) \hat{\tau}_P(H(\gamma g) - T)$$

for  $g \in G$ . More generally, we define a partial mixed truncation by

$$\Lambda_m^{T,P}(\phi \otimes \phi')(g) = \sum_{Q \subset P} (-1)^{\dim \mathfrak{a}_Q^*} \sum_{\delta \in Q(F) \backslash P(F)} \left( \int_{N(F) \backslash N(\mathbb{A}_F)} \phi_Q(n, \delta g) \phi'_Q(n, \delta g) dn \right) \hat{\tau}_Q^P(H(\delta g) - T)$$

for  $\phi, \phi' \in \mathcal{A}(N \rtimes G)$ .

**Lemma 3.1.** *Let  $\phi, \phi' \in \mathcal{A}(N \rtimes G)$ . Then  $\Lambda_m^T(\phi \otimes \phi')$  is rapidly decreasing on  $G(F) \backslash G(\mathbb{A}_F)^1$ .*

For  $(\phi, \phi') \in \mathcal{A}(N \rtimes G)^2$  and  $\phi'' \in \mathcal{A}(G)$ , we consider the following integral

$$(3.1) \quad \int_{G(F) \backslash G(\mathbb{A}_F)^1} \Lambda_m^T(\phi \otimes \phi')(g) \phi''(g) dg.$$

Thanks to Lemma 3.1, this integral converges.

Write  $\rho_0$  for half the sum of positive roots in  $\mathfrak{a}_0^*$  and denote by  $\rho_P$  the projection of  $\rho_0$  to  $\mathfrak{a}_P^*$ . Recall that  $e^{2(\rho_P, H_P(p))} = \delta_P(p)$  for  $p \in P(\mathbb{A}_F)$ . It is known that an automorphic form  $\phi \in \mathcal{A}_P(G)$  admits a finite decomposition

$$\phi(ue^X mk) = \sum_i Q_i(X) \phi_i(mk) e^{\langle \lambda_i + \rho_P, X \rangle}$$

for  $u \in U(\mathbb{A}_F)$ ,  $X \in \mathfrak{a}_P$ ,  $m \in M(\mathbb{A}_F)^1$  and  $k \in K$ , where  $\lambda_i \in \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}$ ,  $Q_i \in \mathbb{C}[\mathfrak{a}_P]$  and  $\phi_i \in \mathcal{A}_P(G)$  satisfies  $\phi_i(e^X g) = \phi_i(g)$  for  $X \in \mathfrak{a}_P$  and  $g \in G$ . (see [10, I.3.2]) We denote the finite set of exponents  $\lambda_i$  appearing in this decomposition by  $\mathcal{E}_P(\phi)$ .

**Proposition 3.2.** *Integral in (3.1) is a function of the form  $\sum_{\lambda} p_{\lambda}(T) e^{\langle \lambda, T \rangle}$ , where  $p_{\lambda}$  is a polynomial in  $T$  and  $\lambda$  can be taken from the set*

$$\bigcup_P \{ \lambda + \lambda' + \lambda'' + \rho_P \mid \lambda \in \mathcal{E}_P(\phi^P), \lambda' \in \mathcal{E}_P(\phi'^P), \lambda'' \in \mathcal{E}_P(\phi''_P) \ (i = 1, 2, 3) \}$$

**Definition 3.3.** Let  $\mathcal{A}_0(N \rtimes G)$  be the subspace of triplets  $(\phi, \phi', \phi'') \in \mathcal{A}(N \rtimes G)^2 \times \mathcal{A}(G)$  such that the polynomial corresponding to the zero exponent of (3.1) is constant. For  $(\phi, \phi', \phi'') \in \mathcal{A}_0(N \rtimes G)$ , we define its regularized period  $\mathcal{P}(\phi, \phi', \phi'')$  as its value  $p_0(T)$ . We also write

$$\mathcal{P}(\phi, \phi', \phi'') = \int_{G(F) \backslash G(\mathbb{A}_F)^1}^* \int_{N(F) \backslash N(\mathbb{A}_F)}^* \phi(n, g) \phi'(n, g) \phi''(g) dg.$$

