Regular symmetries of differential-difference equations and Noether's conservation laws

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Abstract

In a recent paper [11], the author studied continuous symmetries of differential-difference equations and proposed generic symmetry prolongation formulae, which provide essential insights for extending Noether's theorem to differential-difference variational problems. In this note, we will review these results with several examples.

1 Introduction

Symmetries of differential equations, local transformations mapping a solution to another solution, play an important role in understanding solutions and integrability of differential equations. Let $x \in \mathbb{R}^p$ be the independent variables and $u \in \mathbb{R}^q$ be the dependent variables. Partial derivatives of u^α are written in the multi-index form $u^\alpha_{\mathbf{J}}$ where $\mathbf{J} = (j_1, j_2, \dots, j_p)$. For the differential case, each index j_i is a non-negative integer which denotes the number of derivatives with respect to x^i. Namely

$$u^\alpha_{\mathbf{J}} = \frac{\partial^{|\mathbf{J}|} u^\alpha}{\partial (x^1)^{j_1} \partial (x^2)^{j_2} \dots \partial (x^p)^{j_p}},$$

where $|\mathbf{J}| = j_1 + j_2 + \dots + j_p$. Consider a local one-parameter transformation with the following Taylor expansions about the parameter ε:

$$\begin{aligned} \widetilde{x} &= x + \varepsilon \xi(x,u) + O(\varepsilon^2), \\ \widetilde{u} &= u + \varepsilon \phi(x,u) + O(\varepsilon^2). \end{aligned} \tag{1}$$

Its prolongation to derivatives is directly obtained through the chain rule. For instance when $p = q = 1$,

$$\widetilde{u'} := \frac{D_x \widetilde{u}}{D_x \widetilde{x}}.$$

To calculate symmetries, one may alternatively study the corresponding infinitesimal generators

$$\mathbf{v} = \xi^i(x,u)\partial_{x^i} + \phi^\alpha(x,u)\partial_{u^\alpha}.$$

The Einstein summation convension is used in this note. The prolongation of an infinitesimal generator is related to the prolongation of a local transformation (1); it can be conveniently written using the characteristics $Q^\alpha := \phi^\alpha - \xi^i(D_i u^\alpha)$ as follows (see, e.g. [9])

$$\operatorname{pr}\mathbf{v} = \xi^i D_i + \sum_{\alpha,\mathbf{J}} (D_\mathbf{J} Q^\alpha) \frac{\partial}{\partial u^\alpha_\mathbf{J}}. \tag{2}$$

Here the total derivative with respect to x^i is defined by

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha,\mathbf{J}} u^\alpha_{\mathbf{J}+\mathbf{1}_i} \frac{\partial}{\partial u^\alpha_\mathbf{J}}$$

where $\mathbf{1}_i$ is the p-tuple with only one nonzero entry 1 in the i-th place. We also use the shorthand notation $D_\mathbf{J} = D_1^{j_1} D_2^{j_2} \ldots D_p^{j_p}$. Symmetries of a system of differential equations, written as

$$\mathcal{A} = \{F_k(x,[u]) = 0\}_{k=1}^l, \tag{3}$$

can be determined via the so-called linearized symmetry condition:

$$\operatorname{pr}\mathbf{v}(F_k(x,[u])) = 0, \text{ whenever } \{F_k(x,[u]) = 0\}_{k=1}^l \text{ holds}, \tag{4}$$

where $[u]$ is shorthand for u and finitely many of its partial derivatives. Many examples are available in, e.g. [9].

Since 1980s, a great deal of effort has been made in extending symmetry methods to discrete/difference equations, e.g. [1, 3–5, 10, 11, 14, 15]. In particular, Levi & Winternitz and their collaborators made great contributions in symmetry analysis for differential-difference equations, e.g. [6–8]. From now on, we will mainly be focused on DDEs and similar notations as above will be introduced. Let the multidimensional differential and difference variables $x \in \mathbb{R}^{p_1}$ and $n \in \mathbb{Z}^{p_2}$ play as independent variables and let $u \in \mathbb{R}^q$ be the dependent variables. We define derivatives and shifts simultaneously. The forward shift operator (or map) S is defined as

