Spectral geometry of Subriemannian structures on nilmanifolds

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Abstract. This paper is a survey on recent results on spectral geometry for the Sublaplacian on certain compact nilmanifolds. We determine the spectrum of the Sublaplacian on compact quotients of 2-step Carnot groups by a lattice and focus on a class of generalized H-type groups. We prove a Poisson summation formula relating the spectrum of the Sublaplacian and lengths of closed Subriemannian geodesics.

Key words: Sublaplacian, length spectrum, isospectrality, Poisson summation formula, normal geodesic, abnormal geodesic.
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1 Introduction

The classical Poisson summation formula on the torus $\Gamma\backslash\mathbb{R}^n$

$$\sum_{X\in\Gamma^*} e^{-4\pi^2\|X\|^2 t} = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{R}^n)}{(4\pi t)^{n/2}}\sum_{x\in\Gamma} e^{-\frac{\|x\|^2}{4t}}, \quad t>0 \tag{1.1}$$

can be interpreted as an equation that relates the spectrum of the Laplacian and the length spectrum of this manifold.

Hubert Pesce [10] has shown that this formula can be generalized to the case of the Heisenberg manifold. Colin de Verdière [4] used the heat kernel to show that generically, the Laplace spectrum determines the length spectrum. Duistermaat and Guillemin [5] have shown that the singularities of the wave trace $\mathrm{tr}(e^{-t\sqrt{\Delta}})$ are contained in the length spectrum.

In this paper we study the relation between these spectra for a class of equiregular Subriemannian manifolds, on which there is a geometrically defined operator, the so-called Sublaplacian. More precisely, we deal with Subriemannian manifolds of the form $\Gamma\backslash\mathbb{G}$, where $\mathbb{G}$ is a 2-step Carnot group and Γ is a lattice in this group. First we present examples of isospectral (with respect to the Sublaplacian) but non-diffeomorphic manifolds and then study the relation between the Sublaplacian spectrum and lengths of closed Subriemannian geodesics.

In the Subriemannian setting, there are two kinds of geodesics: normal and abnormal. It is interesting to study the effect of the presence of abnormal geodesics. We aim to determine the lengths of such closed geodesics and ask whenever they occur in a summation formula such as (1.1).

Based on an explicit calculation of the Sublaplacian spectrum on $\Gamma\backslash\mathbb{G}$, we prove a formula similar to (1.1) in the Subriemannian setting for H-type Lie groups with arbitrary lattice and more specifically for generalized H-type groups with a standard lattice. In the latter case abnormal geodesics are present.

Furthermore, based on a heat trace formula and the classification of pseudo H-type groups, we construct pairs of isospectral but non-diffeomorphic nilmanifolds with respect to the Sublaplacian. This is a joint work with W. Bauer, K. Furutani and C. Iwasaki.

2 Preliminaries

Definition 2.1. *A Subriemannian manifold is a triple $(M, \mathcal{H}, g)$, where M is a smooth, connected manifold, $\mathcal{H}$ is a smooth vector distribution of rank $k \leq n = dim(M)$ and g is a smooth metric on $\mathcal{H}$.*

Given a smooth measure μ on M, the Sublaplacian (corresponding to μ) is defined as

$$\Delta_{sub} := -\mathrm{div}_\mu \circ \mathrm{grad}_{\mathcal{H}} \quad \text{ on } C^\infty(M).$$

Here $grad_{\mathcal{H}}$ denotes the horizontal gradient. Recall that under additional assumptions, there are natural choince for μ which leads to the notion of an intrinsic Sublaplacian.
Based on the end-point map and the Subriemannian Hamiltonian there are different kinds of Subriemannian geodesics: regular, singular and normal geodesics. If the distribution $\mathcal{H}$ is bracket generating, then by a theorem of Hörmander the Sublaplacian is hypoelliptic and if the manifold is compact, the unique self-adjoint extension of the Sublaplacian has discrete spectrum [6].