Let  $\mathcal{A}(N \rtimes G)^*$  be the space of all triplets  $(\phi, \phi', \phi'') \in \mathcal{A}(N \rtimes G)^2 \times \mathcal{A}(G)$  such that

$$\langle \lambda + \lambda' + \lambda'' + \rho_P, \omega^\vee \rangle \neq 0 \quad (\omega^\vee \in (\hat{\Delta}^\vee)_P, \lambda \in \mathcal{E}_P(\phi^P), \lambda' \in \mathcal{E}_P(\phi'^P), \lambda'' \in \mathcal{E}_P(\phi''_P))$$

for all parabolic subgroups  $P$  of  $G$ . If  $(\phi, \phi', \phi'') \in \mathcal{A}(N \rtimes G)^*$ , then the  $\#$ -integral

$$\mathcal{P}_P^T(\phi, \phi', \phi'') = \int_{P(F) \backslash G(\mathbb{A}_F)^1}^{\#} \Lambda_m^{T,P}(\phi \otimes \phi')(g) \phi''_P(g) \tau_P(H(g) - T) dg$$

is defined as the triple integral

$$\int_K \int_{M(F) \backslash M(\mathbb{A}_F)^1} \int_{\mathfrak{a}_P} \Lambda_m^{T,P}(\phi \otimes \phi')(e^X mk) \phi''_P(e^X mk) e^{-2(\rho_P, X)} \tau_P(X - T) dX dmdk.$$

**Proposition 3.4.** *The following statements hold.*

- (i)  $\mathcal{A}(N \rtimes G)^* \subset \mathcal{A}_0(N \rtimes G)$
- (ii) If  $(\phi, \phi', \phi'') \in \mathcal{A}(N \rtimes G)^*$ , then

$$\mathcal{P}(\phi, \phi', \phi'') = \sum_P \mathcal{P}_P^T(\phi, \phi', \phi'')$$

It says that  $\sum_P \mathcal{P}_P^T$  is independent of  $T$ .

- (iii) The regularized period is a  $G(\mathbb{A})^1$ -invariant linear functional on  $\mathcal{A}(N \rtimes G)^*$ .

Let  $\mathcal{A}(N \rtimes G)^{**}$  be the subspace of all triplets  $(\phi, \phi', \phi'') \in \mathcal{A}(N \rtimes G)^2 \times \mathcal{A}(G)$  such that

$$\langle \lambda + \lambda' + \lambda'' + \rho_P, \omega^\vee \rangle \neq 0 \quad (\omega^\vee \in (\hat{\Delta}^\vee)_Q^P, \lambda \in \mathcal{E}_Q(\phi^Q), \lambda' \in \mathcal{E}_Q(\phi'^Q), \lambda'' \in \mathcal{E}_Q(\phi''^Q))$$

for all pairs of parabolic subgroups  $Q \subset P$  of  $G$ . Clearly  $\mathcal{A}(N \rtimes G)^{**} \subset \mathcal{A}(N \rtimes G)^*$ .

If  $(\phi, \phi', \phi'') \in \mathcal{A}(N \rtimes G)^{**}$ , then the regularized integral

$$\begin{aligned} & \int_{P(F) \backslash G(\mathbb{A})^1}^* \left( \int_{N(F) \backslash N(\mathbb{A}_F)} \phi_P(n, g) \phi'_P(n, g) dn \right) \phi''_P(g) dg \\ = & \int_K \int_{M(F) \backslash M(\mathbb{A}_F)}^* \int_{\mathfrak{a}_P}^\# \left( \int_{N(F) \backslash N(\mathbb{A}_F)} \phi_P(n, e^X mk) \phi'_P(n, e^X mk) dn \right) \phi''_P(e^X mk) \hat{\tau}_P(X-T) e^{-2(\rho_P \cdot X)} dX dmdk \end{aligned}$$

is well defined for every  $P$ .