$$S_j : n \mapsto n + \mathbf{1}_j, \tag{5}$$

while its generalisation to a function $f(n)$ is

$$S_j : f(n) \mapsto f(S_j n). \tag{6}$$

The composite of shift operators using multi-index notation is given by $S_{\mathbf{J}_2} = S_1^{j_1} S_2^{j_2} \ldots S_{p_2}^{j_{p_2}}$, where $\mathbf{J}_2 = (j_1, j_2, \ldots, j_{p_2})$ is a p_2-tuple; hence both $S_{\mathbf{1}_j}$ and S_j are used to denote the forward shift. However, different from the differential multi-index $\mathbf{J}_1$ below, each index of $\mathbf{J}_2$ is an integer. The total derivative in the differential-difference sense is defined as

$$D_i = \partial_{x^i} + \frac{\partial u^\alpha}{\partial x^i} \partial_{u^\alpha} + \cdots + \sum_{\alpha,\mathbf{J}_1,\mathbf{J}_2} u^\alpha_{\mathbf{J}_1+\mathbf{1}_i;\mathbf{J}_2} \partial_{u^\alpha_{\mathbf{J}_1;\mathbf{J}_2}}.$$

Now we are ready to define derivatives and shifts of dependent variables with the following notation

$$u^{\alpha}_{\mathbf{J}_1;\mathbf{J}_2} = D_{\mathbf{J}_1} S_{\mathbf{J}_2} u^{\alpha} = S_{\mathbf{J}_2} D_{\mathbf{J}_1} u^{\alpha}.$$

Namely, the first subindex indicates derivatives while the second subindex indicates shifts. We still use $[u]$ to denote u and finitely many of its derivatives and shifts for differential-difference equations (DDEs).

Consider a vector field

$$\mathbf{v} = \xi^i(x,n,u)\partial_{x^i} + \phi^{\alpha}(x,n,u)\partial_{u^{\alpha}}, \tag{7}$$

which generates a symmetry group for a system of DDEs

$$\mathcal{A} = \{F_k(x,n,[u]) = 0\}_{k=1}^{l}. \tag{8}$$

In this note, we are interested in its prolongations $\operatorname{pr}\mathbf{v}$ which will be used to determine symmetries of DDEs via the linearized symmetry condition

$$\operatorname{pr}\mathbf{v}(F_k(x,n,[u])) = 0, \text{ whenever } \{F_k(x,n,[u]) = 0\}_{k=1}^{l} \text{ holds.} \tag{9}$$

2 Prolongations of infinitesimal generators for DDEs

In the literature, there have been various prolongation formulae used to calculate symmetries of DDEs, e.g., [6–8]. In this note, we will mainly review the prolongation formulae proposed in [11]. In the next section, we will show several illustrative examples, including integrable DDEs of the Volterra type, the Toda lattice and the two-dimensional Toda lattice.

In [11], author of this note proved the prolongation formulae analytically for continuous symmetries of DDEs and in particular presented two extreme cases, depending on how one would define the prolongation $\widetilde{u^{\alpha}_{\mathbf{J}_1;\mathbf{J}_2}}$ for variables $(\widetilde{x}, n, \widetilde{u})$ after the transformation. In particular, n is viewed as a parameter as it is discrete and invariant. In general, the commutativity of derivative and shift breaks, that is $\widetilde{D}S \neq S\widetilde{D}$ where $\widetilde{D}$ is the total derivative with respect to new variables $\widetilde{x}$. These two extreme cases are summarized below, i.e. Theorem 3 and Theorem 4 in [11].

