Realizing the Teichmüller space as a symplectic quotient

Tobias Diez
Delft University of Technology

and

Tudor S. Ratiu
Shanghai Jiao Tong University

Abstract. Given a closed surface endowed with a volume form, we equip the space of compatible Riemannian structures with the structure of an infinite-dimensional symplectic manifold. We show that the natural action of the group of volume-preserving diffeomorphisms by push-forward has a group-valued momentum map that assigns to a Riemannian metric the canonical bundle. We then deduce that the Teichmüller space and the moduli space of Riemann surfaces can be realized as symplectic orbit reduced spaces.

Key words: symplectic structure on moduli spaces of geometric structures, infinite-dimensional symplectic geometry, momentum maps, differential characters, Teichmüller space
MSC(2010): 53D20, (58D27, 53C08, 32G15)

1 Introduction

In their seminal work, Atiyah and Bott [AB83] have shown that the curvature yields a momentum map for the action of the group of gauge transformations on the space of connections on a principal bundle over a closed surface. Thus, the moduli space of flat connections, and the related moduli space of central Yang–Mills connections, are realized as symplectic reduced spaces. This observation has inspired a lot of research to reformulate problems in differential geometry in the language of infinite-dimensional symplectic geometry.

For example, Donaldson [Don03] observed that the scalar curvature produces a momentum map $\mathcal{J}_{\mathrm{ex}}$ for the action of the group $\mathrm{Diff}_{\mu,\mathrm{ex}}(M)$ of *exact* volume-preserving diffeomorphisms of a surface (M, μ) on the space $\mathcal{M}_\mu$ of all compatible Riemannian structures. Even though the symplectic quotient $\mathcal{J}_{\mathrm{ex}}^{-1}(c)/\mathrm{Diff}_{\mu,\mathrm{ex}}(M)$ is related to the Riemann moduli space, they do not coincide because the latter is the quotient by the full group $\mathrm{Diff}_\mu(M)$ of all volume-preserving diffeomorphisms. According to Donaldson [Don03, p. 181], the "difficulty is that there is no way to extend the moment[um map] $\mathcal{J}_{\mathrm{ex}}$ to an equivariant moment[um] map for the full action of $\mathrm{Diff}_\mu(M)$". In this note, we will show that the action of $\mathrm{Diff}_\mu(M)$ admits a momentum map in a generalized sense. This result allows us to realize the Riemann moduli space as a symplectic orbit quotient. The main result is the following.

Main Theorem *Let M be a closed surface endowed with a volume form μ. The space $\mathcal{M}_\mu$ of Riemannian metrics on M compatible with μ is endowed with the symplectic form*

$$\Omega_g(h_1, h_2) = -\frac{1}{2}\int_M \mathrm{Tr}\Big(\big(g^{-1}h_1\big)\big(g^{-1}\mu\big)\big(g^{-1}h_2\big)\Big)\,\mu\,, \tag{1.1}$$

where $g \in \mathcal{M}_\mu$ and $h_1, h_2 \in \mathrm{T}_g\mathcal{M}_\mu$. The left action of $\mathrm{Diff}_\mu(M)$ on $\mathcal{M}_\mu$ by push-forward preserves Ω and has a group-valued momentum map

$$\mathcal{J} \colon \mathcal{M}_\mu \to \hat{\mathrm{H}}^2(M, \mathrm{U}(1)), \quad g \mapsto \mathrm{K}_g M, \tag{1.2}$$

where $\hat{\mathrm{H}}^2(M, \mathrm{U}(1))$ is the Abelian group of gauge equivalence classes of circle bundles with connection over M and $\mathrm{K}_g M$ is the canonical circle bundle. ♢

We need to explain the terminology and notation of the theorem.

1. A Riemannian metric g on M is *compatible* with the volume form μ on M if the volume form induced by g coincides with μ. For such metrics, we have $\nabla\mu = 0$, where ∇ is the Levi-Civita connection defined by g.

   The space $\mathcal{M}_\mu$ of Riemannian metrics compatible with μ is identified with the space of sections of the associated bundle $\mathrm{L}M \times_{\mathrm{SL}} \big(\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)\big)$, where $\mathrm{L}M$ denotes the $\mathrm{SL}(2,\mathbb{R})$-frame bundle induced by the volume form μ. As such, $\mathcal{M}_\mu$ naturally comes with the structure of an infinite-dimensional Fréchet manifold. The space of g-trace-free symmetric covariant 2-tensors is the tangent space to $\mathcal{M}_\mu$ at g. In particular, in formula (1.1), h_1 and h_2 are trace-free symmetric covariant 2-tensors.

