SUBRIEMANNIAN GEODESIC FLOW ON $\mathbb{S}^7$

W. BAUER AND D. TARAMA

Abstract. We consider three non-isometric trivializable subriemannian structures on the Euclidean 7-sphere $\mathbb{S}^7$ of rank 4,5, and 6 which are induced by a Clifford module structure of $\mathbb{R}^8$. In this paper we explain the geometric setting and start the analysis of the corresponding subriemannian geodesic flow. We derive the geodesic flow equations and present an equivalent but more symmetric form of ODEs. In some cases we derive normal subriemannian geodesics as the projections of solutions to either of these systems and we construct some first integrals of the geodesic flow.

1. Introduction

One aspect of subriemannian geometry concerns the study on the motions of a particle under non-holonomic constraints and, in particular, the structure of subriemannian geodesic curves [7, 13, 16]. Although in general abnormal geodesics may exist on a subriemannian manifold (see [16, Chapter 3]) it often suffices to study the subriemannian geodesic flow and its projections to the manifold (normal geodesics). From an analytic point of view one may ask how the solutions to the geodesic flow are linked to the subelliptic heat kernel or spectrum of the induced hypoelliptic sub-Laplace operators. Such questions have been studied in the literature for different examples but are not discussed here.

In the present paper we start the analysis of a subriemannian geodesic flow on the seven-dimensional Euclidean sphere $\mathbb{S}^7$. This example serves as a model case which is not of Lie group type and to some extend is accessible to explicit calculations. More precisely, we study three different subriemannian structures on $\mathbb{S}^7$ which were previously introduced in [3]. The underlying bracket generating distributions are trivial as vector bundles of ranks 4,5 and 6 and are induced from a Clifford module structure on $\mathbb{R}^8$. We remark that $\mathbb{S}^7$ can naturally be treated as a subriemannian manifold in different ways [14, 15, 16]. Being an odd-dimensional sphere it carries a contact structure which defines a bracket generating distribution as the kernel of the contact one-form. Moreover, a lift of a Riemannian structure on the base $\mathbb{S}^4$ in the quaternionic Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^7 \to \mathbb{S}^4$ induces a subriemannian structure on $\mathbb{S}^7$ of rank four which seems to differ from the rank-4-structure in this paper. We remark that $\mathbb{S}^{15}$ can be equipped with a trivializable subriemannian structure of rank 8 (see [3]) which may be treated in a similar way. However, due to the fact that the tangent bundle of $\mathbb{S}^{15}$ is not trivial we cannot choose the same type of coordinates as in Section 2 in the case of $\mathbb{S}^7$.

In Section 2 we recall some basic results in subriemannian geometry and we define three trivializable subriemannian geometries on $\mathbb{S}^7$ which are induced from a Clifford module structure on $\mathbb{R}^8$, cf. [3]. According to a theorem of L. Hörmander in [10],

2010 *Mathematics Subject Classification.* 53C17, 53C22, 37K05.
Key words and phrases. subriemannian geometry, Poisson structure, first integrals.

these structures induce three second order subelliptic sub-Laplace operators Δ_j^{sub} on $\mathbb{S}^7$. Their spectral theory has been studied before in [3].

A choice of anti-commuting skew-symmetric matrices $A_1, \dots, A_7 \in \mathbb{R}(8)$ induces a trivialization of the tangent bundle $T\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7$, cf. [1, 12]. In Section 3, via the natural identification of the tangent and cotangent bundles through the induced Riemannian metric from $\mathbb{R}^8$ and the above trivialization, we translate the symplectic structure of $T^*\mathbb{S}^7$ to the product manifold $M = \mathbb{S}^7 \times \mathbb{R}^7$. With respect to the coordinates of M we express the symplectic form, Hamiltonian vector fields and the induced Poisson bracket. In Proposition 3.8 we describe the Hamilton system (geodesic flow equations) induced from the principal symbols of the three different sub-Laplace operators Δ_j^{sub} subordinate to the above subriemannian structures of rank $4, 5$, and 6. Finally, for each $j \in \{4, 5, 6\}$ we introduce a second system of ODEs $(HS)_j$ which is equivalent to the previous one in the sense that it produces the same normal subriemannian geodesics.

An advantage of the system $(HS)_j$ seems its symmetry with respect to the space and dual variables. Based on the observations in Section 3, we use $(HS)_j$ to calculate some subriemannian geodesic curves (in the case $j = 6$) and a set of first integrals of the subriemannian geodesic flow. These geodesic curves have been detected by different techniques already in [14, 16].

2. Trivializable subriemannian structures on $\mathbb{S}^7$

A subriemannian manifold is a triple $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$, where M is a smooth connected oriented manifold (without boundary) and $\mathcal{H}$ is a subbundle of the tangent bundle. Moreover, $\langle \cdot, \cdot \rangle$ denotes a smoothly varying inner product on the fibers $\mathcal{H}_x \subset T_xM$. Let $\Gamma(\mathcal{H})$ denote the space of all vector fields that take values in $\mathcal{H}$. The subbundle $\mathcal{H}$ is called *bracket generating* or *completely non-holonomic* if at any point $x \in M$ the evaluations of vector fields in a finite sum of iterated commutators

$$\Gamma(\mathcal{H}) + \big[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})\big] + \Big[\Gamma(\mathcal{H}), \big[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})\big]\Big] + \cdots$$

span the tangent space T_xM. An absolutely continuous curve $\gamma : [0,1] \to M$ is called *horizontal* or *admissible* if $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$ for a.e. $t \in [0,1]$. A basic result in subriemannian geometry concerns global connectivity by horizontal curves:

Theorem 2.1 (W.L. Chow [8], P.K. Rashevskii [17])**.** *Let $\mathcal{H}$ be bracket generating and M connected. Then any two points on M can be connected by a piecewise smooth horizontal curve.*

The metric on $\mathcal{H}$ induces a length functional defined on horizontal curves γ as above:

$$\ell(\gamma) = \int_0^1 \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\, dt.$$

In a standard way one defines a distance (see [16]):

$$d(x,y) := \inf \big\{ \ell(\gamma) \ : \ \gamma \textit{ smooth horizontal with } \gamma(0) = x \textit{ and } \gamma(1) = y \big\}.$$

Then, an absolutely continuous path γ which locally realizes the distance is called a *subriemannian geodesic*.

In [3] a class of subriemannian structures on $\mathbb{S}^7 := \{x \in \mathbb{R}^8 \ : \ |x| = 1\}$ with trivial bracket generating distribution has been obtained and we shortly recall the construction. We choose a set $X_1, \dots, X_7$ of vector fields on $\mathbb{R}^8$ whose restriction to

$\mathbb{S}^7$ are orthonormal at any point. More precisely, let $\mathbb{R}(8)$ denote the algebra of 8×8 real matrices and choose $A_1, A_2, \dots, A_7 \in \mathbb{R}(8)$ skew-symmetric with the Clifford anti-commutation relations

$$A_jA_k + A_kA_j = -2\delta_{jk}, \qquad j,k = 1,\dots,7. \tag{2.1}$$

These matrices A_j will be fixed throughout the paper. With respect to the standard coordinates $x = (x_1,\dots,x_8)^t$ of $\mathbb{R}^8$ we define the linear vector fields

(2.2)
$$X_j := X(A_j) = \sum_{i=1}^{8}\sum_{\ell=1}^{8}(A_j)_{i,\ell}x_\ell\frac{\partial}{\partial x_i} = \sum_{i=1}^{8}(A_jx)_i\frac{\partial}{\partial x_i}, \qquad \textit{where} \quad j = 1,\dots,7.$$

As is well-known, these vector fields restricted to $\mathbb{S}^7$ form a global orthonormal frame of $T\mathbb{S}^7$. The following bracket relations can easily be checked (cf. [3]):

(a) Let $i \neq j$, then $[X_i, X_j] = -2X(A_iA_j)$,
(b) Let i,j,k be pairwise distinct, then $[X_i,[X_j,X_k]] = 0$,
(c) Let $i \neq j$, then $[X_i,[X_i,X_j]] = -4X_j$.

As a consequence, all iterated commutators of vector fields $X_1,\dots,X_7$ are contained in

$$\mathfrak{V} := \Big\{X_i = X(A_i), X(A_jA_k) \;:\; 1 \leq i \leq 7 \textit{ and } 1 \leq j < k \leq 7\Big\}.$$

We also need the following observation:

Lemma 2.2. *Let X, Y, Z be vector fields on $\mathbb{S}^7$. With the standard Riemannian metric g on $\mathbb{S}^7$ we then have*

$$g\big(X,[Y,Z]\big) = g\big([Z,X],Y\big).$$

Proof. We set $X = \sum_{r=1}^{7}\alpha_rX_r$, $Y = \sum_{\ell=1}^{7}\beta_\ell X_\ell$ and $Z = \sum_{j=1}^{7}\eta_jX_j$. At $x \in \mathbb{S}^7$, one easily verifies that

$$g\Big(X_r, X(A_\ell A_j)\Big)_x = \langle A_rx, A_\ell A_jx\rangle_{\mathbb{R}^8}, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^8}$ denotes the standard Euclidean inner product on $\mathbb{R}^8$. Inserting the above expansions and (a) gives

$$g\big(X,[Y,Z]\big)_x = \sum_{r,\ell,j=1}^{7}\alpha_r\beta_\ell\eta_j g\Big(X_r,[X_\ell,X_j]\Big)_x = -2\sum_{\substack{r,\ell,j=1\\ \ell\neq j}}^{7}\alpha_r\beta_\ell\eta_j g\Big(X_r, X(A_\ell A_j)\Big)_x.$$