2-step Carnot groups: We write $\mathbb{R}^{r,s}$ for the Euclidean space $\mathbb{R}^{r+s}$ equipped with the non-degenerate scalar product

$$\langle x,\, y\rangle_{r,s} := \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}.$$

Let $C\ell_{r,s}$ denote the Clifford algebra induced by this scalar product (see [9]). We call a $C\ell_{r,s}$-module V *admissible*, if there is a non-degenerate bilinear form (= scalar product) $\langle \bullet, \bullet \rangle_V$ on V satisfying the following conditions:

(a) There is a *Clifford module action* $J : C\ell_{r,s} \times V \to V : (z, X) \mapsto J_z X$, i.e.

$$J_z J_{z'} + J_{z'} J_z = -2\langle z, z'\rangle_{r,s} I \quad \textit{for all} \quad z, z' \in \mathbb{R}^{r,s}.$$

(b) For all $z \in \mathbb{R}^{r,s}$ the map J_z is skew-symmetric on V with respect to $\langle \bullet, \bullet \rangle_V$.

The pseudo H-type group $\mathbb{G}_{r,s}(V)$ is defined as the simply connected 2-step nilpotent Lie group associated to the nilpotent Lie algebra $\mathcal{N}_{r,s}(V) = V \oplus_\perp \mathbb{R}^{r,s}$ with center $\mathbb{R}^{r,s}$ and Lie brackets given by:

$$\big\langle J_z(X),\, Y \big\rangle_V = \big\langle z, [X, Y] \big\rangle_{r,s}, \quad z \in \mathbb{R}^{r,s}, \text{ and } X, Y \in V.$$

We denote by $\mathbb{G}_{r,s}$ the pseudo H-type groups constructed from a minimal admissible module.
In the same way we define generalized H-type groups by the condition

$$J_z J_{z'} + J_{z'} J_z = -2\langle z, z'\rangle_{\mathbb{R}^d} S^2 \quad \textit{for all} \quad z, z' \in \mathbb{R}^d.$$

where S is a symmetric, non-negative linear map on $\mathbb{R}^{2N+s}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ denotes the Euclidean inner product in $\mathbb{R}^d$.
If $S = \mathrm{Id}_{2N,s}$, where $\mathrm{Id}_{2N,s}$ is the diagonal matrix of order $2N+s$ with the first $2N$ entries equals 1 and the remaining s entries equals 0, we denote the associated generalized H-type group with $\mathbb{G}^d_{N,s}$.
Every 2-step Carnot group $\mathbb{G}$ can be endowed with a natural left invariant Subriemannian structure.

3 Results

3.1 Pairs of isospectral, but non-diffeomorphic nilmanifolds

From an integral basis $\{X_i, Z_k\}$ of $\mathcal{N}_{r,s}(V) = V \oplus \mathbb{R}^{r,s}$ we define a *lattice* in the pseudo H-type group $\mathbb{G}_{r,s}(V)$ by

$$\Gamma_{r,s}(V) := \exp\left\{ \sum_{m_i \in \mathbb{Z}} m_i X_i + \frac{1}{2} \sum_{k_j \in \mathbb{Z}} k_j Z_j \right\}.$$

We call $\Gamma_{r,s}(V)$ (resp. $\Gamma_{r,s}$) a *standard integral lattice* in $\mathbb{G}_{r,s}(V)$ (resp. $\mathbb{G}_{r,s}$).
If we denote with $\mathbb{A}$ the center of the group $\mathbb{G}_{r,s}(V)$, then an element $\mathbf{n}$ in the dual lattice $\left[\Gamma_{r,s} \cap \mathbb{A}\right]^*$ can be expressed as

$$\mathbf{n} = 2\left(\sum_{i=1}^{r} m_i Z_i + \sum_{j=1}^{s} n_j Z_{r+j} \right) = 2(\mu + \nu) \quad \textit{where} \quad (m, n) \in \mathbb{Z}^{r,s}.$$