2. Given a symmetric covariant 2-tensor h and a Riemannian metric g on M, the notation $g^{-1}h$ is the $(1,1)$-tensor defined by $(g^{-1}h)(\alpha, X) = h(g^{-1}\alpha, X)$ for a 1-form $\alpha \in \Omega^1(M)$ and a vector field $X \in \mathfrak{X}(M)$, where $g^{-1} \colon \Omega^1(M) \to \mathfrak{X}(M)$ is the isomorphism induced by the Riemannian metric. In coordinates, this amounts to the operation of raising the first index by the Riemannian metric g, i.e., $(g^{-1}h)^i{}_j = g^{ik}h_{kj}$ with $[g^{ij}] = [g_{ij}]^{-1}$.

   Hence, in (1.1), the integrand is the trace of the product of three matrices, i.e., $\mathrm{Tr}\Big((g^{-1}h_1)\,(g^{-1}\mu)\,(g^{-1}h_2)\Big) = (h_1)^i{}_j\,\mu^j{}_k\,(h_2)^k{}_i\,.$

3. Let I be the almost complex structure on M induced by the symplectic form μ and the Riemannian metric g on M. The complex line bundle $\Lambda^{1,0}M$ of holomorphic forms is called the *canonical line bundle*. The form μ is a non-vanishing section of $\Lambda^{1,1}M = \Lambda^{1,0}M \otimes \Lambda^{0,1}M$ and thus induces a Hermitian metric on $\Lambda^{1,0}M$. The associated Hermitian frame bundle $\mathrm{K}_g M$ is a principal circle bundle, called the *canonical circle bundle*. The Levi–Civita connection of g naturally induces a connection in $\mathrm{K}_g M$.

4. The set $\hat{\mathrm{H}}^2(M, \mathrm{U}(1))$ of gauge equivalence classes of circle bundles with connection is an Abelian group relative to the following addition introduced in Kobayashi [Kob56]. Let P and $\tilde{P}$ be two principal $\mathrm{U}(1)$-bundles, form their fiber product $P \times_M \tilde{P}$, and identify points which differ by the $\mathrm{U}(1)$-action $(p, \tilde{p}) \cdot z \colon = (p \cdot z, \tilde{p} \cdot z^{-1})$, for $p \in P$, $\tilde{p} \in \tilde{P}$, and $z \in \mathrm{U}(1)$. This defines a new principal $\mathrm{U}(1)$-bundle $P + \tilde{P} := (P \times_M \tilde{P})/\,\mathrm{U}(1)$, where the $\mathrm{U}(1)$-action on $P + \tilde{P}$ is translation on the first factor.

This operation is associative and commutative. The trivial bundle is the identity element, i.e., $P + (M \times \mathrm{U}(1))$ is isomorphic to P. Given a principal bundle P, denote by $-P$ the $\mathrm{U}(1)$-bundle having the same underlying bundle structure as P but carrying the opposite $\mathrm{U}(1)$-action $p * z := p \cdot z^{-1}$, where the right side is the given $\mathrm{U}(1)$-action on P. Then $P + (-P)$ is isomorphic to the trivial bundle. Connections A on P and $\tilde{A}$ on $\tilde{P}$ induce the connection $\mathrm{pr}_1^* A + \mathrm{pr}_2^* \tilde{A}$ on $P \times_M \tilde{P}$, where $\mathrm{pr}_1 \colon P \times_M \tilde{P} \to P$ and $\mathrm{pr}_2 \colon P \times_M \tilde{P} \to \tilde{P}$ are the projections on the two factors, which descends to a connection on $P + \tilde{P}$, denoted by $A + \tilde{A}$. The curvature of $A + \tilde{A}$ is the sum of the corresponding curvatures.

Moreover, $\hat{\mathrm{H}}^2(M, \mathrm{U}(1))$ is an Abelian *Fréchet Lie* group with Lie algebra $\Omega^1(M)/\mathrm{d}\Omega^0(M)$; see [BSS17, Appendix A].

5. The concept of a group-valued momentum map is inspired by the notion of a momentum map in Poisson geometry as introduced by Lu and Weinstein [Lu90; LW90] and will be discussed in detail below.