Using (2.3) together with the skew-symmetry of the matrices A_k implies

$$g\big(X,[Y,Z]\big)_x = -2\sum_{r,\ell,j=1}^{7}\alpha_r\beta_\ell\eta_j\Big\langle A_jA_\ell A_rx, x\Big\rangle_{\mathbb{R}^8}. \tag{2.4}$$

Note that $\langle A_jA_\ell A_rx, x\rangle_{\mathbb{R}^8} = 0$ if $j = \ell$. Similarly we have

$$g\big(Y,[Z,X]\big)_x = -2\sum_{r,\ell,j=1}^{7}\alpha_r\beta_\ell\eta_j\Big\langle A_rA_jA_\ell x, x\Big\rangle_{\mathbb{R}^8}. \tag{2.5}$$

From the anti-commutation relations (2.1) one concludes that (2.4) and (2.5) coincide. $\square$

With these notation we consider the distributions $\mathcal{H}_j$:

$$(\mathcal{H}_j)_x := \mathrm{span}_{\mathbb{R}}\big\{\,(X_\ell)_x \;:\; \ell = 1,\dots,j\big\} \subset T_x\mathbb{S}^7 \;\; at \;\; x \in \mathbb{S}^7, \;\; where \;\; j = 4,5,6.$$

Let $\langle\cdot,\cdot\rangle_{\mathrm{sub},j}$ denote the restriction of the standard metric on $\mathbb{S}^7$ to $\mathcal{H}_j$. In other words for each $j = 4,5,6$ the metric is defined in such a way that $[X_\ell(x) \;:\; \ell = 1,\dots,j]$ are orthonormal at any point $x \in \mathbb{S}^7$. We obtain three subriemannian structures on $\mathbb{S}^7$ of rank 4, 5, and 6, respectively:

$$\Big(\mathbb{S}^7, \mathcal{H}_j, \langle\cdot,\cdot\rangle_{\mathrm{sub},j}\Big) \quad where \quad j = 4,5,6. \tag{2.6}$$

The following result has been shown in [3, Theorem 4.4]:

Theorem 2.3. *The distributions $\mathcal{H}_j$ are bracket generating if and only if $j \in \{4,5,6,7\}$.*

To each of the subriemannian structures (2.6) we can associate a sum-of-squares operator which coincides with the intrinsic sub-Laplacian (see [2] for a definition):

$$\Delta_j^{\mathrm{sub}} = -\sum_{\ell=1}^{j} X_\ell^2, \qquad j = 4,5,6. \tag{2.7}$$

Based on the bracket generating condition in Theorem 2.3 and *Hörmander's Theorem* in [10], it is known that Δ_j^{sub} defines a subelliptic second order differential operator on $\mathbb{S}^7$.

3. Geometric setting and Hamilton's equations

Recall that the tangent bundle $T\mathbb{S}^7$ is trivial and a trivialization can be obtained via the above choice of vector fields:

$$T_x\mathbb{S}^7 = \Big\{\sum_{j=1}^{7} \alpha_j X_j(x) \;:\; \alpha = (\alpha_1,\cdots,\alpha_7)^t \in \mathbb{R}^7\Big\} \subset \mathbb{R}^8, \qquad x \in \mathbb{S}^7.$$

Namely, we have a (bundle) map

$$\Phi : T\mathbb{S}^7 \to \mathbb{S}^7 \times \mathbb{R}^7 : \Big(x, \sum_{j=1}^{7} \alpha_j X_j(x)\Big) \mapsto \big(x, \alpha_1, \dots, \alpha_7\big). \tag{3.1}$$

Note that the inclusion $T\mathbb{S}^7 \hookrightarrow \mathbb{S}^7 \times \mathbb{R}^8$ allows us to pull back the product Riemannian metric on $\mathbb{S}^7 \times \mathbb{R}^8$ to induce a Riemannian metric on $T\mathbb{S}^7$. Then Φ is an isometric diffeomorphism, as $X_1(x), \dots, X_7(x)$ form an orthonormal basis of $T_x\mathbb{S}^7$ at each $x \in \mathbb{S}^7$.

Let $(x,v) \in T\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7$. We can identify the double tangent space $T_{(x,v)}(T\mathbb{S}^7)$ with

$$T_{(x,v)}(T\mathbb{S}^7) \cong T_x\mathbb{S}^7 \times T_v\mathbb{R}^7 \cong \mathbb{R}^7 \times \mathbb{R}^7.$$

Let $\pi : T\mathbb{S}^7 \to \mathbb{S}^7$ denote the canonical projection and let $g(\cdot,\cdot)$ be the standard Riemannian metric on $\mathbb{S}^7$. We identify the tangent and the cotangent bundles via the natural diffeomorphism $\varphi : T\mathbb{S}^7 \to T^*\mathbb{S}^7$ defined by $\varphi(v) := g(v,\cdot)$. Recall that via this identification the canonical one-form Θ on $T\mathbb{S}^7$ is defined through

$$\Theta(\xi) = g\big(\alpha, d\pi(\xi)\big)_x \quad where \quad (x,\alpha) \in T_x\mathbb{S}^7 \quad and \quad \xi \in T_{(x,\alpha)}\big(T\mathbb{S}^7\big).$$

Fix $(x,\alpha) \in \mathbb{S}^7 \times \mathbb{R}^7 \cong T\mathbb{S}^7$ and let $\gamma : (-\varepsilon, \varepsilon) \to T\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7$ with

$$\gamma(s) = \big(x(s), \alpha(s)\big)$$

denote a smooth curve on $T\mathbb{S}^7$ such that $\gamma(0) = (x(0), \alpha(0)) = (x,\alpha)$ and

$$\xi = \frac{d}{ds}\gamma(s)\Big|_{s=0} = \big(\dot{x}(0), \dot{\alpha}(0)\big) \in T_x\mathbb{S}^7 \times \mathbb{R}^7 \cong T_{(x,\alpha)}\big(T\mathbb{S}^7\big).$$

If we choose ξ in the form

$$\xi = \Big(\sum_{j=1}^{7} \beta_j X_j(x), w\Big) \cong (\beta, w) \in T_x\mathbb{S}^7 \times T_\alpha\mathbb{R}^7 \quad \textit{where} \quad \beta_j \in \mathbb{R},\ j = 1, \cdots, 7,$$

then we obtain for the differential

$$d\pi(\xi) = \dot{x}(0) = \sum_{j=1}^{7} \beta_j X_j(x) \in T_x\mathbb{S}^7.$$

Hence, the canonical one-form is expressed as

$$\Theta(\xi) = g\big(\alpha, d\pi(\xi)\big)_x = g\big(\alpha, \dot{x}(0)\big)_x = \sum_{j=1}^{7} \alpha_j \beta_j = \big\langle \alpha, \beta \big\rangle, \tag{3.2}$$

where $\alpha = \sum_{j=1}^{7} \alpha_j X_j(x)$. In the last equality we have identified α with $(\alpha_1, \ldots, \alpha_7)^t \in \mathbb{R}^7$ and β with $(\beta_1, \ldots, \beta_7)^t \in \mathbb{R}^7$. We write $\langle \cdot, \cdot \rangle$ for the Euclidean inner product on $\mathbb{R}^7$.

Let Y_ℓ for $\ell = 1, 2$ be vector fields on $T\mathbb{S}^7$. Then we have

$$Y_\ell : T\mathbb{S}^7 \to T(T\mathbb{S}^7) \cong T\big(\mathbb{S}^7 \times \mathbb{R}^7\big) \cong T\mathbb{S}^7 \times T\mathbb{R}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{R}^7.$$

Moreover, for each $(x,\alpha) \in \mathbb{S}^7 \times \mathbb{R}^7 \cong T\mathbb{S}^7$ we have

$$Y_\ell(x,\alpha) \in T_{(x,\alpha)}(T\mathbb{S}^7) \cong T_x\mathbb{S}^7 \times T_\alpha\mathbb{R}^7 \cong \mathbb{R}^7 \times \mathbb{R}^7.$$

With this identification we write $Y_\ell(x,\alpha) = (Y_{\ell,1}(x,\alpha), Y_{\ell,2}(x,\alpha)) \in \mathbb{R}^7 \times \mathbb{R}^7$.

Remark 3.1. Note that for each $\alpha \in \mathbb{R}^7$ we obtain a vector field on $\mathbb{S}^7$ via

$$Y_\ell^\alpha := Y_{\ell,1}(\cdot, \alpha) : \mathbb{S}^7 \to T\mathbb{S}^7.$$

Moreover, at each point $x \in \mathbb{S}^7$ we identify $(Y_\ell^\alpha)_x$ with $\alpha = (\alpha_1, \ldots, \alpha_7)^t \in \mathbb{R}^7$ where $Y_\ell^\alpha(x) = \sum_{\ell=1}^{7} \alpha_\ell X_\ell(x)$. ♦

Let $(x,\alpha) \in T\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7$ and denote by $\varphi_s = (\varphi_s^{(1)}, \varphi_s^{(2)})$ the local flow of the vector field $Y_1 = (Y_{1,1}, Y_{1,2})$ in some region containing (x,α) for $s \in (-\epsilon, \epsilon)$, where