The heat trace of the Sublaplacian $\Delta_{\mathrm{sub}}^{\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)}$ on the nilmanifolds $\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)$ is given by decomposing the Sublaplacian into an infinite sum of elliptic operators $\mathcal{D}^{(n)}$ acting on some line bundles $E^{(n)}$:

Theorem 3.1 (W. Bauer, K. Furutani, C. Iwasaki, A. Laaroussi, [3],[2])**.**
We have

$$\mathbf{tr}\left(e^{-\Delta_{\mathrm{sub}}^{\Gamma_{r,s}(V)\backslash\mathbb{G}_{r,s}(V)}}\right) = \sum_{\mathbf{n} \in [\Gamma \cap \mathbb{A}]^*} \mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right).$$

Where
(1) If $\mathbf{n} = 0$*, then the trace of the operator* $e^{-t\mathcal{D}^{(0)}}$ *is given by*

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(0)}}\right) = \frac{1}{(2\pi\, t)^N} \sum_{\ell \in \mathbb{Z}^{2N}} e^{-\frac{\|\ell\|^2}{2t}}. \tag{3.1}$$

(2) Assume that $\mathbf{n} \in \big[\Gamma_{r,s} \cap \mathbb{A}\big]^*$ *with*

$$\sum_{i=1}^{r} {m_i}^2 = \sum_{j=1}^{s} {n_j}^2,$$

and let $d_0 > 0$ *be the greatest common divisor of* $(\mu,\, \nu) = (m_1, \cdots, m_r, n_1, \cdots, n_s)$*. Define integers* m_i' *and* n_i' *through the equations* $m_i = {m_i}' d_0$ *and* $n_j = {n_j}' d_0$*. It holds*

$$\mathbf{tr}\Big(e^{-t\mathcal{D}^{(\mathbf{n})}}\Big) = \frac{1}{(\pi\, t)^{N/2}} \sum_{\ell \in \mathbb{Z}^{2N}} e^{-\frac{\|\mu\|^2\, \|\ell\|^2}{{d_0}^2\, t}} \left(\frac{2\|\mu\|}{\sinh\big(8\pi\, t\|\mu\|\big)}\right)^{N/2}. \tag{3.2}$$

(3) For $\mathbf{n} = 2(\mu + \nu)$ *with* $\|\mu\| \neq \|\nu\|$ *it holds*

$$\mathbf{tr}\Big(e^{-t\mathcal{D}^{(\mathbf{n})}}\Big) = 2^N \left(\frac{\|\mu\|^2 - \|\nu\|^2}{\sinh\{4\pi\, t(\|\mu\| + \|\nu\|)\}\sinh\{4\pi\, t(\|\mu\| - \|\nu\|)\}}\right)^{N/2.}. \tag{3.3}$$

From the above heat trace formula we see that the manifolds $\Gamma_{r,s}\backslash\mathbb{G}_{r,s}$ and $\Gamma_{s,r}\backslash\mathbb{G}_{s,r}$ are isospectral with respect to the Sublaplacian if the dimension of their admissible modules coincide.
If there is a diffeomorphism between nilmanifolds $\Gamma\backslash\mathbb{G}$ and $\Gamma'\backslash\mathbb{G}'$, then their fundamental groups $\pi_1(\Gamma\backslash\mathbb{G}) \simeq \Gamma$ and $\pi_1(\Gamma'\backslash\mathbb{G}') \simeq \Gamma'$ are isomorphic and we can extend an isomorphism from the lattices to the whole groups. Using the classification of pseudo H-type algebras [7] :