The curvature of the canonical bundle $\mathrm{K}_g M$ is given by $-S_g \mu$, where S_g denotes the scalar curvature of g. Hence, symplectic reduction at the subset $\mathrm{curv}^{-1}(\mu)$ of all bundles with constant curvature μ yields the Riemann moduli space:

$$\mathcal{J}^{-1}(\mathrm{curv}^{-1}(\mu))/\mathrm{Diff}_\mu(M) = \Big\{ g \in \mathcal{M}_\mu : \; S_g = -1 \Big\} / \mathrm{Diff}_\mu(M). \tag{1.3}$$

Note that, in contrast to classical symplectic reduction, we take the inverse image of a set and not just of a point. However, one can show that $\mathrm{Diff}_\mu(M)$ acts (infinitesimally) transitively on $\mathrm{curv}^{-1}(\mu)$, so that the reduction is a *symplectic orbit reduction* [OR03, Section 6.3].

Instead of taking the quotient with respect to $\mathrm{Diff}_\mu(M)$, we can also restrict attention to the connected component of the identity $\mathrm{Diff}_\mu(M)^\circ$. The action of $\mathrm{Diff}_\mu(M)^\circ$ is free and its momentum map is given by the same formula (1.2). Thus, the symplectic quotient with respect to the $\mathrm{Diff}_\mu(M)^\circ$-action is a smooth manifold that coincides with the Teichmüller space. Moreover, the expression (1.1) for the symplectic form on the space of Riemannian metrics implies that the reduced symplectic form is proportional to the Weil–Petersson symplectic form on the Teichmüller space.

Remark 1.1 In [DR], we consider the general setting given by a symplectic fiber bundle $F \to M$. Then the space of sections of F carries a natural symplectic structure induced by the fiber-symplectic structure on F and we determine the group-valued momentum for the action of the automorphism group of F. From this perspective, the Main Theorem is deduced as a special case of the theory developed in [DR]. The results of that paper also imply that the symplectic geometry of $\mathcal{M}_\mu$ and the momentum map $\mathcal{J}$ are largely determined by the finite-dimensional coadjoint orbit $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. It follows that the Teichmüller space has two close relatives corresponding to the hyperbolic and the parabolic coadjoint orbits of $\mathrm{SL}(2,\mathbb{R})$; we refer to [DR] for details. ♢

Remark 1.2 Besides the scalar curvature as a geometric datum, the group-valued momentum map $\mathcal{J}$ contains topological information in the form of the Chern class of M. Such discrete topological data cannot be encoded in classical momentum maps. In particular, the momentum map $\mathcal{J}_{\mathrm{ex}}$ for the action of the group $\mathrm{Diff}_{\mu,\mathrm{ex}}(M)$ of exact volume-preserving diffeomorphisms does not contain topological information.

On the other hand, it is a generic feature of group-valued momentum maps that they capture geometric as well as topological data. For example, in [DR] we show that the group-valued momentum map can recover the Liouville class of a Lagrangian embedding or the integral helicity of fluid configurations as additional topological information. ♢

2 Group-valued momentum maps

In order to handle the full groups of volume-preserving diffeomorphisms, we introduce the concept of a group-valued momentum map.

Our starting point is the notion of a momentum map in Poisson geometry as introduced by Lu and Weinstein [Lu90; LW90]. Let (G, ϖ_G) be a (finite-dimensional) Poisson Lie group with dual group G^* and let (M, ϖ_M) be a (finite-dimensional) Poisson manifold. Recall that a left action of G on M is called a Poisson action if the action map $G \times M \to M$ is a Poisson map, where $G \times M$ is endowed with the product Poisson structure $\varpi_G \times \varpi_M$. A smooth map $J\colon M \to G^*$ is called a *momentum map* of this action if

$$A^* + \varpi_M\left(\cdot, J^* A^l\right) = 0 \tag{2.1}$$

holds for all $A \in \mathfrak{g}$, where A^* denotes the fundamental vector field on M induced by the infinitesimal action of A and $A^l \in \Omega^1(G^*)$ is the left-invariant extension of A seen as a functional on $\mathfrak{g}^*$. If the Poisson structure ϖ_M is induced by a symplectic form ω on M, then (2.1) is equivalent to

$$A^* \lrcorner\, \omega + \langle A, \delta J \rangle = 0, \tag{2.2}$$

where $\delta J \in \Omega^1(M, \mathfrak{g}^*)$ is the left-logarithmic derivative of J defined by $(\delta J)_m(X_m) = J(m)^{-1} . \mathrm{T}_m J(X_m)$. Note that this equation does no longer use the fact that G is a Poisson Lie group. Indeed, this identity still makes sense if the momentum map is replaced by a smooth map $J\colon M \to H$ with values in an arbitrary Lie group H, as long as there is a duality between the Lie algebras of G and H. This observation leads to our generalization of Lu's momentum map.