$\epsilon > 0$. From (3.2) we obtain

$$\begin{aligned} Y_1\Theta\big(Y_2\big)\big|_{(x,\alpha)} &= \frac{d}{ds}\Theta\Big(Y_2 \circ \varphi_s(x,\alpha)\Big)\Big|_{s=0} \\ &= \frac{d}{ds}\Big\langle \varphi_s^{(2)}(x,\alpha), Y_{2,1}\circ\varphi_s(x,\alpha)\Big\rangle\Big|_{s=0} \\ &= \frac{d}{ds}\Big\langle \alpha, Y_{2,1}\circ\varphi_s(x,\alpha)\Big\rangle\Big|_{s=0} + \Big\langle Y_{1,2}(x,\alpha), Y_{2,1}(x,\alpha)\Big\rangle \\ &= \Big\langle \alpha, \mathcal{L}_{Y_1}Y_{2,1}(x,\alpha)\Big\rangle + \Big\langle Y_{1,2}(x,\alpha), Y_{2,1}(x,\alpha)\Big\rangle \\ &= \Big\langle \alpha, [Y_1, Y_{2,1}]_1(x,\alpha)\Big\rangle + \Big\langle Y_{1,2}(x,\alpha), Y_{2,1}(x,\alpha)\Big\rangle, \end{aligned}$$

where $\mathcal{L}_{Y_1}Y_{2,1}(x,\alpha)$ denotes the Lie derivative of $Y_{2,1}$ along the vector field Y_1.[1] By interchanging the roles of Y_1 and Y_2 we also find

$$Y_2\Theta\big(Y_1\big)\big|_{(x,\alpha)} = \Big\langle \alpha, [Y_2, Y_{1,1}]_1(x,\alpha)\Big\rangle + \Big\langle Y_{2,2}(x,\alpha), Y_{1,1}(x,\alpha)\Big\rangle.$$

Next, consider the commutator $[Y_1, Y_2] \in T(T\mathbb{S}^7)$ with components $[Y_1, Y_2]_1$ and $[Y_1, Y_2]_2$ according to the above notation. From (3.2) and Remark 3.1 we have

$$\Theta\Big(\big[Y_1, Y_2\big]\Big)_{(x,\alpha)} = \langle \alpha, [Y_1, Y_2]_1\rangle.$$

We obtain a symplectic form on $T\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7$ induced by the one on $T^*\mathbb{S}^7$:

$$\begin{aligned} &\Omega\big(Y_1, Y_2\big)\big|_{(x,\alpha)} = -\,d\Theta\Big(Y_1, Y_2\Big)\Big|_{(x,\alpha)} = -\,\Big[Y_1\Theta(Y_2) - Y_2\Theta(Y_1) - \Theta\Big(\big[Y_1, Y_2\big]\Big)\Big]\Big|_{(x,\alpha)} \\ &= -\,\Big[\big\langle \alpha, [Y_1, Y_{2,1}]_1\big\rangle + \big\langle Y_{1,2}, Y_{2,1}\big\rangle - \big\langle \alpha, [Y_2, Y_{1,1}]_1\big\rangle - \big\langle Y_{2,2}, Y_{1,1}\big\rangle - \big\langle \alpha, \big[Y_1, Y_2\big]_1\big\rangle\Big]\Big|_{(x,\alpha)} \\ &= \big\langle Y_{2,2}, Y_{1,1}\big\rangle - \big\langle Y_{1,2}, Y_{2,1}\big\rangle - \big\langle \alpha, \big[Y_{1,1}, Y_{2,1}\big]\big\rangle. \end{aligned}$$

Lemma 3.2. *The symplectic form on $T\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7$ is given as*

$$\Omega(Y_1, Y_2)_{|(x,\alpha)} = \big\langle Y_{2,2}, Y_{1,1}\big\rangle - \big\langle Y_{1,2}, Y_{2,1}\big\rangle - \big\langle \alpha, \big[Y_{1,1}, Y_{2,1}\big]\big\rangle.$$

Recall that the Hamiltonian vector field X_F and the gradient vector field grad F of a smooth function $F : \mathbb{S}^7 \times \mathbb{R}^7 \to \mathbb{R}$ are defined by the following equations for all $V \in T_{(x,\alpha)}(T\mathbb{S}^7)$:

$$dF_{(x,\alpha)}(V) = \Omega_{(x,\alpha)}\Big(X_F, V\Big) \quad \text{and} \quad dF_{(x,\alpha)}(V) = \tilde{g}\Big(\mathrm{grad}_{(x,\alpha)}F, V\Big)_{(x,\alpha)}.$$

Here $\tilde{g}(\cdot,\cdot)$ denotes the Riemannian metric on $T\mathbb{S}^7$. In what follows, we calculate the Hamiltonian vector field in terms of the gradient vector field.

Set $\mathrm{grad}_{(x,\alpha)}F = Y = (Y_1, Y_2) \in T_x\mathbb{S}^7 \times \mathbb{R}^7$. With an arbitrary element

$$V = (V_1, V_2) \in \mathbb{R}^7 \times \mathbb{R}^7 \cong T_x\mathbb{S}^7 \times T_\alpha\mathbb{R}^7 \cong T_{(x,\alpha)}(T\mathbb{S}^7)$$

[1] Note that $[Y_1, Y_2]_1 = [Y_{1,1}, Y_{2,1}]$, where the left-hand side is the first component of the Lie bracket of the vector fields on $T\mathbb{S}^7$ and the right-hand side is the Lie bracket of the vector fields on $\mathbb{S}^7$.

we obtain

$$\begin{aligned}\big\langle Y_1, V_1\big\rangle + \big\langle Y_2, V_2\big\rangle &= \tilde{g}\big(\mathrm{grad}_{(x,\alpha)}F, V\big)\\ &= \Omega_{(x,\alpha)}\big(X_F, V\big)\\ &= \big\langle V_2, (X_F)_1\big\rangle - \big\langle (X_F)_2, V_1\big\rangle - \big\langle \alpha, \big[(X_F)_1, V_1\big]\big\rangle.\end{aligned}$$

In particular, by choosing $V_1 = 0$ we obtain

$$(X_F)_1 = \big(\mathrm{grad}_{(x,\alpha)}F\big)_2 = \sum_{i=1}^{7} \big(\mathrm{grad}_{(x,\alpha)}F\big)_{2,i} X_i. \tag{3.3}$$

If we set $V_2 = 0$ and use the identification $\mathbb{R}^7 \ni \alpha \cong \displaystyle\sum_{j=1}^{7} \alpha_j X_j$, then we have

$$\begin{aligned}\big\langle (X_F)_2, V_1\big\rangle &= -\big\langle \alpha, \big[(X_F)_1, V_1\big]\big\rangle - \big\langle \big(\mathrm{grad}_{(x,\alpha)}F\big)_1, V_1\big\rangle\\ &= -g\Big(V_1, \Big[\alpha, (X_F)_1\Big]\Big) - g\Big(\big(\mathrm{grad}_{(x,\alpha)}F\big)_1, V_1\Big),\end{aligned}$$

where in the second equality Lemma 2.2 has been applied. Hence:

$$\big(X_F\big)_2 = \Big(\Big\langle \big[(X_F)_1, \alpha\big] - \big(\mathrm{grad}_{(x,\alpha)}F\big)_1, X_1\Big\rangle, \dots, \Big\langle \big[(X_F)_1, \alpha\big] - \big(\mathrm{grad}_{(x,\alpha)}F\big)_1, X_7\Big\rangle\Big)^t.$$

With (3.3) and the above identifications we obtain the following lemma:

Lemma 3.3. *Let $F : \mathbb{S}^7 \times \mathbb{R}^7 \cong T^*\mathbb{S}^7 \cong T\mathbb{S}^7$ be a smooth function. Then the Hamiltonian vector field has the form*

$$X_F = \Big(Y_2, \Big(\Big\langle \big[Y_2, \alpha\big] - Y_1, X_\ell\Big\rangle\Big)_{\ell=1,\dots,7}\Big)^t \in T\mathbb{S}^7 \times \mathbb{R}^7,$$

where $\mathrm{grad}_{(x,\alpha)}F = (Y_1, Y_2)$.

Recall that the Poisson bracket of two smooth functions $F, G : T^*\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7 \to \mathbb{R}$ is defined through

$$\big\{F, G\big\}(x,\alpha) = \Omega_{(x,\alpha)}\big(X_F, X_G\big).$$

With the splitting of the gradient $\mathrm{grad}_{(x,\alpha)}G = (W_1, W_2)$ we obtain:

$$\begin{aligned}\big\{F, G\big\}(x,\alpha) &= \Omega_{(x,\alpha)}\Big(\big(Y_2, [Y_2,\alpha] - Y_1\big), \big(W_2, [W_2,\alpha] - W_1\big)\Big)\\ &= \big\langle \big[W_2, \alpha\big] - W_1, Y_2\big\rangle - \big\langle \big[Y_2, \alpha\big] - Y_1, W_2\big\rangle - \big\langle \alpha, \big[Y_2, W_2\big]\big\rangle\\ &= \big\langle Y_1, W_2\big\rangle - \big\langle W_1, Y_2\big\rangle + \big\langle \alpha, \big[Y_2, W_2\big]\big\rangle. \qquad (3.4)\end{aligned}$$

In the last equation we have used Lemma 2.2 to simplify the expression.