8	$\cong$	$\cong$	$\cong$	$\ncong$					
7	$\ncong$	$\ncong$	$\ncong$	$\ncong$					
6	$\ncong$	$\cong$	$\cong$	$\ncong$	$\cong$				
5	$\ncong$	$\cong$	$\cong$	$\ncong$	$\cong$				
4	$\cong$	$\ncong$	$\ncong$	$\ncong$		$\cong$	$\cong$		
3	$\ncong$	$\ncong$	$\ncong$		$\ncong$	$\ncong$	$\ncong$	$\ncong$	$\ncong$
2	$\cong$	$\ncong$		$\ncong$	$\ncong$	$\cong$	$\cong$	$\ncong$	$\cong$
1	$\cong$		$\ncong$	$\ncong$	$\ncong$	$\cong$	$\cong$	$\ncong$	$\cong$
0		$\cong$	$\cong$	$\ncong$	$\cong$	$\ncong$	$\ncong$	$\ncong$	$\cong$
s/r	0	1	2	3	4	5	6	7	8

Classification of pseudo H-type Lie algebras defined via minimal admissible modules.

Here $\cong$ and $\ncong$ mean that the associated Lie algebras are isomorphic or not.
We can detect pairs of isospectral, but non-diffeomorphic nilmanifolds.

Theorem 3.2 (W. Bauer, K. Furutani, C. Iwasaki, A. Laaroussi [3])**.**
Denote by $N_{r,s}$ *the nilmanifold* $\Gamma_{r,s}\backslash\mathbb{G}_{r,s}$ *constructed from a minimal admissible module. Then the following pairs of nilmanifolds are isospectral and non-diffeomorphic:*

1. $(N_{r,s}, N_{s,r})$ *for* $r \equiv 3 \bmod 8$ *and* $s \equiv 1, 2, 7 \bmod 8$.
2. $(N_{r+4k,s+4k}, N_{s+4k,r+4k})$ *for* $(r, s) \in \{(3, 1), (3, 2), (3, 7)\}$ *and* $k \in \mathbb{N}_0$.

3.2 Poisson summation formula for the Sublaplacian

Let $\mathbb{G}^d_{N,s}$ be a generalized H-type group and let Γ be a uniform lattice in $\mathbb{G}^d_{N,0} =: \mathbb{G}^d_N$ or the standard lattice for $s > 0$. The set of lengths of closed normal Subriemannian geodesics $L(\Gamma\backslash\mathbb{G}^d_{N,s})$ in $\Gamma\backslash\mathbb{G}^d_{N,s}$ is given by

Theorem 3.3 (A. Laaroussi [8])**.**

1. *If $s = 0$, then*

$$L(\Gamma\backslash\mathbb{G}^d_N) = \{\|X\| : \ X \in \Gamma_{\mathcal{H}}\} \cup \{\sqrt{4k\pi\|V\|} : \ k \in \mathbb{N}, V \in \Gamma_{\mathcal{V}}\}.$$

2. *If $s > 0$, then*

$$L(\Gamma\backslash\mathbb{G}^d_{N,s}) = \{\|X\| : \ X \in \Gamma_{\mathcal{H}}\} \cup \{\sqrt{\|X\|^2 + 4k\pi\|V\|} : \ k \in \mathbb{N}, X \in \Gamma_d, V \in \Gamma_{\mathcal{V}}\backslash\{0\}\}.$$

Here $\Gamma_{\mathcal{H}}$, $\Gamma_{\mathcal{V}}$ and Γ_d denote the induced horizontal, vertical and degenerate lattices (see [8]).

From the explicit spectral data of the Sublaplacian we obtain the following Poisson summation formula:

Theorem 3.4. *Let $\mathbb{G}^d_N$ be an H-type group with uniform lattice Γ. We assume that d is odd. Then for the heat trace of the Sublaplacian on $\Gamma\backslash\mathbb{G}^d_N$ it holds:*

$$tr(e^{-t\Delta_{sub}}) = \sum_{w \in L\left(\Gamma\backslash\mathbb{G}^d_N\right)\cup\{0\}} \varphi_w\left(\frac{1}{t}\right) e^{-\frac{w^2}{4t}},$$

where for all $w \in L\left(\Gamma\backslash\mathbb{G}^d_N\right)$, φ_w is a polynomial and

$$\varphi_0\left(\frac{1}{t}\right) = \frac{2^d vol(\Gamma\backslash\mathbb{G}^d_N)}{(4\pi t)^{N+d}} \int_{\mathbb{R}^d} \left(\frac{\|\xi\|}{\sinh\|\xi\|}\right)^N d\xi$$