A *dual pair of Lie algebras* (not necessarily finite-dimensional) consists of two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, which are in duality through a given (weakly) non-degenerate bilinear map $\kappa\colon \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$. Using notation stemming from functional analysis, we write the dual pair as $\kappa(\mathfrak{g}, \mathfrak{h})$. Two Lie groups G and H are said to be *dual* to each other if there exists a non-degenerate bilinear form $\kappa\colon \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$ relative to which the associated Lie algebras are in duality. We use the notation $\kappa(G, H)$ in this case.

Definition 2.1 Let M be G-manifold endowed with a symplectic form ω. A *group-valued momentum map* is a pair (J, κ), where $\kappa(G, H)$ is a dual pair of Lie groups and $J\colon M \to H$ is a smooth map satisfying

$$A^* \lrcorner\, \omega + \kappa(A, \delta J) = 0 \tag{2.3}$$

for all $A \in \mathfrak{g}$. ♢

We emphasize that the concept of a group-valued momentum map is a vast generalization of many notions of momentum maps appearing in the literature including circle-valued, cylinder-valued, and Lie algebra-valued momentum maps. Most of the well-known results about the classical momentum map (such as equivariance properties and the Bifurcation Lemma) generalize in a natural way to group-valued momentum maps; see [Die19; DR].

For our purposes, the following dual pair is of particular relevance.

Example 2.2 Let M be a compact manifold endowed with a volume form μ. The group $G = \mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms is a Fréchet Lie group with Lie algebra consisting of μ-divergence-free vector fields X on M:

$$\mathfrak{g} = \mathfrak{X}_\mu(M) \simeq \{X \in \mathfrak{X}(M) : \mathrm{d}(X \lrcorner \mu) = 0\}. \tag{2.4}$$

Hence, $\mathfrak{X}_\mu(M)$ can be identified with the space $\Omega^{\dim M - 1}_{\mathrm{cl}}(M)$ of closed sub-top forms so that $\mathfrak{h} := \Omega^1(M)/\,\mathrm{d}\Omega^0(M)$ is the regular dual with respect to the weakly non-degenerate integration paring

$$\kappa(X, [\alpha]) := \int_M (X \lrcorner \alpha)\, \mu. \tag{2.5}$$

We now observe that a 1-form α on M can be seen as a connection on the trivial principal circle bundle $M \times \mathrm{U}(1) \to M$. From this point of view, $\mathfrak{h}$ parametrizes gauge equivalence classes of connections on the trivial circle bundle. Thus, it is natural to think of it as the Lie algebra of the Abelian group $H := \hat{\mathrm{H}}^2(M, \mathrm{U}(1))$ of all principal circle bundles with connections, modulo gauge equivalence. This heuristic argument can be made rigorous using the theory of Cheeger–Simons differential characters [BB13]; see [DR]. Summarizing, we get a dual pair $\kappa\Big(\mathrm{Diff}_\mu(M), \hat{\mathrm{H}}^2(M, \mathrm{U}(1))\Big)$ of Lie groups. In other words, a group-valued momentum map for an action of $\mathrm{Diff}_\mu(M)$ takes values in $\hat{\mathrm{H}}^2(M, \mathrm{U}(1))$. ♢

3 Proof of the Main theorem

In the sequel, we will prove the Main Theorem by means of two lemmas, which compute the two terms in the momentum map relation (2.3).

Lemma 3.1 *In the setting of the Main Theorem, we have*

$$\Omega_g(X \,.\, g, h) = -\int_M X^i \Big(\mu_{ik} \nabla_j h^{kj}\Big)\, \mu\,, \tag{3.1}$$

♢

where $X \,.\, g = -\,\mathfrak{L}_X g$ *denotes the fundamental vector field induced by the action of the divergence-free vector field* $X \in \mathfrak{X}_\mu(M)$*, evaluated at* $g \in \mathcal{M}_\mu$*, and* $h \in \mathrm{T}_g\mathcal{M}_\mu$ *is a trace-free symmetric covariant* 2*-tensor.*