Case $\widetilde{D}S$. The prolongation formula reads

$$\operatorname{pr}\mathbf{v} = \xi^i D_i + \sum_{\alpha,\mathbf{J}_1,\mathbf{J}_2} \left(D_{\mathbf{J}_1} Q^{\alpha}_{\mathbf{J}_2}\right) \partial_{u^{\alpha}_{\mathbf{J}_1;\mathbf{J}_2}}, \tag{10}$$

where

$$Q^{\alpha}_{\mathbf{J}_2} := S_{\mathbf{J}_2}\phi^{\alpha} - \xi^i u^{\alpha}_{\mathbf{1}_i;\mathbf{J}_2}.$$

Case $S\widetilde{D}$**.** Now the prolongation formula can be expressed in terms of the functions $Q^\alpha = \phi^\alpha(x,n,u) - \xi^i(x,n,u)D_i u^\alpha$ as

$$\operatorname{pr}\mathbf{v} = \xi^i\partial_{x^i} + \sum_{i,\mathbf{I}}(S_\mathbf{I}\xi^i)\left(D_{i;\mathbf{I}} - \partial_{x^i}\right) + \sum_{\alpha,\mathbf{J}_1,\mathbf{J}_2}(D_{\mathbf{J}_1}S_{\mathbf{J}_2}Q^\alpha)\partial_{u^\alpha_{\mathbf{J}_1;\mathbf{J}_2}}, \tag{11}$$

where

$$D_{i;\mathbf{I}} := \partial_{x^i} + \frac{\partial u^\alpha_{\mathbf{0};\mathbf{I}}}{\partial x^i}\partial_{u^\alpha_{\mathbf{0};\mathbf{I}}} + \cdots + \sum_{\alpha,\mathbf{J}_1} u^\alpha_{\mathbf{J}_1+\mathbf{1}_i;\mathbf{I}}\partial_{u^\alpha_{\mathbf{J}_1;\mathbf{I}}}.$$

Note that both formulae can equivalently be written as an evolutionary form when $\xi = \xi(x)$, which is called a *regular symmetry* or a *regular vector field* in [11] to distinguish from an intrinsic one for $\xi = \xi(x,u)$.

To make things easier, we will only consider regular symmetries with infinitesimal generators $\mathbf{v} = \xi^i(x)\partial_{x^i} + \phi^\alpha(x,n,u)\partial_{u^\alpha}$ or higher-order cases, when their prolongations can be equivalently written as evolutionary representations

$$\operatorname{pr}\mathbf{v} = \xi^i D_i + \sum_{\alpha,\mathbf{J}_1,\mathbf{J}_2}(D_{\mathbf{J}_1}S_{\mathbf{J}_2}Q^\alpha)\partial_{u^\alpha_{\mathbf{J}_1;\mathbf{J}_2}}, \tag{12}$$

where the characteristics are defined as $Q^\alpha = \phi^\alpha - \xi^i D_i u^\alpha$ again. A regular vector field generates a group of (divergence) **variational symmetries** for a differential-difference Lagrangian $L(x,n,[u])$ if there exists a $(p_1;p_2)$-tuple $(P_1(x,n,[u]);P_2(x,n,[u]))$ subject to

$$\operatorname{pr}\mathbf{v}(L) + L(D_i\xi^i) = \operatorname{Div}P_1 + \operatorname{Div}^\triangle P_2. \tag{13}$$

Noether's theorem assures that the symmetry characteristics Q are also characteristics of conservation laws for the corresponding Euler–Lagrange equations. Namely there exists another $(p_1;p_2)$-tuple $(P_1(x,n,[u]);P_2(x,n,[u]))$ such that

$$\operatorname{Div}P_1 + \operatorname{Div}^\triangle P_2 = Q^\alpha \mathbf{E}_\alpha(L), \tag{14}$$

where the **differential-difference Euler operator E** is defined by

$$\mathbf{E}_\alpha := \sum_{\mathbf{J}_1,\mathbf{J}_2}(-D)_{\mathbf{J}_1}S_{-\mathbf{J}_2}\frac{\partial}{\partial u^\alpha_{\mathbf{J}_1;\mathbf{J}_2}}. \tag{15}$$

Here $(-D)_{\mathbf{J}_1} = (-1)^{|\mathbf{J}_1|}D_{\mathbf{J}_1}$ is the adjoint of $D_{\mathbf{J}_1}$. In the next section, we will recall several examples.

Note that in [11], Noether's theorem was only proved for regular symmetries of differential-difference variational problems. In fact, regular symmetries of DDEs have actually been well understood for quite some time. For general symmetries, we also believe that an evolutionary representative should exist although a clear explanation is not yet available.