Similarly, if $s > 0$ we have:

Theorem 3.5 (A. Laaroussi [8])**.**
Let $\mathbb{G}^d_{N,s}$ be a generalized H-type group with standard lattice Γ. We assume that d is odd. Then for the heat trace of the Sublaplacian on $\Gamma\backslash\mathbb{G}^d_{N,s}$ it holds:

$$tr(e^{-t\Delta_{sub}}) = \sum_{w \in L\left(\Gamma\backslash\mathbb{G}^d_{N,s}\right)\cup\{0\}} \varphi_w\left(\frac{1}{t}\right) e^{-\frac{w^2}{4t}},$$

where for all $w \in L\left(\Gamma\backslash\mathbb{G}^d_{N,s}\right)$, φ_w is a polynomial and

$$\varphi_0\left(\frac{1}{t}\right) = \frac{2^d vol(\Gamma\backslash\mathbb{G}^d_{N,s})}{(4\pi t)^{N+s/2+d}} \int_{\mathbb{R}^d} \left(\frac{\|\xi\|}{\sinh\|\xi\|}\right)^N d\xi.$$

Here $N + s/2 + d$ is half the Hausdroff-dimension of the manifold considered as a metric space with the Carnot Caratheodory distance.

From the Poisson summation formula above we also obtain short-time asymptotic of the heat trace:

$$\operatorname{tr}(e^{-t\Delta_{sub}}) = \varphi_0\left(\frac{1}{t}\right) + O(t^\infty) \text{ as } t \to 0^+.$$

i.e. only the first term contributes to the short-time asymptotic and all length of closed geodesics contribute to the remainder-term. Furthermore, knowing the spectrum of the Sublaplacian we can entirely recover the lengths of closed Subriemannian geodesics.

References

[1] W. Bauer, K. Furutani, C. Iwasaki, *Spectral zeta function of the sub-Laplacian on two step nilmanifolds*, J. Math. Pures Appl. (9), 97 no. 3, 2012.

[2] W. Bauer, K. Furutani, C. Iwasaki, *Spectral zeta function on pseudo H-type nilmanifolds* , Indian Journal of Pure and Applied Mathematics, 46(4), 539-582, 2015.

[3] W. Bauer, K. Furutani, C. Iwasaki, A. Laaroussi, *Isospectral but non-diffeomorphic nilmanifolds attached to Clifford modules*, Preprint, 2019.

[4] Y. Colin de Verdière, *Spectre du Laplacien et longueurs des géodésiques périodiques. II*, Compositio Mathematica 27.2, 1973.

[5] J.J. Duistermaat, V.W. Guillemin, *On the spectrum of positive elliptic operators and periodic bicharacteristics.*, Fourier Integral Operators and Partial Differential Equations, Springer, 1975.

[6] V. Franceschi, D. Prandi, L. Rizzi, *On the essential self-adjointness of singular sub-Laplacians*, Potential Analysis, Springer, 2019.

[7] K. Furutani, I. Markina, *Complete classification of pseudo H-type algebras: I, II*, Geometriae Dedicata : 1-32, 2017.

[8] A. Laaroussi, *Spectrum of the Sublaplacian and lengths of closed Subriemannian geodesics on nilmanifolds*, Preprint, 2019.

[9] H. B. Lawson, M.L. Michelsohn, *Spin geometry* , Princeton university press, 2016.

[10] H. Pesce, *Une formule de Poisson pour les variétés de Heisenberg*, Duke Mathematical Journal 73.1, 1994.