**Proposition 3.5.** *If  $(\phi, \phi', \phi'') \in \mathcal{A}(N \rtimes G)^{**}$ , then*

$$\begin{aligned} & \int_{G(F) \backslash G(\mathbb{A}_F)^1} \Lambda_m^T(\phi \otimes \phi')(g) \phi''(g) dg \\ = & \sum_P (-1)^{\dim \mathfrak{a}_P} \int_{P(F) \backslash G(\mathbb{A}_F)^1}^* \left( \int_{N(F) \backslash N(\mathbb{A}_F)} \phi_P(n, g) \phi'_P(n, g) dn \right) \phi''_P(g) \hat{\tau}_P(H(g) - T) dg. \end{aligned}$$

#### 4. Jacquet module corresponding to Fourier-Jacobi character

In this section,  $E/F$  can be either quadratic extension of number fields or a non-archimedean quadratic extension of local fields whose characteristics are zero. In the local field case,  $\psi$  and  $\mu$  denote a nontrivial character of  $F$  and  $E^\times$  respectively. Write  $|\cdot|$  and  $|\cdot|_E$  for the normalized absolute value on  $F$  and  $E$  respectively, viewed as a character of general linear group composed with  $\det$ .

Let  $(W_n, (\cdot, \cdot))$  be a skew-hermitian space over  $E$  of dimension  $n$  and let  $G_n$  its unitary group. Let  $a$  be the dimension of a maximal totally isotropic subspace of  $W_n$  and we assume  $a > 0$ . We fix maximal totally isotropic subspaces  $X$  and  $Y$  of  $W_n$ , in duality, with respect to  $(\cdot, \cdot)$ . Fix a complete flag in  $X$

$$0 = X_0 \subset X_1 \subset \dots \subset X_a = X,$$

and choose a basis  $\{e_1, e_2, \dots, e_a\}$  of  $X_a$  such that  $\{e_1, \dots, e_k\}$  is a basis of  $X_k$  for  $1 \leq k \leq a$ . Let  $\{f_1, f_2, \dots, f_a\}$  be the basis of  $X^*$  which is dual to the fixed basis of  $X$ , i.e.,  $(e_i, f_j) = \delta_{ij}$  for  $1 \leq i, j \leq r$ , where  $\delta_{i,j}$  denotes the Kronecker delta. We write  $X_k^*$  for the subspace of  $X^*$  spanned by  $\{f_1, f_2, \dots, f_k\}$  and  $W_{n-2k}$  for the orthogonal complement of  $X_k + X_k^*$  in  $W_n$ .

Denote by  $P_{n,k}$  the parabolic subgroup of  $G_n$  stabilizing  $X_k$ , by  $U_{n,k}$  its unipotent radical and  $M_{n,k}$  the Levi subgroup of  $P_{n,k}$  stabilizing the above decomposition. Then  $M_{n,k} \simeq GL(X_k) \times G_{n-2k}$ . (Here, we regard  $GL(X_k) \simeq GL_k$  as the subgroup of  $M_{n,k}$  which acts as the identity map on  $W_{n-2k}$ .)

For a smooth representation  $\sigma$  of  $GL(X_k)$  and a smooth representation  $\pi$  of  $G_{n-2k}$ , we denote by  $\text{Ind}_{P_{n,k}}^{G_n}(\sigma \boxtimes \pi)$  the normalized induced representation of  $G_n$  and by  $\text{ind}_{P_{n,k}}^{G_n}(\sigma \boxtimes \pi)$  the unnormalized induction. For  $1 \leq i \leq a - k$ , we write  $\sigma^{(i)}$  for the Bernstein-Zelevinski  $(i)$ -th derivative of  $\sigma$ . (For the definition of Bernstein-Zelevinski derivative, refer to [2, Section 4.3].)

For  $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ , we write  $N_{n,k}$  (resp.  $\mathcal{N}_k$ ) for the unipotent radical of the parabolic subgroup of  $G_n$  (resp.  $GL(X_k)$ ) stabilizing the flag  $\{0\} \subset X_1 \subset \dots \subset X_k$ . If we regard  $\mathcal{N}_a$  as a subgroup of  $M_{n,a} \simeq GL(X) \times G_{n-2a}$ , it acts on  $U_{n,a}$  and so  $N_{n,a} = U_{n,a} \rtimes \mathcal{N}_a$ .