In [8], the authors proposed an approach via the semi-continuum limit of symmetry prolongations for purely difference equations, in particular for $1+1$-dimensional DDEs. We are seeking for an analytic and systematic proof of an evolutionary representative for general symmetries, which should be free from the differential or purely difference pictures. We will also extend Noether's theorem to include all variational symmetries as well as to prove Noether's second theorem for DDEs; these results will be presented in [12].

3 Illustrative examples

In this section, we will derive regular symmetries of several DDEs using the linearized symmetry condition as well as regular symmetries of differential-difference variational problems and conservation laws of the underlying Euler–Lagrange equations. We will present the main results without providing computational details; many of the examples were included in [11]; see also [8].

3.1 Volterra-type equations

The first family of equations we consider are the so-called Volterra-type equations

$$u' = f(x, n, u_{-1}, u, u_1).$$

Now $p_1 = p_2 = q = 1$; let x and n be the continuous and discrete independent variables respectively and let u be the dependent variable.

One of the simplest examples is the Volterra equation

$$u' = u(u_1 - u_{-1}). \tag{16}$$

The following regular infinitesimal generators are obtained (e.g. [8, 11])

$$\mathbf{v}_1 = -x\partial_x + u\partial_u, \qquad \mathbf{v}_2 = \partial_x,$$

where $c(n)$ is an arbitrary function. Introduce a new variable via

$$u = \exp(v_1 - v_{-1}), \tag{17}$$

and we have a new differential-difference equation

$$v_1' - v_{-1}' = \exp(v_2 - v) - \exp(v - v_{-2}), \tag{18}$$

which admits a differential-difference Lagrangian

$$L = v(v_1' - v') + \exp(v_2 - v). \tag{19}$$

In the following table, we show the conservation laws

$$D_tP_1 + (S - \mathrm{id})P_2 = Q\mathbf{E}(L), \tag{20}$$

corresponding to the variational symmetries ∂_v, $(-1)^n\partial_v$ and $f(t)\partial_v$, respectively. Here $f(t)$ is an arbitrary function of t.

Characteristics	Conservation laws
$Q = 1$	$P_1 = v_1 - v_{-1}$ $P_2 = -\exp(v_1 - v_{-1}) - \exp(v - v_{-2})$
$Q = (-1)^n$	$P_1 = (-1)^n(v_1 - v_{-1})$ $P_2 = (-1)^n \exp(v_2 - v) - (-1)^n \exp(v - v_{-2})$
$Q = f(t)$	$P_1 = 0$ $P_2 = f(t)\left(v' + v'_{-1} - \exp(v_1 - v_{-1}) - \exp(v - v_{-2})\right)$

A second example is a special YdKN equation (e.g. [15])

$$u' = \frac{u^2u_1u_{-1}}{u_1 - u_{-1}}.$$

The linearized symmetry condition leads to the infinitesimal generators

$$\mathbf{v}_1 = (-1)^n u\partial_u, \quad \mathbf{v}_2 = x\partial_x - \frac{1}{2}u\partial_u, \quad \mathbf{v}_3 = \partial_x, \quad \mathbf{v}_4 = u^2\partial_u.$$

These are consistent with [8].

3.2 Semi-discretisations of the KdV equation

As a final example, consider the KdV equation

$$u_t + uu_x + u_{xxx} = 0, \tag{21}$$

which can be rewritten as

$$v_{tx} + v_xv_{xx} + v_{xxxx} = 0, \tag{22}$$

by introducing $v_x = u$. The latter is governed by a Lagrangian

$$L = -\frac{v_tv_x}{2} - \frac{v_x^3}{6} + \frac{v_{xx}^2}{2}, \tag{23}$$

which admits the following symmetries

$$Q_1 = 1,\ Q_2 = v_x,\ Q_3 = v_x^2 + 2v_{xxx},\ Q_4 = t. \tag{24}$$

Hence they contribute to four distinct conservation laws. The first three can be changed back to conservation laws of the original equation using the same transformation $v_x = u$

and they are the conservation of mass, the conservation of momentum and the conservation of energy:

$$
\begin{aligned}
D_t u + D_x \left(\frac{1}{2}u^2 + u_{xx}\right) &= F,\\
D_t \left(\frac{1}{2}u^2\right) + D_x \left(\frac{1}{3}u^3 + u u_{xx} - \frac{1}{2}u_x^2\right) &= uF, \qquad (25)\\
D_t \left(\frac{1}{3}u^3 - u_x^2\right) + D_x \left(\frac{1}{4}u^4 + u^2 u_{xx} + 2u_x u_t + u_{xx}^2\right) &= \left(u^2 + 2u_{xx}\right) F,
\end{aligned}
$$

where $F = u_t + uu_x + u_{xxx}$. However, the last one with characteristic $Q_4 = t$ can not be transformed back because its flux depends on v.