Proof. Let ∇ be the Levi–Civita connection associated to the metric g. Since g is parallel with respect to ∇, we have

$$(\mathfrak{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i \tag{3.2}$$

for every vector field X. Moreover, the tensor $\mu_{ik} h^k{}_j$ is symmetric in the indices i and j, because h is trace-free and thus Hamiltonian with respect to the symplectic form μ. Hence, we obtain

$$\begin{aligned} \int_M \mathrm{Tr}\Big(\Big(g^{-1}(\mathfrak{L}_X g)\Big)\Big(g^{-1}\mu\Big)\Big(g^{-1}h\Big)\Big)\,\mu &= \int_M \Big(\nabla^i X_j + \nabla_j X^i\Big)\mu^j{}_k h^k{}_i\, \mu \\ &= 2\int_M \Big(\nabla_j X^i\Big)\mu_{ik} h^{kj}\, \mu \\ &= -2\int_M X^i \mu_{ik} \nabla_j h^{kj}\, \mu\,, \end{aligned} \tag{3.3}$$

where we integrated by parts and used $\nabla\mu = 0$. □

Lemma 3.2 *In the setting of the Main Theorem, the map*

$$\mathcal{J}\colon \mathcal{M}_\mu \to \hat{\mathrm{H}}^2(M, \mathrm{U}(1)), \quad g \mapsto \mathrm{K}_g M, \tag{3.4}$$

has logarithmic derivative $(\delta\mathcal{J})_g\colon \mathrm{T}_g\mathcal{M}_\mu \to \Omega^1(M)/\,\mathrm{d}\Omega^0(M)$ *given by*

$$(\delta\mathcal{J})_g(h)_i = \mu_{ik}\nabla_j h^{kj} \mod \mathrm{d}\Omega^0(M), \tag{3.5}$$

where $g \in \mathcal{M}_\mu$ *and* $h \in \mathrm{T}_g\mathcal{M}_\mu$. ♢

Proof. This follows from the general results of [DR]. To keep this paper self-contained, we give here a direct proof in our particular case under the additional assumption that the first singular homology group of M is trivial.

We need to determine the derivative of the holonomy map of $\mathrm{K}_g M$ with respect to g. For this, let γ be a closed loop at $m \in M$. As $\mathrm{H}_1(M, \mathbb{Z})$ is trivial, there exists a smooth contraction $\Sigma\colon [0,1] \times [0,1] \to M$ of γ to its base point m. Let $\mathrm{Hol}_g(\gamma)$ be the holonomy of γ relative to the connection on the canonical bundle $\mathrm{K}_g M$ induced by the Levi–Civita connection of g. The Stokes theorem implies

$$\mathcal{J}(g)(\gamma) = \mathrm{Hol}_g(\gamma) = \exp \int_{[0,1]\times[0,1]} \Sigma^* \left(-S_g \mu\right) \equiv \exp\Big(- \int_\Sigma S_g \mu\Big), \tag{3.6}$$

because the connection on $\mathrm{K}_g M$ has curvature $-S_g\mu$. According to [MEF72, Lemma 2.4.1], the derivative of the map $S\colon g \mapsto S_g$ in the direction h is given by

$$\mathrm{T}_g S(h) = \triangle(h^i{}_i) + \nabla_i\nabla_j h^{ij} - R_{ij}h^{ij}. \tag{3.7}$$

As M is 2-dimensional, the Ricci curvature R_{ij} of g satisfies $R_{ij} = \frac{S_g}{2} g_{ij}$. Thus, the last term in (3.7) is proportional to the trace of h. However, if h is a tangent vector at g to $\mathcal{M}_\mu$, then its trace vanishes and we then get

$$\mathrm{T}_g S(h) = \nabla_i\nabla_j h^{ij}. \tag{3.8}$$

Hence, for the logarithmic derivative of the holonomy of γ, we obtain

$$\Big(\delta\,\mathrm{Hol}(\gamma)\Big)_g(h) = -\int_\Sigma \nabla_i\nabla_j h^{ij}\,\mu \tag{3.9}$$

for $g \in \mathcal{M}_\mu$ and $h \in \mathrm{T}_g\mathcal{M}_\mu$.