Lemma 3.4. *Let $F, G : T\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7 \to \mathbb{R}$ be smooth functions with gradients* $\mathrm{grad}_{(x,\alpha)}F = (Y_1, Y_2)$ *and* $\mathrm{grad}_{(x,\alpha)}G = (W_1, W_2)$*. Then the Poisson bracket of F and G is given as*

$$\big\{F, G\big\}(x,\alpha) = \big\langle Y_1, W_2\big\rangle - \big\langle W_1, Y_2\big\rangle + \big\langle \alpha, \big[Y_2, W_2\big]\big\rangle.$$

In the next step we calculate the gradient of F in coordinates. This will lead to a coordinate form of the Poisson bracket. Fix a smooth function

$$F : T\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7 \to \mathbb{R} : (x,\alpha) \mapsto F(x,\alpha),$$

and let $(x, \alpha) \in \mathbb{S}^7 \times \mathbb{R}^7$ be fixed. Consider the local flows $\varphi_t^{(1)}, \ldots, \varphi_t^{(7)}$ of the vector fields $X_1, \ldots, X_7$ around $x \in \mathbb{S}^7$. Then we obtain smooth curves

$$\psi_{\cdot,\alpha}^{(j)} : (-\varepsilon, \varepsilon) \ni t \mapsto \psi_{t,\alpha}^{(j)} := \big(\varphi_t^{(j)}, \alpha\big) \in \mathbb{S}^7 \times \mathbb{R}^7, \quad \textit{with} \quad \frac{d}{dt}\Big|_{t=0} \psi_{t,\alpha}^{(j)} = \big(X_j(x), 0\big).$$

Define the translations $\rho_t^{(j)} := \alpha + te_j^7$ on $\mathbb{R}^7$ where $[e_j^k \,:\, j = 1, \ldots, k]$ denotes the standard orthonormal basis of $\mathbb{R}^k$. Then we have

$$dF\big(X_j(x), 0\big) = \frac{d}{dt}\Big|_{t=0} f\big(\varphi_t^{(j)}, \alpha\big) = X_j f(\cdot, \alpha)|_x = \Big\langle \mathrm{grad}_{(x,\alpha)} f, e_j^{14} \Big\rangle,$$

$$dF\Big(0, \frac{\partial}{\partial \alpha_j}\Big) = \frac{d}{dt}\Big|_{t=0} f\big(x, \rho_t^{(j)}\big) = \frac{\partial}{\partial \alpha_j} f(x, \alpha) = \Big\langle \mathrm{grad}_{(x,\alpha)} f, e_{7+j}^{14} \Big\rangle.$$

From these relations we find:

$$\mathrm{grad}_{(x,\alpha)} F = \Big(X_1 F, \ldots, X_7 F, \frac{\partial F}{\partial \alpha_1}, \ldots, \frac{\partial F}{\partial \alpha_7}\Big)^t\Big|_{x=w} \in \mathbb{R}^7 \times \mathbb{R}^7 \cong T_{(x,\alpha)}(T\mathbb{S}^7).$$

In combination with (3.4) we can express the Poisson bracket on $T\mathbb{S}^7$ with respect to the above identifications.

Lemma 3.5. *Let $F, G : T\mathbb{S}^7 \cong \mathbb{S}^7 \times \mathbb{R}^7 \to \mathbb{R}$ be smooth functions. Then the Poisson bracket induced from $T^*\mathbb{S}^7$ and expressed in the above coordinates is given by:*

$$\big\{F, G\big\}(x, \alpha) = \sum_{j=1}^{7} \big(X_j F\big) \cdot \frac{\partial G}{\partial \alpha_j} - \frac{\partial F}{\partial \alpha_j} \cdot \big(X_j G\big) - 2 \sum_{r,j,\ell=1}^{7} \alpha_j \frac{\partial G}{\partial \alpha_r} \frac{\partial F}{\partial \alpha_\ell} \langle A_\ell A_j A_r x, x \rangle_{\mathbb{R}^8}.$$

Proof. We use Lemma 3.4 and calculate the last term $\langle W_2, [\alpha, Y_2] \rangle$ using the above observation. With the Riemannian metric g on $\mathbb{S}^7$ we obtain

$$\begin{aligned} \big\langle W_2, \big[\alpha, Y_2\big]\big\rangle &= g\left(\sum_{r=1}^{7} \frac{\partial G}{\partial \alpha_r} X_r, \Big[\sum_{j=1}^{7} \alpha_j X_j, \sum_{\ell=1}^{7} \frac{\partial}{\partial \alpha_\ell} X_\ell\Big]\right) \\ &= \sum_{r,j,\ell=1}^{7} \alpha_j \frac{\partial G}{\partial \alpha_r} \frac{\partial F}{\partial \alpha_\ell} g\big(X_r, \big[X_j, X_\ell\big]\big)_x. \end{aligned}$$

Finally, relation (a) of Section 2 implies

$$g\big(X_r, \big[X_j, X_\ell\big]\big)_x = -2\big\langle A_r x, A_j A_\ell x\big\rangle_{\mathbb{R}^8} = -2\big\langle A_\ell A_j A_r x, x\big\rangle_{\mathbb{R}^8}.$$

□

We reconsider the sub-Laplacians Δ_j^{sub} for $j = 4, 5, 6$ defined in (2.7). In the following we write $p_j^{\mathrm{ps}}(x, \alpha)$ for its principal symbol. Let $w = (w_1, \ldots, x_8)^t \in \mathbb{S}^7$ be fixed and $j \in \{1, \ldots, 7\}$. We consider the functions $\Psi_{j,w} \in C^\infty(\mathbb{R}^8)$ defined at $x = (x_1, \ldots, x_8)^t \in \mathbb{R}^8$ by

$$\Psi_{j,w}(x) := \big\langle A_j w, x\big\rangle_{\mathbb{R}^8} = \sum_{q,r=1}^{8} (A_j)_{r,q} w_q x_r.$$

Then,

$$X_j^*(w) := d\Psi_{j,w}(w) = \sum_{q,r=1}^{8} (A_j)_{r,q} w_q dx_r \in T_w^* \mathbb{R}^8, \quad j = 1, \cdots, 7,$$

are the dual elements to $[X_j(w) \,:\, j = 1\dots,7]$ in $T_w\mathbb{R}^8$ and can be considered as a basis of $T_w^*\mathbb{S}^7$ in global coordinates. In fact, from (2.1) it follows

$$X_j^*(w)\Big(X_\ell(w)\Big) = \sum_{q,r=1}^{8}(A_j)_{r,q}w_q dx_r\left(\sum_{i,m=1}^{8}(A_\ell)_{i,m}w_m\frac{\partial}{\partial x_i}\right)$$
$$= -\sum_{q,m,i=1}^{8}(A_j)_{q,i}(A_\ell)_{i,m}w_q w_m = -\big\langle A_jA_\ell w, w\big\rangle_{\mathbb{R}^8} = \delta_{j,\ell}|w|^2 = \delta_{j,\ell}.$$

More generally, with $\xi = (\xi_1,\dots,\xi_7)^t \in \mathbb{R}^7$ and fixed $w \in \mathbb{S}^7$ we put

$$\Psi_w(x,\xi) := \sum_{q=1}^{7}\xi_q\Psi_{q,w}(x) \in C^\infty\big(\mathbb{R}^7\times\mathbb{R}^7\big).$$

Applying the differential d with respect to the x-variable shows

$$d\Psi_w(w,\xi) = \sum_{q=1}^{7}\xi_q X_q^*(w) \in T_w^*\mathbb{S}^7.$$

Let $j = 1,\dots,7$, then it follows by the same calculation as above:

$$X_j\Psi_{q,w}(w) = \sum_{i,\ell=1}^{8}\big(A_j\big)_{i,\ell}w_\ell\frac{\partial}{\partial x_i}\left(\sum_{n,m=1}^{8}\big(A_q\big)_{m,n}w_n x_m\right)\Bigg|_{x=w}$$
$$= -\sum_{n,i,\ell=1}^{8}w_\ell w_n\big(A_j\big)_{\ell,i}\big(A_q\big)_{i,n} = -\big\langle A_jA_q w, w\big\rangle_{\mathbb{R}^8} = \delta_{j,q}.$$

Therefore, it follows that $X_j\Psi_w(\cdot,\xi)\big|_{x=w} = \xi_j$. The principal symbol of $\Delta_j^{\rm sub}$ for $j = 4,5,6$ at a point $(w,\sum_{q=1}^{7}\xi_q X_q^*(w))$ takes the value:

$$p_j^{\rm ps}\left(w,\sum_{q=1}^{7}\xi_q X_q^*(w)\right) = \lim_{t\to\infty}\frac{1}{t^2}\exp\Big(-t\Psi_w(w,\xi)\Big)\Delta_j^{\rm sub}\exp\Big(t\Psi_w(x,\xi)\Big)\Bigg|_{x=w}$$
$$= -\lim_{t\to\infty}\frac{1}{t^2}\exp\Big(-t\Psi_w(w,\xi)\Big)\sum_{\ell=1}^{j}X_\ell^2\exp\Big(t\Psi_w(x,\xi)\Big)\Bigg|_{x=w} = \sum_{\ell=1}^{j}\xi_\ell^2.$$

According to the above identification $T^*\mathbb{S}^7 \cong \mathbb{S}^7\times\mathbb{R}^7$ we express the principal symbol of $\Delta_j^{\rm sub}$ on $\mathbb{S}^7\times\mathbb{R}^7$ in the form

$$p_j^{\rm ps}(w,\alpha) = \sum_{\ell=1}^{j}\alpha_\ell^2.$$

Lemma 3.6. *The Hamiltonian vector field $X_{p_j^{\rm ps}}$ associated to the principal symbol of the sub-Laplacian $\Delta_j^{\rm sub}$ acts on functions $F \in C^\infty(\mathbb{S}^7\times\mathbb{R}^7)$ in the following form:*

$$X_{p_j^{\rm ps}}(F)(x,\alpha) = 2\sum_{\ell=1}^{j}\alpha_\ell\big(X_\ell F\big) - \sum_{\ell=1}^{7}B_\ell(x,\alpha)\cdot\frac{\partial F}{\partial\alpha_\ell}, \tag{3.5}$$

where the coefficient functions B_ℓ in the second sum are given by:

$$B_\ell(x,\alpha) := 4\sum_{m=j+1}^{7}\sum_{r=1}^{j}\alpha_m\alpha_r\big\langle A_\ell A_m A_r x, x\big\rangle_{\mathbb{R}^8}.$$