For any  $0 < k < \frac{n}{2}$ , let  $\mathcal{H}_{n-2k}$  be the Heisenberg group of skew hermitian space  $W_{n-2k}$  over  $E$  and  $\Omega_{\psi^{-1}, \mu^{-1}, W_{n-2k}}$  be the Weil representation of  $\mathcal{H}_{n-2k} \rtimes G_{n-2k}$  with respect to  $\psi^{-1}, \mu^{-1}$ . Then since  $U_{n,k-1} \backslash U_{n,k} \simeq \mathcal{H}_{n-2k}$ , we can pull back  $\Omega_{\psi^{-1}, \mu^{-1}, W_{n-2k}}$  to  $U_{n,k} \rtimes G_{n-2k}$  and denote it by the same symbol  $\Omega_{\psi^{-1}, \mu^{-1}, W_{n-2k}}$ . We define a character  $\lambda_k : \mathcal{N}_k \rightarrow \mathbb{C}^\times$  by

$$\lambda_k(n) = \psi((Tr_{E/F}^*(n_{1,2} + n_{2,3} + \dots + n_{k-1,k})), \quad u \in \mathcal{N}_k.$$

Here,  $n_{i,i+1}$  is the  $(i, i + 1)$ -component of  $n$  when we regard  $n$  as an element in  $GL_k$  and

$$Tr_{E/F}^* = \begin{cases} Tr_{E/F} & , \quad \text{local fields case} \\ Tr_{\mathbb{A}_E/\mathbb{A}_F} & , \quad \text{number fields case.} \end{cases}$$

Put  $\nu_{\psi^{-1}, \mu^{-1}, W_{n-2k}} = \Omega_{\psi^{-1}, \mu^{-1}, W_{n-2k}} \otimes \lambda_k$  and denote  $H_{n,k} = N_{n,k} \rtimes G_{n-2k}$ . We can embed  $H_{n,k}$  into  $G_n \times G_{n-2k}$  by inclusion on the first factor and projection on the second factor. Then  $\nu_{\psi^{-1}, \mu^{-1}, W_{n-2k}}$  is a smooth representation of  $H_{n,k} = N_{n,k} \rtimes G_{n-2k}$  and upto conjugation of the normalizer of  $H_{n,k}$  in  $G_n \times G_{n-2k}$ , it is uniquely determined by  $\psi$  modulo  $\text{Nm}_{E/F} E^\times$  and  $\mu$ . We shall denote by  $\omega_{\psi^{-1}, \mu^{-1}, W_{n-2k}}$  the restriction of  $\nu_{\psi^{-1}, \mu^{-1}, W_{n-2k}}$  to  $G_{n-2k}$ .

For  $0 \leq l \leq \frac{n-2}{2}$ , we define a character  $\psi_l$  of  $N_{n,l+1}$ , which factors through the quotient  $n : N_{n,l+1} \rightarrow U_{n,l+1} \backslash N_{n,l+1} \simeq \mathcal{N}_{l+1}$ , by setting

$$\psi_l(u) = \lambda_{l+1}(n(u)).$$

In the local fields case, for a smooth representation  $\pi'$  of  $G_n$ , we write  $J_{\psi_l}(\pi' \otimes \Omega_{\psi^{-1}, \mu^{-1}, W_{n-2l-2}})$  for the Jacquet module of  $\pi' \otimes \Omega_{\psi^{-1}, \mu^{-1}, W_{n-2l-2}}$  with respect to the group  $N_{n,l+1}$  and its character  $\psi_l$ , regarded as a representation of the unitary group  $G_{n-2l-2}$ .

**Lemma 4.1.** *Let  $n, m, a$  be positive integers such that  $n - m \geq 0$  and even. Write  $q$  for the residual characteristic of  $E$ . Let  $\mathcal{E}, \sigma$  and  $\pi$  be smooth representations of finite lengths of  $G_{n+2a}, GL(X_a)$  and  $G_m$ , respectively. Then*

$$\dim_{\mathbb{C}} \text{Hom}_{G_{m+2a}}(\mathcal{E} \otimes \nu_{\psi^{-1}, \mu^{-1}, W_{m+2a}} \otimes \text{ind}_{P_{m,a}}^{G_{m+2a}}(\sigma | \cdot |_E^s \boxtimes \pi))$$

is equal or less than

$$\dim_{\mathbb{C}} \sigma^{(a)} \cdot \dim_{\mathbb{C}} \text{Hom}_{G_m}(J_{\psi_{\frac{n-m}{2}+a-1}}(\mathcal{E} \otimes \Omega_{\psi^{-1}, \mu^{-1}, W_m}), \pi^\vee))$$

except for finitely many  $q^{-s}$ .