Next we consider semi-discretisations of the KdV equation preserving multiple symmetries and/or multiple conservation laws simultaneously. Start with semi-discretisations of the Lagrangian (23), for instance

$$L_1 = -\frac{v'}{2}(v_1 - v) - \frac{(v_1 - v)^3}{6} + \frac{(v_1 - 2v + v_{-1})^2}{2}. \qquad (26)$$

Now $v' = v_t$. The underlying DDE (i.e. the Euler–Lagrange equation $\mathbf{E}(L_1) = 0$) is

$$\frac{v_1' - v_{-1}'}{2} + \frac{(v_1 - v)^2 - (v - v_{-1})^2}{2} + v_2 - 4v_1 + 6v - 4v_{-1} + v_{-2} = 0. \qquad (27)$$

It becomes a semi-discretisation of the original KdV equation, introducing $v - v_{-1} = u$, and it reads

$$\frac{u_1' + u'}{2} + \frac{u_1^2 - u^2}{2} + u_2 - 3u_1 + 3u - u_{-1} = 0. \qquad (28)$$

In this case, symmetries with characteristics $Q_1 = 1$ and $Q_4 = t$ are preserved, namely they are still variational symmetries of L_1 and hence contributes to conservation laws of the Euler–Lagrange equation. The first one becomes a conservation law of the semi-discretised equation (28):

$$D_t\left(\frac{u_1 + u}{2}\right) + (S - \mathrm{id})\left(\frac{1}{2}u^2 + u_1 - 2u + u_{-1}\right) = F_1, \qquad (29)$$

where F_1 is the left hand side of (28).

Alternatively, let us consider semi-discretisations by discretising time t. For instance, consider the following differential-difference Lagrangian

$$L_2 = -\frac{v_1 - v}{2}\frac{v_1' + v'}{2} - \frac{(v')^3}{6} + \frac{(v'')^2}{2}. \qquad (30)$$

Now 'dash' denotes derivatives with respect to x, for example $v' = v_x$ and so forth, while n is the discretised time. Its Euler–Lagrange equation is

$$\frac{v_1' - v_{-1}'}{2} + v'v'' + v'''' = 0, \qquad (31)$$

which becomes a semi-discretisation of the original KdV equation using $v' = u$, namely

$$\frac{u_1 - u_{-1}}{2} + uu' + u''' = 0. \tag{32}$$

Now symmetries with characteristics Q_1, Q_2, Q_4 are preserved and they become

$$Q_1 = 1, \quad Q_2 = v', \quad Q_4 = n. \tag{33}$$

They yield three conservation laws of the Euler–Lagrange equation; the first two become conservation laws of the DDE (32):

$$\begin{aligned} (S - \mathrm{id})\left(\frac{u_1 + u}{2}\right) + D_x\left(\frac{1}{2}u^2 + u''\right) &= F_2, \\ (S - \mathrm{id})\left(\frac{uu_{-1}}{2}\right) + D_x\left(\frac{1}{3}u^3 + uu'' - \frac{1}{2}(u')^2\right) &= uF_2. \end{aligned} \tag{34}$$

Here F_2 is the left hand side of (32).

4 Conclusions and further remarks

In this note, we reviewed the main results of the paper [11], that is continuous symmetries of DDEs and Noether's first theorem for deriving conservation laws of DDEs governed by differential-difference Lagrangians. Several examples were provided to illustrate the theory, in particular for regular symmetries. In our next paper [12], we will show how the current results can be generalised to general symmetries by proving an equivalent evolutionary representative for symmetry prolongations as well as extending Noether's two theorems to DDEs.

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