Since $\mathrm{H}^1(M,\mathbb{R}) = 0$, the exterior differential $\mathrm{d}\colon \Omega^1(M) \to \Omega^2(M)$ yields an isomorphism of $\Omega^1(M)/\,\mathrm{d}\Omega^0(M)$ with $\mathrm{d}\Omega^1(M) \subseteq \Omega^2(M)$. It suffices to show that the 1-form $\alpha_i = \mu_{ik}\nabla_l h^{kl}$ satisfies $\mathrm{d}\alpha = -\nabla_k\nabla_l h^{kl}\,\mu$, because then

$$\int_\gamma (\delta\mathcal{J})_g(h) = \Big(\delta\,\mathrm{Hol}(\gamma)\Big)_g(h) = \int_\Sigma \mathrm{d}\alpha = \int_\gamma \alpha \tag{3.10}$$

and so $(\delta\mathcal{J})_g(h) = \alpha \mod \mathrm{d}\Omega^0(M)$ as γ was an arbitrary closed loop. For this, note that every vector field Y^k satisfies

$$\nabla_i(Y^k\mu_{kj}) - \nabla_j(Y^k\mu_{ki}) = \Big(\mathrm{d}(Y \lrcorner\, \mu)\Big)_{ij} = (\mathfrak{L}_Y\mu)_{ij} = (\operatorname{div} Y)\,\mu_{ij} = (\nabla_k Y^k)\,\mu_{ij}\,. \tag{3.11}$$

Applying this identity with $Y^k = \nabla_l h^{kl}$, we find

$$\begin{aligned}(\mathrm{d}\alpha)_{ij} &= \nabla_i\alpha_j - \nabla_j\alpha_i \\ &= \nabla_i(\mu_{jk}\nabla_l h^{kl}) - \nabla_j(\mu_{ik}\nabla_l h^{kl}) \\ &= -(\nabla_k\nabla_l h^{kl})\,\mu_{ij}\end{aligned} \tag{3.12}$$

and the claim follows. □

References

[AB83] M. F. Atiyah and R. Bott. "The Yang-Mills equations over Riemann surfaces". *Philos. Trans. Roy. Soc. London Ser. A* 308.1505 (1983), pp. 523–615.

[BB13] C. Bär and C. Becker. "Differential characters and geometric chains" (2013), pp. 1–90. arXiv: `1303.6457 [math.DG]`.

[BSS17] C. Becker, A. Schenkel, and R. J. Szabo. "Differential cohomology and locally covariant quantum field theory". *Rev. Math. Phys.* 29.1 (2017). arXiv: `1406.1514 [hep-th]`.

[Die19] T. Diez. "Singular symplectic reduction for Fréchet group actions and the Cauchy problem for the Yang–Mills equation". PhD thesis. Universität Leipzig, 2019.

[DR] T. Diez and T. Ratiu. "Group-valued momentum maps for actions of automorphism groups". In preparation.

[Don03] S. K. Donaldson. "Moment maps in differential geometry". In: *Surveys in differential geometry.* Vol. 8. Surv. Differ. Geom. 2003, pp. 171–189.

[Kob56] S. Kobayashi. "Principal fibre bundles with the 1-dimensional toroidal group". *Tohoku Math. J.* 2nd ser. 8.1 (1956), pp. 29–45.

[Lu90] J.-H. Lu. "Multiplicative and affine Poisson structures on Lie groups". PhD thesis. University of California, Berkeley, 1990, p. 74.

[LW90] J.-H. Lu and A. Weinstein. "Poisson Lie groups, dressing transformations, and Bruhat decompositions". *J. Differential Geom.* 31.2 (1990), pp. 501–526.

[MEF72] J. E. Marsden, D. G. Ebin, and A. E. Fischer. "Diffeomorphism groups, hydrodynamics and relativity". In: *Proceedings of the Thirteenth Biennial Seminar of the Canadian Mathematical Congress Differential Geometry and Applications.* Vol. 1. 1972, pp. 135–279.

[OR03] J.-P. Ortega and T. S. Ratiu. *Momentum Maps and Hamiltonian Reduction.* Birkhäuser Boston, 2003.

Tobias Diez
Max Planck Institute for Mathematics in the Sciences,
04103 Leipzig, Germany,
Institute of Applied Mathematics,
Delft University of Technology,
2628 XE Delft, Netherlands
E-mail address: `T.Diez@tudelft.nl`
Supported by the NWO grant 639.032.734.

Tudor S. Ratiu
School of Mathematical Sciences,
Shanghai Jiao Tong University, Minhang District,
800 Dongchuan Road, 200240 China,

Section de Mathématiques, Université de Genève,
1211 Genève 4,
Ecole Polytechnique Fédéralede Lausanne,
1015 Lausanne, Switzerland
E-mail address: ratiu@sjtu.edu.cn, tudor.ratiu@epfl.ch
Partially supported by National Natural Science Foundation of China grant 11871334 and NCCR SwissMAP grant of the Swiss National Science Foundation.