*Proof.* The proof is similar with [15, Lemma 4.1] except for the symplectic group is replaced by unitary group. □

5. Residual representation

For an irreducible cuspidal automorphic representations  $\pi$  of  $G_n(\mathbb{A}_F)$  and  $\sigma$  of  $GL_a(\mathbb{A}_F)$ , we write  $L(s, \sigma \times \pi)$  for the Rankin-Selberg  $L$ -function  $L(s, \sigma \times BC(\pi))$ . We also write  $L(s, \sigma, As^+)$  for the Asai  $L$ -function of  $\sigma$  and  $L(s, \sigma, As^-)$  for the  $\mu$ -twisted Asai  $L$ -function  $L(s, \sigma \otimes \mu, As^+)$ . (cf. [5, Section 7])

**Proposition 5.1** ([9], Proposition 5.3). *Let  $\pi$  be an irreducible globally generic cuspidal automorphic representation of  $G_n(\mathbb{A}_F)$  and  $\sigma$  an irreducible cuspidal automorphic representation of  $GL_a(\mathbb{A}_F)$ . For  $\phi \in \mathcal{A}_{P_{n,a}}^{\sigma \boxtimes \pi}(G_{n+2a})$ , the Eisenstein series  $E(\phi, z)$  has at most a simple pole at  $z = \frac{1}{2}$  and  $z = 1$ . Moreover, it has a pole at  $z = \frac{1}{2}$  as  $\phi$  varies if and only if  $L(s, \sigma \times \pi^\vee)$  is non-zero at  $s = \frac{1}{2}$  and  $L(s, \sigma, As^{(-1)^n})$  has a pole at  $s = 1$ . Furthermore, it has a pole at  $z = 1$  as  $\phi$  varies if and only if  $L(s, \sigma \times \pi^\vee)$  has a pole at  $s = 1$ .*

For  $\phi \in \mathcal{A}_{P_a}^{\sigma \boxtimes \pi}(G_{n+2a})$ , we define the residue of the Eisenstein series to be the limit

$$\mathcal{E}^0(\phi) = \lim_{z \rightarrow \frac{1}{2}} (z - \frac{1}{2})E(\phi, z), \quad \mathcal{E}^1(\phi) = \lim_{z \rightarrow 1} (z - 1)E(\phi, z).$$

For  $i = 0, 1$ , let  $\mathcal{E}^i(\sigma, \pi)$  be the residual representations of  $G_{n+2a}(\mathbb{A}_F)$  generated by  $E^i(\phi)$ .

The assumption that  $\pi$  is globally generic ensures the existence of the weak base change  $BC(\pi)$  and we can write it as an isobaric sum of the form  $\sigma_1 \boxplus \dots \boxplus \sigma_t$ , where  $\sigma_1, \dots, \sigma_t$  are distinct irreducible cuspidal automorphic representations of the general linear groups such that the (twisted) Asai  $L$ -function  $L(s, \sigma_i, As^{(-1)^{n-1}})$  has a pole at  $s = 1$ .

*Remark 5.2.* Since  $L(s, \sigma \times \pi^\vee) = \prod_{i=1}^t L(s, \sigma \times \sigma_i^\vee)$ , Proposition 5.1 implies that  $\mathcal{E}^1(\sigma, \pi)$  is non-zero if and only if  $\sigma \simeq \sigma_i$  for some  $1 \leq i \leq t$ .