Proof. According to Lemma 3.5, we have

$$\begin{aligned}
X_{p_j^{\mathrm{ps}}}(F)(x,\alpha) &= \Big\{F, p_j^{\mathrm{sc}}\Big\} = \Big\{F, \sum_{\ell=1}^{j}\alpha_j^2\Big\}\\
&= \sum_{\ell=1}^{7}\big(X_\ell F\big)\frac{\partial p_j^{\mathrm{ps}}}{\partial\alpha_\ell} - \frac{\partial F}{\partial\alpha_\ell}\underbrace{\big(X_\ell p_\ell^{\mathrm{ps}}\big)}_{=0} - 2\sum_{r,m,\ell=1}^{7}\alpha_m\frac{\partial p_j^{\mathrm{sc}}}{\partial\alpha_r}\frac{\partial F}{\partial\alpha_\ell}\big\langle A_\ell A_m A_r x, x\big\rangle_{\mathbb{R}^8}\\
&= 2\sum_{\ell=1}^{j}\alpha_\ell\big(X_\ell F\big) - 4\sum_{m,\ell=1}^{7}\sum_{r=1}^{j}\alpha_m\alpha_r\frac{\partial F}{\partial\alpha_\ell}\big\langle A_\ell A_m A_r x, x\big\rangle_{\mathbb{R}^8}.
\end{aligned}$$

From the Clifford relations (2.1) we find that

$$\sum_{m=1}^{j}\sum_{r=1}^{j}\alpha_m\alpha_r\big\langle A_\ell A_m A_r x, x\big\rangle_{\mathbb{R}^8} = 0,$$

and therefore (3.5) follows. □

Remark 3.7. In case of the standard Laplace-Beltrami operator $\Delta = \Delta_7^{\mathrm{sub}}$ on $\mathbb{S}^7$ we obtain that $X_{p_7^{\mathrm{ps}}}(F) = 2\sum_{\ell=1}^{7}\alpha_\ell\big(X_\ell F\big)$ since $B_\ell = 0$ for $\ell = 1,\dots,7$. ♦

Proposition 3.8. *For $j = 4,\dots,6$ the Hamiltonian system corresponding to the principal symbol p_j^{ps} of Δ_j^{sub} is given by*

$$\big(\dot{x},\dot{\alpha}\big) = X_{p_j^{\mathrm{ps}}}(x,\alpha) = \Big(2\sum_{\ell=1}^{j}\alpha_\ell X_\ell(x), -B_1(x,\alpha),\dots,-B_7(x,\alpha)\Big). \tag{3.6}$$

Note that a projection to $\mathbb{S}^7$ of a solution to (3.6) defines a horizontal curve on (2.6).

Example 3.9. Let $j = 6$, then we obtain for $\ell = 1,\dots,7$:

$$B_\ell(x,\alpha) = 4\alpha_7\sum_{r=1}^{6}\alpha_r\big\langle A_\ell A_7 A_r x, x\big\rangle_{\mathbb{R}^8}. \tag{3.7}$$

In particular, $B_7 = 0$ since $A_7^2 = -I$ and A_r is skew-symmetric. Let $q \in \mathbb{S}^7$ and choose $\rho = (\rho_1,\dots,\rho_7)^t \in \mathbb{R}^7$ with $|\rho| = 1$. Consider the following curve on $\mathbb{S}^7$:

$$x_q(t) = \exp\big(-t\rho_7 A_7\big)\exp\big(tA(\rho)\big)q \quad \textit{where} \quad A(\rho) := \sum_{\ell=1}^{7}\rho_\ell A_\ell.$$

We construct $\alpha_q(t)$ such that $(x_q(t), \alpha_q(t))$ for $t \in (-\epsilon,\epsilon)$, where $\epsilon > 0$, solves (3.6) and hence $x_q(t)$ is a normal subriemannian geodesic curve. In fact, consider

$$\alpha_q(t) = \frac{1}{2}\Big(\big\langle x_{A(\rho)q}(t), A_1 x_q(t)\big\rangle_{\mathbb{R}^8},\dots,\big\langle x_{A(\rho)q}(t), A_7 x_q(t)\big\rangle_{\mathbb{R}^8}\Big).$$

A direct calculation shows that

$$\dot{x}_q = -\rho_7 A_7 x_q + x_{A(\rho)q} \quad \textit{and} \quad \big\langle \dot{x}_q, A_7 x_q\big\rangle_{\mathbb{R}^8} = 0. \tag{3.8}$$

The first equation in (3.6) is obtained from the fact that $[A_\ell x_q : \ell = 1, \dots, 7]$ forms an orthonormal basis of $T_{x_q}\mathbb{S}^7$ and

$$2\sum_{\ell=1}^{6}(\alpha_q)_\ell A_\ell x_q = \sum_{\ell=1}^{6}\big\langle x_{A(\rho)q}, A_\ell x_q\big\rangle_{\mathbb{R}^8} A_\ell x_q = x_{A(\rho)q} - \big\langle x_{A(\rho)q}, A_7 x_q\big\rangle_{\mathbb{R}^8} A_7 x_q \tag{3.9}$$

together with $\langle x_{A(\rho)q}, A_7 x_q\rangle_{\mathbb{R}^8} = \rho_7$. We verify the second equation in (3.6). From (3.8) and the observation that $A(\rho)^2 = -|\rho|^2 I = I$ it follows:

$$\begin{aligned}
\big(\dot\alpha_q\big)_\ell &= \frac{1}{2}\big\langle \dot x_{A(\rho)q}, A_\ell x_q\big\rangle_{\mathbb{R}^8} + \frac{1}{2}\big\langle x_{A(\rho)q}, A_\ell \dot x_q\big\rangle_{\mathbb{R}^8}\\
&= \frac{1}{2}\Big\langle -\rho_7 A_7 x_{A(\rho)q} - x_q, A_\ell x_q\Big\rangle_{\mathbb{R}^8} + \frac{1}{2}\Big\langle x_{A(\rho)q}, -\rho_7 A_\ell A_7 x_q + A_\ell x_{A(\rho)q}\Big\rangle_{\mathbb{R}^8}\\
&= -\frac{\rho_7}{2}\Big\langle A_7 x_{A(\rho)q}, A_\ell x_q\Big\rangle_{\mathbb{R}^8} - \frac{\rho_7}{2}\Big\langle x_{A(\rho)q}, A_\ell A_7 x_q\Big\rangle_{\mathbb{R}^8}\\
&= \begin{cases} -\rho_7\Big\langle x_{A(\rho)q}, A_\ell A_7 x_q\Big\rangle_{\mathbb{R}^8}, & \textit{if} \quad \ell \neq 7,\\ 0, & \textit{if} \quad \ell = 7.\end{cases}
\end{aligned}$$

In particular, $(\dot\alpha_q)_7 = 0 = B_7$. It follows that

$$\big(\alpha_q\big)_7 \equiv (\alpha_q)_7(0) = \frac{1}{2}\Big\langle A(\rho)q, A_7 q\Big\rangle_{\mathbb{R}^8} = \frac{\rho_7}{2}. \tag{3.10}$$

Finally, in the case of $\ell \in \{1, \dots, 6\}$ it follows from (3.7), (3.9) and (3.10) that:

$$\begin{aligned}
B_\ell(x_q, \alpha_q) &= 4(\alpha_q)_7\Big\langle A_\ell A_7 \sum_{r=1}^{6}(\alpha_q)_r A_r x_q, x_q\Big\rangle_{\mathbb{R}^8}\\
&= 2(\alpha_q)_7\Big\langle A_\ell A_7\big(x_{A(\rho)q} - \rho_7 A_7 x_q\big), x_q\Big\rangle_{\mathbb{R}^8}\\
&= -\rho_7\Big\langle x_{A(\rho)q}, A_\ell A_7 x_q\Big\rangle_{\mathbb{R}^8} = \big(\dot\alpha_q\big)_\ell.
\end{aligned}$$

♦

4. On a Hamiltonian system and normal geodesics

We calculate families of normal subriemannian geodesics on $\mathbb{S}^7$ for the subriemannian structures introduced above, using a Hamiltonian system on $\mathbb{R}^8 \times \mathbb{R}^8$. In the case where the distribution has co-rank 1 (i.e. $j = 6$) the formulas we present have been previously derived in [13, Proposition 5] based on more general results in [16, Theorem 1.26].

With $j \in \{4, 5, 6\}$ consider the subriemannian Hamiltonians defined by:

$$H_{j,\mathrm{sub}} : \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R} : H_{j,\mathrm{sub}}(x, \xi) = \frac{1}{2}\sum_{\ell=1}^{j}\big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8}^2.$$

Correspondingly we obtain three Hamiltonian systems:

$$(HS)_j \quad \begin{cases} \dot x &= \dfrac{\partial H_{j,\mathrm{sub}}}{\partial \xi} = \displaystyle\sum_{\ell=1}^{j}\big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8} A_\ell x,\\ \dot\xi &= -\dfrac{\partial H_{j,\mathrm{sub}}}{\partial x} = \displaystyle\sum_{\ell=1}^{j}\big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8} A_\ell \xi.\end{cases}$$

If $(x(t), \xi(t))$ for $t \in (-\epsilon, \epsilon)$, where $\epsilon > 0$, is a solution to $(HS)_j$ then the "dual curve" $\xi(t)$ is tangent to a sphere and starting at a point $\xi_0 = \xi(0) \in \mathbb{S}^7$ it defines a curve on $\mathbb{S}^7$. In relation to the subriemannian geodesic flow (3.6), we have the following theorem.

Theorem 4.1. *The Hamiltonian system $(HS)_j$ can be restricted to $T\mathbb{S}^7 \subset \mathbb{R}^8 \times \mathbb{R}^8$. The restriction of $(HS)_j$ to $T\mathbb{S}^7$ is equivalent to the system (3.6) of the subriemannian geodesic flow induced from the standard symplectic structure on $T^*\mathbb{S}^7$.*

Proof. Recall that $T\mathbb{S}^7$ is regarded as the submanifold of $\mathbb{R}^8 \times \mathbb{R}^8$ defined through $\langle x, x\rangle_{\mathbb{R}^8} = 1, \langle x, \xi\rangle_{\mathbb{R}^8} = 0$. Let $(x, \xi) : [0, 1] \to \mathbb{S}^7 \times \mathbb{R}^8$ be a solution to $(HS)_j$ and for $\ell = 1, \ldots, 7$ define

$$\alpha = (\alpha_1, \ldots, \alpha_7)^t \quad \textit{where} \quad \alpha_\ell := \frac{1}{2}\big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8}.$$

Then the first equation in $(HS)_j$ implies that $\dot{x} = 2\sum_{\ell=1}^{j} \alpha_\ell A_\ell x$ which gives the first equation in (3.6). Moreover, for $r = 1, \cdots, 7$, we have

$$\begin{aligned}
\dot{\alpha}_r &= \frac{1}{2}\big\langle A_r \dot{x}, \xi\big\rangle_{\mathbb{R}^8} + \frac{1}{2}\big\langle A_r x, \dot{\xi}\big\rangle_{\mathbb{R}^8} \\
&= \frac{1}{2}\sum_{\ell=1}^{j}\Big[\big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8}\big\langle A_r A_\ell x, \xi\big\rangle_{\mathbb{R}^8} + \big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8}\big\langle A_r x, A_\ell \xi\big\rangle_{\mathbb{R}^8}\Big] \\
&= -\sum_{\substack{\ell=1 \\ \ell\neq r}}^{j}\big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8}\big\langle A_\ell x, A_r \xi\big\rangle_{\mathbb{R}^8}.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
B_\ell(x, \alpha) &= -\sum_{m=1}^{7}\sum_{\substack{r=1 \\ \ell\neq r}}^{j}\big\langle A_m x, \xi\big\rangle_{\mathbb{R}^8}\big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8}\big\langle A_\ell A_r A_m x, x\big\rangle_{\mathbb{R}^8} \\
&= -\sum_{\substack{r=1 \\ \ell\neq r}}^{j}\big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8}\Big\langle A_\ell A_r \sum_{m=1}^{7}\langle A_m x, \xi\rangle A_m x, x, \Big\rangle_{\mathbb{R}^8} \\
&= -\sum_{\substack{r=1 \\ \ell\neq r}}^{j}\big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8}\big\langle A_\ell A_r \xi, x\big\rangle_{\mathbb{R}^8}.
\end{aligned}$$

By skew-symmetry of A_ℓ we obtain that $\dot{\alpha}_\ell = -B_\ell(x, \alpha)$ for $\ell = 1, \ldots, 7$ which implies the second set of equations in (3.6).

Conversely, assume that a solution $(x, \alpha) : [0, 1] \to \mathbb{S}^7 \times \mathbb{R}^7$ to (3.6) is given. Then we define a curve $\xi : [0, 1] \to \mathbb{R}^8$ via:

$$\xi := 2\sum_{\ell=1}^{j} \alpha_\ell A_\ell x.$$

Note that $\langle \xi, x\rangle_{\mathbb{R}^8} \equiv 0$ due to the skew-symmetry of A_ℓ. Since $2\alpha_\ell = \langle A_\ell x, \xi\rangle_{\mathbb{R}^8}$ we immediately obtain the first equation in $(HS)_j$:

$$\dot{x} = 2\sum_{\ell=1}^{j} \alpha_\ell A_\ell x = \sum_{\ell=1}^{j} \big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8} A_\ell x.$$

Finally, note that

$$\begin{aligned}\dot{\xi} &= 2\sum_{\ell=1}^{j} \Big[\dot{\alpha}_\ell A_\ell x + \alpha_\ell A_\ell \dot{x}\Big] \\ &= 2\sum_{\ell=1}^{j} \Big[- B_\ell(x,\alpha) A_\ell x + 2\alpha_\ell A_\ell \sum_{q=1}^{j} \alpha_q A_q x\Big].\end{aligned} \tag{4.1}$$

We calculate the first sum:

$$\begin{aligned}\sum_{\ell=1}^{j} B_\ell(x,\alpha)A_\ell x &= -\sum_{\ell=1}^{j}\sum_{m=1}^{7}\sum_{r=1}^{j} \big\langle A_m x, \xi\big\rangle_{\mathbb{R}^8}\big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8}\big\langle A_\ell A_r A_m x, x\big\rangle_{\mathbb{R}^8} A_\ell x \\ &= -\sum_{\substack{r,\ell=1 \\ \ell\neq r}}^{j} \big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8}\big\langle A_\ell A_r \xi, x\big\rangle_{\mathbb{R}^8} A_\ell x \\ &= -2\sum_{r,\ell=1}^{j} \alpha_r \big\langle A_\ell \xi, A_r x\big\rangle_{\mathbb{R}^8} A_\ell x = 0,\end{aligned}$$

as $A_\ell A_r = -A_r A_\ell$ if $r \neq \ell$ and $\langle A_\ell \xi, A_\ell x\rangle_{\mathbb{R}^8} = \langle \xi, x\rangle_{\mathbb{R}^8}$. According to our definition of ξ we conclude from (4.1):

$$\dot{\xi} = 4\sum_{\ell,q=1}^{j} \alpha_\ell \alpha_q A_\ell A_q x = 2\sum_{\ell=1}^{j} \alpha_\ell A_\ell \xi = \sum_{\ell=1}^{j} \big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8} A_\ell \xi,$$

which proves the second equations in $(HS)_j$. □

Let $\zeta(t) = (x(t), \xi(t))$ be a solution to the system $(HS)_j$ with $\zeta(0) = (x_0, \xi_0) \in \mathbb{S}^7 \times \mathbb{R}^7$. Then the projection $x(t)$ is called a *normal subriemannian geodesic*. Note that $x(t)$ is a horizontal curve, i.e. $x(t) \in \mathbb{S}^7$ with $\dot{x}(t) \in (\mathcal{H}_j)_{x(t)}$ for each t. More precisely, sufficiently short arcs of $x(t)$ are length minimizing subriemannian geodesics. By the form of $(HS)_j$, $x(t), \xi(t)$ are horizontal curves on $\mathbb{S}^7$ if $x(0), \xi(0) \in \mathbb{S}^7$, respectively.

Lemma 4.2. *Let $J \in \mathbb{R}(8)$ be an arbitrary matrix and $\zeta(t) = (x(t), \xi(t))$ be a solution to the system $(HS)_j$. Then*

$$\frac{d}{dt}\big\langle Jx, \xi\big\rangle_{\mathbb{R}^8} = \sum_{\ell=1}^{j} \big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8}\big\langle [J, A_\ell]x, \xi\big\rangle_{\mathbb{R}^8}.$$

In particular, $t \mapsto \langle Jx(t), \xi(t)\rangle$ is constant if $[J, A_\ell] = 0$ for all $\ell = 1, \dots, j$.

Proof. The statement follows by a direct calculation. In fact:

$$\frac{d}{dt}\big\langle Jx, \xi\big\rangle_{\mathbb{R}^8} = \big\langle J\dot{x}, \xi\big\rangle_{\mathbb{R}^8} + \big\langle Jx, \dot{\xi}\big\rangle_{\mathbb{R}^8}.$$

The assertion follows by inserting the equations in $(HS)_j$ into the right hand side and using the skew-symmetry of A_ℓ. □

Example 4.3. Choose $J = I =$ identity. Then $\langle x, \xi\rangle_{\mathbb{R}^8}$ is constant along a solution of $(HS)_j$. In particular, $x(t)$ and $\xi(t)$ remain orthogonal if $x(0)$ and $\xi(0)$ are orthogonal. ♦

We start by calculating the simplest type of normal subriemannian geodesics.

Let $q \in \mathbb{S}^7$ and $\alpha = (\alpha_1, \dots, \alpha_j)^t \in \mathbb{R}^j$. Consider the great circle on $\mathbb{S}^7$ defined by

$$\gamma_\alpha(t) := \exp\Big(tA(\alpha)\Big)q \quad \textit{where} \quad A(\alpha) := \sum_{\ell=1}^{j} \alpha_\ell A_\ell.$$

From the anti-commutation relation (2.1) we conclude that $A(\alpha)$ is skew-symmetric with $A(\alpha)^2 = -|\alpha|^2 I$. Therefore, we have

$$\gamma_\alpha(t) = \cos\big(t|\alpha|\big)\cdot q + \frac{\sin\big(t|\alpha|\big)}{|\alpha|}\cdot A(\alpha)q,$$

where $|\alpha|^2 = \sum_{\ell=1}^{j} \alpha_\ell^2$. Note that $\gamma_\alpha(t)$ is horizontal on $\mathbb{S}^7$ with respect to $\mathcal{H}_j$. Put

$$\xi_\alpha(t) := \gamma_{\alpha, A(\alpha)q}(t) = \exp\Big(tA(\alpha)\Big)A(\alpha)q.$$

Then we obtain

$$\begin{aligned}\sum_{\ell=1}^{j}\big\langle A_\ell\gamma_\alpha(t), \xi_\alpha(t)\big\rangle_{\mathbb{R}^8} A_\ell\gamma_\alpha(t) &= \sum_{\ell=1}^{j}\Big\langle A_\ell \exp(tA(\alpha))q, A(\alpha)\exp\big(tA(\alpha)\big)q\Big\rangle_{\mathbb{R}^8} A_\ell\gamma_\alpha(t)\\ &= \sum_{\ell=1}^{j}\alpha_\ell A_\ell\gamma_\alpha(t) = A(\alpha)\exp\big(tA(\alpha)\big)q = \dot{\gamma}_\alpha(t).\end{aligned}$$

Hence the pair $(x(t), \xi(t)) = (\gamma_\alpha(t), \xi_\alpha(t))$ solves the first equation of $(HS)_j$. A similar calculation shows that the second equation in $(HS)_j$ is fulfilled, as well.