*Remark 5.3.* Let  $c$  be the automorphism of  $GL_n(E)$  induced by  $\bar{\cdot} : E \rightarrow E$  and for a representation  $\sigma$  of  $GL_n(\mathbb{A}_E)$ , we define  $\sigma^c := \sigma \circ c$ . Note that  $L(s, \sigma, As^\pm)$  are nonzero at  $s = 1$  by [12, Theorem 5.1]. Thus if  $L(s, \sigma, As^{(-1)^{n-1}})$  has a pole at  $s = 1$ , the Rankin-Selberg  $L$ -function

$$L(s, \sigma \times \sigma^c) = L(s, \sigma, As^+) \cdot L(s, \sigma, As^-)$$

has a simple pole at  $s = 1$  and so  $\sigma^c \simeq \sigma^\vee$ .

6. Lemmas

In this section,  $E/F$  denotes a quadration extension of number fields.

Let  $W_m \subset W_n$  be two skew-hermitian spaces over  $E$  of dimension  $m, n$  such that  $W_n = X \oplus W_m \oplus X^*$ . Let  $V$  be the  $\text{Res}_{E/F}(W_m)$ , which is the restriction of scalar of  $W_m$  to  $F$ . Write  $n - m = 2a$ . Let  $V = Y + Y^*$  be the complete polarization of  $V$ . Then the global Weil representation  $\Omega_{\psi^{-1}, \mu^{-1}, W_m}$  of  $\mathcal{N}(X) \rtimes G_m$  has a realization on the Schrodinger model  $\mathcal{S}(Y(\mathbb{A}_F))$ . For  $f \in \mathcal{S}(Y(\mathbb{A}_F))$ , we define theta function  $\Theta_{\psi^{-1}, \mu^{-1}}(\cdot, f)$  on  $H(\mathbb{A}_F) = N_{n,a}(\mathbb{A}_F) \rtimes G_m(\mathbb{A}_F)$  by

$$\Theta_{\psi^{-1}, \mu^{-1}}(h, f) = \sum_{x \in Y(F)} (\nu_{\psi^{-1}, \mu^{-1}, W_m}(h)f)(x) = \sum_{x \in Y(F)} \psi_{a-1}(n) \cdot (\Omega_{\psi^{-1}, \mu^{-1}, W_m}(u, g)f)(x)$$

where  $h = ((u, n), g) \in (U_{n,a} \rtimes \mathcal{N}(X)) \rtimes G_m$ . Then  $\Theta_{\psi^{-1}, \mu^{-1}}(f) \in \mathcal{A}(H)$  and the space of these theta functions  $\{\Theta_{\psi^{-1}, \mu^{-1}}(\cdot, f) \mid f \in \mathcal{S}(Y(\mathbb{A}_F))\}$  is another realization of Weil representation  $\nu_{\psi^{-1}, \mu^{-1}, W_m}$  of  $H(\mathbb{A}_F)$ .

Since we have fixed  $\mu, \psi$ , we simply write  $\nu_{W_m}$  for  $\nu_{\psi^{-1}, \mu^{-1}, W_m}$  and its associated theta function  $\Theta_{\psi^{-1}, \mu^{-1}}(\cdot, f)$  as  $\Theta(\cdot, f)$ .

**Lemma 6.1.** *Let  $\sigma$  be an irreducible globally generic cuspidal automorphic representation of  $GL_a(\mathbb{A}_E)$ ,  $\pi_1, \pi_2$  an irreducible globally generic cuspidal automorphic representation of  $G_n(\mathbb{A}_F)$  and  $G_m(\mathbb{A}_F)$  respectively. We write  $BC(\pi_1)$  as an isobaric sum  $\sigma_1 \boxplus \cdots \boxplus \sigma_t$ , where  $\sigma_1, \dots, \sigma_t$  are distinct irreducible cuspidal automorphic representations of the general linear groups such that the (twisted) Asai  $L$ -function  $L(s, \sigma_i, As^{(-1)^{n-1}})$  has a pole at  $s = 1$ . If  $\sigma \simeq \sigma_i$  for some  $1 \leq i \leq t$ , then  $\mathcal{P}(\varphi, \Theta(f), E(\phi, z)) = 0$  for all  $\varphi \in \mathcal{E}^1(\sigma, \pi_1), \phi \in \mathcal{A}_{P_{m,a}}^{\mu \cdot \sigma^c \boxtimes \pi_2}(G_{m+2a})$  and  $f \in \nu_{W_{m+2a}}$ .*