Proposition 4.4. *For $j \in \{4, 5, 6\}$ and each $\alpha \in \mathbb{R}^j$ the great circle $\gamma_\alpha(t)$ defines a normal subriemannian geodesic on $\mathbb{S}^7$ with respect to the subriemannian structure $(\mathbb{S}^7, \mathcal{H}_j, \langle\cdot,\cdot\rangle_{\mathrm{sub},j})$.*

In order to enlarge the family of geodesics obtained in Proposition 4.4, we consider the cases $j \in \{4, 5, 6\}$ separately. If $j = 6$, then normal geodesics can be constructed via [16, Theorem 1.26] (see [13, Proposition 5]). From

$$\begin{aligned}x &= \sum_{\ell=0}^{7}\langle x, A_\ell\xi\rangle_{\mathbb{R}^8} A_\ell\xi = -\sum_{\ell=1}^{7}\langle A_\ell x, \xi\rangle_{\mathbb{R}^8} A_\ell\xi + \langle x, \xi\rangle_{\mathbb{R}^8}\xi,\\ \xi &= \sum_{\ell=0}^{7}\langle \xi, A_\ell x\rangle_{\mathbb{R}^8} A_\ell x = \sum_{\ell=1}^{7}\langle \xi, A_\ell x\rangle_{\mathbb{R}^8} A_\ell x + \langle x, \xi\rangle_{\mathbb{R}^8} x,\end{aligned}$$

and $(HS)_6$ we obtain the equivalent equations

$$\begin{cases}\dot{\xi} &= -x - \langle A_7 x, \xi\rangle_{\mathbb{R}^8} A_7\xi + \langle x, \xi\rangle_{\mathbb{R}^8}\xi,\\ \dot{x} &= \xi - \langle A_7 x, \xi\rangle_{\mathbb{R}^8} A_7 x - \langle x, \xi\rangle_{\mathbb{R}^8} x.\end{cases}$$

If we assume that the initial data $(x_0, \xi_0) = (x(0), \xi(0))$ are orthogonal, then $\langle x, \xi\rangle_{\mathbb{R}^8} = 0$ according to Example 4.3 and we arrive at the simplified system

$$\begin{cases} \dot{\xi} &= -x - \langle A_7 x, \xi\rangle_{\mathbb{R}^8} A_7\xi, \\ \dot{x} &= \xi - \langle A_7 x, \xi\rangle_{\mathbb{R}^8} A_7 x, \end{cases} \tag{4.2}$$

which we can solve explicitly (see Example 3.9). Let $q \in \mathbb{S}^7$ and fix $\alpha = (\alpha_1, \dots, \alpha_7)^t \in \mathbb{R}^7$. Consider the following curve starting at q:

$$\gamma_{6,\alpha,q}(t) := \exp\big(-t\alpha_7 A_7\big)\exp\big(tA(\alpha)\big)q \quad \textit{where} \quad A(\alpha) = \sum_{\ell=1}^{7} \alpha_\ell A_\ell.$$

In Example 3.9 we already have seen that $\gamma_{6,\alpha,q}$ defines a subriemannian geodesic starting at q. Here we reprove the result using the Hamilton system $(HS)_6$. First, we show the following lemma.

Lemma 4.5. *$\gamma_{6,\alpha,q}(t)$ is horizontal with respect to $\mathcal{H}_6$.*

Proof. It suffices to show that $\dot{\gamma}_{6,\alpha,q}(t)$ is orthogonal to $A_7\gamma_{6,\alpha,q}(t)$ at $t \in \mathbb{R}$. Note that

$$\dot{\gamma}_{6,\alpha,q}(t) = -\alpha_7 A_7 \gamma_{6,\alpha,q}(t) + \gamma_{6,\alpha,A(\alpha)q}(t). \tag{4.3}$$

Hence, we obtain:

$$\begin{aligned} \Big\langle \dot{\gamma}_{6,\alpha,q}(t), A_7\gamma_{6,\alpha,q}(t)\Big\rangle_{\mathbb{R}^8} &= -\alpha_7 + \Big\langle \exp\big(-t\alpha_7 A_7\big)A(\alpha)\exp\big(tA(\alpha)\big)q, A_7\gamma_{6,\alpha,q}(t)\Big\rangle_{\mathbb{R}^8} \\ &= -\alpha_7 + \Big\langle A(\alpha)\exp\big(tA(\alpha)\big)q, A_7\exp\big(tA(\alpha)\big)q\Big\rangle_{\mathbb{R}^8} \\ &= -\alpha_7 + \alpha_7\big|A_7\exp\big(tA(\alpha)\big)q\big|^2 = 0. \end{aligned}$$

This shows that assertion. □

In order to verify that $\gamma_{6,\alpha,q}(t)$ for each $\alpha \in \mathbb{R}^7$ defines a subriemannian geodesic for the structure induced by the distribution $\mathcal{H}_6$ we define:

$$\xi_{6,\alpha,q}(t) := \gamma_{6,\alpha,A(\alpha)q}(t) = \exp\big(-t\alpha_7 A_7\big)\exp\big(tA(\alpha)\big)A(\alpha)q.$$

Proposition 4.6. *For each $\alpha \in \mathbb{R}^7$ with $|\alpha| = 1$ the curve $\Gamma_{6,\alpha,q}(t) := (\gamma_{6,\alpha,q}(t), \xi_{6,\alpha,q}(t))$ solves the Hamiltonian system $(HS)_6$ under the initial condition $\Gamma_{6,\alpha,q}(0) = (q, A(\alpha)q)$. In particular, $\gamma_{6,\alpha,q}$ defines a subriemannian geodesic with respect to $\mathcal{H}_6$.*

Proof. According to Lemma 4.5 the curve $\gamma_{6,\alpha,q}(t)$ is horizontal with

$$\dot{\gamma}_{6,\alpha,q} = -\alpha_7 A_7\gamma_{6,\alpha,q} + \gamma_{6,\alpha,A(\alpha)q} = -\alpha_7 A_7 \gamma_{6,\alpha,q} + \xi_{6,\alpha,q}. \tag{4.4}$$

It follows that

$$\dot{\gamma}_{6,\alpha,q} = \sum_{\ell=1}^{6} \big\langle \dot{\gamma}_{6,\alpha,q}, A_\ell\gamma_{6,\alpha,q}\big\rangle_{\mathbb{R}^8} A_\ell\gamma_{6,\alpha,q} = \sum_{\ell=1}^{6} \big\langle \xi_{6,\alpha,q}, A_\ell\gamma_{6,\alpha,q}\big\rangle_{\mathbb{R}^8} A_\ell\gamma_{6,\alpha,q}.$$

Hence the first equation of $(HS)_6$ is fulfilled with $x = \gamma_{\alpha,q}$ and $\xi = \xi_{\alpha,q}$. Applying Lemma 4.5 again gives:

$$\begin{aligned} \dot{\xi}_{6,\alpha,q} = \dot{\gamma}_{6,\alpha,A(\alpha)q} &\in \operatorname{span}\big\{A_\ell\gamma_{6,\alpha,A(\alpha)q} \,:\, \ell = 1,\dots,6\big\} \\ &= \operatorname{span}\big\{A_\ell\xi_{6,\alpha,q} \,:\, \ell = 1,\dots,6\big\}. \end{aligned}$$

Hence,

$$\dot{\xi}_{6,\alpha,q} = \sum_{\ell=1}^{6} \big\langle \dot{\xi}_{6,\alpha,q}, A_\ell \xi_{6,\alpha,q} \big\rangle_{\mathbb{R}^8} A_\ell \xi_{6,\alpha,q}. \tag{4.5}$$

From (4.3) we have

$$\begin{aligned}\dot{\xi}_{6,\alpha,q} = \dot{\gamma}_{6,\alpha,A(\alpha)q} &= -\alpha_7 A_7 \xi_{6,\alpha,q} + \gamma_{6,\alpha,A^2(\alpha)q} \\ &= -\alpha_7 A_7 \xi_{6,\alpha,q} - |\alpha|^2 \gamma_{6,\alpha,q} = -\alpha_7 A_7 \xi_{6,\alpha,q} - \gamma_{6,\alpha,q}.\end{aligned}$$

By inserting the last relation into (4.5) we observe that the second equation in $(HS)_6$ is also solved by $(x,\xi) = (\gamma_{6,\alpha,q}, \xi_{6,\alpha,q})$. □

Now let $j = 5$, then we can rewrite $(HS)_5$ in the form:

$$(\widetilde{HS})_5 : \begin{cases} \dot{\xi} &= -x + \langle x, A_6\xi\rangle_{\mathbb{R}^8} A_6 \xi + \langle x, A_7 \xi\rangle_{\mathbb{R}^8} A_7 \xi + \langle x, \xi\rangle_{\mathbb{R}^8} \xi, \\ \dot{x} &= \xi - \langle \xi, A_6 x\rangle_{\mathbb{R}^8} A_6 x - \langle \xi, A_7 x\rangle_{\mathbb{R}^8} A_7 x - \langle x, \xi\rangle_{\mathbb{R}^8} x. \end{cases}$$

Remark 4.7. Note that $A_1, \ldots, A_5$ commute with

$$J \in \{I, A_6A_7, A_1A_2\cdots A_5, A_1A_2\cdots A_7\}$$

and based on Lemma 4.2 we obtain four constants of motion. ♦

Let (x,ξ) be a solution to $(HS)_5$, then (x,ξ) solve $(\widetilde{HS})_5$ and we obtain:

$$\begin{aligned}
&\frac{d}{dt}\sum_{r=6}^{7} \big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8}^2 = 2\sum_{r=6}^{7} \big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8} \Big[\big\langle A_r \dot{x}, \xi\big\rangle_{\mathbb{R}^8} + \big\langle A_r x, \dot{\xi}\big\rangle_{\mathbb{R}^8}\Big] \\
&= 2\sum_{r=6}^{7} \big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8} \Big[- \Big\langle \xi - \langle \xi, A_6 x\rangle_{\mathbb{R}^8} A_6 x - \big\langle \xi, A_7 x\big\rangle_{\mathbb{R}^8} A_7 x - \langle x, \xi\rangle_{\mathbb{R}^8} x, A_r \xi \Big\rangle_{\mathbb{R}^8} + \\
&\qquad\qquad + \Big\langle A_r x, -x + \langle x, A_6 \xi\rangle_{\mathbb{R}^8} A_6 \xi + \langle x, A_7\xi\rangle_{\mathbb{R}^8} A_7 \xi + \langle x, \xi\rangle_{\mathbb{R}^8} \xi \Big\rangle_{\mathbb{R}^8}\Big] \\
&= 2\sum_{r=6}^{7} \big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8} \Big[\langle \xi, A_6 x\rangle_{\mathbb{R}^8} \langle A_6 x, A_r \xi\rangle_{\mathbb{R}^8} + \langle \xi, A_7 x\rangle_{\mathbb{R}^8} \langle A_7 x, A_r\xi\rangle_{\mathbb{R}^8} + \\
&\qquad\qquad + \langle x, A_6\xi\rangle_{\mathbb{R}^8} \langle A_r x, A_6 \xi\rangle_{\mathbb{R}^8} + \langle x, A_7 \xi\rangle_{\mathbb{R}^8} \langle A_r x, A_7\xi\rangle_{\mathbb{R}^8}\Big] \\
&= 4 \sum_{r,\ell=6\; r\neq \ell}^{7} \big\langle A_r x, \xi\big\rangle_{\mathbb{R}^8} \big\langle A_\ell x, \xi\big\rangle_{\mathbb{R}^8} \big\langle A_\ell x, A_r \xi\big\rangle_{\mathbb{R}^8} = 0.
\end{aligned}$$

Hence, we have the following lemma:

Lemma 4.8. *The function*

$$C : \mathbb{S}^7 \times \mathbb{R}^8 \to \mathbb{R} : F(q,\xi) := \sum_{r=6}^{7} \big\langle A_r q, \xi\big\rangle_{\mathbb{R}^8}^2$$

defines a first integral of the subriemannian geodesic flow on $\mathbb{S}^7$ with respect to $\mathcal{H}_5$.

Remark 4.9. With the notation in (2.2) consider the second order operator on $\mathbb{S}^7$:

$$L_5 := -X(A_6)^2 - X(A_7)^2 = \Delta_{\mathbb{S}^7} - \Delta_5^{\text{sub}}.$$

Since each vector field $X(A_j)$ is induced by a flow of isometries it commutes with the Laplace-Beltrami operator $\Delta_{\mathbb{S}^7}$ on $\mathbb{S}^7$. From this it follows that

$$[L_5, \Delta_5^{\mathrm{sub}}] = [L_5, \Delta_{\mathbb{S}^7} - L_5] = 0.$$

Moreover, the function C in Lemma 4.8 is the symbol of L_5 and hence Poisson commutes with the subriemannian Hamiltonian $H_{5,\mathrm{sub}}$. By this observation one obtains a second proof of Lemma 4.8. ♦

As before, if $x(0)$ and $\xi(0)$ are orthogonal, then $\langle x, \xi\rangle_{\mathbb{R}^8} \equiv 0$ and we obtain the reduces system of equations

$$\begin{cases} \dot{\xi} &= -q + \langle x, A_6\xi\rangle_{\mathbb{R}^8} A_6\xi + \langle x, A_7\xi\rangle_{\mathbb{R}^8} A_7\xi, \\ \dot{x} &= \xi - \langle \xi, A_6x\rangle_{\mathbb{R}^8} A_6x - \langle \xi, A_7x\rangle_{\mathbb{R}^8} A_7x. \end{cases} \tag{4.6}$$

5. Open problems

Above we have obtained solutions to the system $(HS)_j$ and first integral of the subriemannian geodesic flow in some special cases, cf. Proposition 4.4, 4.6 or Lemma 4.8. We will postpone a more systematic discussion of the geodesic equations to a future work. Finally, we mention some open problems in the analysis of the subriemannian geodesic flow on $\mathbb{S}^7$ described in this paper. For $j = 4, 5, 6$:

(A) Derive explicit solutions to $(HS)_j$. In particular, given two points $A, B \in \mathbb{S}^7$, can we find a normal subriemannian geodesic explicitly induced by $(HS)_j$ and connecting A and B? A solution to this problem may, at least locally, lead to an explicit form of the Carnot-Carathéodory distance on $\mathbb{S}^7$ with respect to $(HS)_j$. We mention that such an expression could be also derived from the small time expansion of the heat kernels for the corresponding subelliptic sub-Laplacians (2.7). However, to our knowledge, an explicit expression of these heat kernels is unknown in the cases $j \neq 6, 7$.

(B) Decide whether the subriemannian geodesic flow induced by $(HS)_j$ is *completely integrable* in the sense of Liouville. In particular, can we find seven, linearly independent, and Poisson commuting first integrals explicitly?

(C) Compare the subriemannian structures on $\mathbb{S}^7$ considered in the present paper with the one induced by the quaternionic Hopf fibration [14, 15, 16].

Acknowledgements The first author acknowledges support through the DFG project BA 3793/6-1 in the framework of the SPP 2026 *Geometry at Infinity*. The second author is partially supported by Grant for Basis Science Research Projects from The Sumitomo Foundation and by JSPS KAKENHI Grant-in-Aid for Young Scientists Grant Number 19K14540.

References

[1] J.F. Adams, *Vector fields on spheres*, Ann. of Math. 75 (1962), 603-632.

[2] A. Agrachev, U. Boscain, J.-P. Gauthier, F. Rossi, *The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups*. J. Funct. Anal. 256 (2009), no. 8, 2621 - 2655.

[3] W. Bauer, K. Furutani, C. Iwasaki, *Trivializable sub-Riemannian structures on spheres*, Bull. Sci. math. 137 (2013), 361-385.

[4] A. V. Brailov, *Construction of completely integrable geodesic flows on compact symmetric spaces*, Math. USSR Izv. 29 19-31, (1987).

[5] A. V. Bolsinov, B. Jovanović, *Integrable geodesic flows on Riemannian manifolds: construction and obstructions*, Contemporary geometry and related topics, 57 - 103, World Sci. Publ., River Edge, NJ, 2004.

[6] A. V. Bolsinov, *Integrable geodesic flows on Riemannian manifolds*, (Russian) Sovrem. Mat. Prilozh. No. 1, Topol., Anal. Smezh. Vopr. (2003), 19 - 32; translation in J. Math. Sci. (N.Y.) 123 (2004), no. 4, 4185 - 4197.

[7] O. Calin, D.-C. Chang, and I. Markina *Subriemannian Geometry on the Sphere* S^3, Canad. J. Math. Vol. 61 (4) (2009), 721 - 739.

[8] W.L. Chow *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. 117(1) (1939), 98 - 105.

[9] C. E. Durán, *On the complete integrability of the geodesic flow of manifolds all of whose geodesics are closed*, Ergod. Th. & Dynam. Sys. 17 (1997), 1359 - 1370.

[10] L. Hörmander, *Hypo-elliptic second order differential equations*, Acta Math. 119 (1967), 147 - 171.

[11] L. Hörmander, *The analysis of partial differential operators I*, Springer 1990.

[12] I. M. James, *The topology of Stiefel manifolds*, London Math. Soc. Lecture Note Ser., vol. 24, Cambridge Univ. Press, 1977.

[13] I. Markina, *Geodesics in geometry with constraints and applications*, in: Quantization, PDEs, and geometry, 153 - 314, Oper. Theory Adv. Appl., 251, Adv. Partial Differ. Equ. (Basel), Birkhäuser/Springer, Cham, 2016.

[14] I. Markina, M. Godoy Molina, *Sub-Riemannian geodesics and heat operator on odd dimensional spheres*, Anal. Math. Phys. 2, (2012), 123 - 147.

[15] I. Markina, M. Godoy Molina, *Sub-Riemannian geometry of parallelizable spheres*, Rev. Mat. Iberoamericana 27 (2011), no. 3, 997 - 1022.

[16] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, 91. AMS, Providence, RI, 2002.

[17] P.K. Rashevskii, *About connecting two points of complete nonholonomic space by admissible curve*, Uch. Zapiski ped. inst. Libknexta 2, (1938), 83 - 94 (in Russian).

Wolfram Bauer
Institut für Analysis, Leibniz Universität Hannover
Welfengarten 1, 30167 Hannover, Germany
E-mail address: bauer@math.uni-hannover.de

Daisuke Tarama
Department of Mathematical Sciences, Ritsumeikan University,
1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577, Japan
E-mail address: dtarama@fc.ritsumei.ac.jp