**Lemma 6.2.** *With the same notation as in Lemma 6.1, we assume  $\sigma \simeq \sigma_i$  for some  $1 \leq i \leq t$ . If  $\varphi \in \mathcal{E}^1(\sigma, \pi_1), \phi \in \mathcal{A}_{P_{m,a}}^{\mu \cdot \sigma^c \boxtimes \pi_2}(G_{m+2a})$  and  $f \in \nu_{W_{m+2a}}$ , then*

$$\mathcal{P}(\varphi, \Theta(f), \mathcal{E}(\phi)) = \int_{K_{m+2a}} \int_{M_{m+2a}(F) \backslash M_{m+2a}(\mathbb{A})^1} \phi(mk) \left( \int_{N_{n+2a,r}(F) \backslash N_{n+2a,r}(\mathbb{A}_F)} \varphi_{P_a}(nmk) \Theta_{P_a}((n, mk), f) dn \right) dmdk.$$

*Proof.* The proof is almost same with [15, Proposition 6.3]. □

**Lemma 6.3.** *With the same notation as in Lemma 6.1, we assume  $\sigma \simeq \sigma_i$  for some  $1 \leq i \leq t$ . If there are  $\xi_1 \in \pi_1, \xi_2 \in \pi_2$  and  $\xi \in \nu_{W_m}$  such that  $\mathcal{FJ}(\xi_1, \xi_2, \xi) \neq 0$ , then there are  $\varphi \in \mathcal{E}^1(\sigma, \pi_1), \phi \in \mathcal{A}_{P_a}^{\mu \cdot \sigma^c \boxtimes \pi_2}(G_{m+2a})$  and  $f \in \nu_{W_{m+2a}}$  such that*

$$\int_{K_{m+2a}} \int_{M_{m+2a}(F) \backslash M_{m+2a}(\mathbb{A})^1} \phi(mk) \left( \int_{N_{n+2a,r}(F) \backslash N_{n+2a,r}(\mathbb{A}_F)} \varphi_{P_a}(nmk) \Theta_{P_a}^\psi((n, mk), f) dn \right) dmdk \neq 0.$$

7. Main theorem

**Theorem 7.1.** *Let  $\pi_1, \pi_2$  be an irreducible globally generic cuspidal automorphic representations of  $G_n(\mathbb{A}_F)$  and  $G_m(\mathbb{A}_F)$  respectively. If there are  $\varphi_1 \in \pi_1, \varphi_2 \in \pi_2$  and  $f \in \nu_{W_m}$  such that  $\mathcal{FJ}_{\psi, \mu}(\varphi_1, \varphi_2, f) \neq 0$ , then  $L(\frac{1}{2}, BC(\pi_1) \times BC(\pi_2) \otimes \mu^{-1}) \neq 0$ .*

*Proof.* Since  $\pi_1$  is globally generic,  $BC(\pi_1)$  is an isobaric sum of the form  $\sigma_1 \boxplus \cdots \boxplus \sigma_t$ , where  $\sigma_1, \dots, \sigma_t$  are distinct irreducible cuspidal automorphic representations of the general linear groups such that the (twisted) Asai  $L$ -function  $L(s, \sigma_i, As^{(-1)^{n-1}})$  has a pole at  $s = 1$ . Then for each  $1 \leq i \leq t$ ,  $L(s, \mu^{-1} \cdot \sigma_i, As^{(-1)^n})$  has a pole at  $s = 1$ . On the other hand,  $\mathcal{E}^0(\mu \cdot \sigma_i^c, \pi_2)$  is nonzero by Lemma 6.2 and Lemma 6.3. Thus by Proposition 5.1, we have  $L(\frac{1}{2}, BC(\pi_2) \times \mu \cdot \sigma_i^c) \neq 0$  and so  $L(\frac{1}{2}, BC(\pi_2) \times \mu^{-1} \sigma_i) \neq 0$ . Thus

$$L(\frac{1}{2}, BC(\pi_1) \times BC(\pi_2) \otimes \mu^{-1}) = \prod_{i=1}^t L(\frac{1}{2}, BC(\pi_2) \times \mu^{-1} \sigma_i) \neq 0.$$

□

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