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Homogenization and penalization of Hamilton-Jacobi equations with integral terms

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1. Introduction

We consider the functional partial differential equation

$$u^\varepsilon(x,\xi)+H\left(\frac{x}{\varepsilon},Du^\varepsilon(x,\xi),\xi\right) = \frac{1}{\delta(\varepsilon)}\int_I k(\xi,\eta)\,[u^\varepsilon(x,\eta)-u^\varepsilon(x,\xi)]\,d\eta \qquad (\mathrm{E})_\varepsilon$$
$$\text{for } (x,\xi)\in\mathbf{R}^n\times I,$$

where ε and $\delta(\varepsilon)$ are a positive parameter and a positive parameter satisfying $\delta(\varepsilon)\to 0$ as $\varepsilon\searrow 0$ respectively, I is a finite interval of $\mathbf{R}$, H is a Borel measurable function on $\mathbf{R}^{2n}\times I$ such that for each $\xi\in I$ the function $H(\cdot,\xi)$ is continuous on $\mathbf{R}^{2n}$, and k is a bounded, positive, Borel measurable function on $I\times I$.

Equation $(\mathrm{E})_\varepsilon$ appears as a fundamental equation in optimal control of the system whose states are described by ordinary differential equations, subject to random changes of states in I and to control which induce the integral term in $(\mathrm{E})_\varepsilon$ and the nonlinearity of H, respectively.

An evolution equation similar to $(\mathrm{E})_\varepsilon$ was considered in Ishii-Shimano[11]. They proved a convergence theorem in which the limit equation is identified with a nonlinear parabolic PDE. The second and third terms of $(\mathrm{E})_\varepsilon$ indicate the effects of homogenization and penalization, respectively. Our motivation is to study the interaction in the asymptotics between the effects of the almost periodic homogenization and penalization in $(\mathrm{E})_\varepsilon$.

In this paper we deal with the almost periodic homogenization. In [8], Ishii studied the almost periodic homogenization of Hamilton-Jacobi equations. There are many references concerning the homogenization of Hamilton-Jacobi equations. However most of these deal with the periodic homogenization. See e.g., [1,4,5,6,7,10]. Except for the periodic and almost periodic cases, Souganidis studied stochastic homogenization for the Cauchy problem for first-order PDE in [12], and Arisawa dealt with the quasi-periodic homogenization for second-order Hamilton-Jacobi-Bellman equations in [3].

Our plan is the following. In Section 2 we explain some properties for the integral operator of $(\mathrm{E})_\varepsilon$ and give our definition of viscosity solutions. In Section 3 we consider three cell problems. These cell problems play important parts in proofs of our main theorems. In Section 4 we state convergence theorems which are our main theorems. Our main theorems, Theorems 4.2, 4.3 and 4.4, say that the equation, which the limit function of the viscosity solution u^ε of $(\mathrm{E})_\varepsilon$, as $\varepsilon \to 0$, varies according to the ranges of $\gamma := \lim_{\varepsilon\to 0} \delta(\varepsilon)/\varepsilon$, $\gamma = 0$, $0 < \gamma < \infty$, or $\gamma = \infty$. In Section 5 we deal with functional first-order PDE including two positive parameters. Theorem 5.2 says that in the case where $\gamma = 0$ $(\mathrm{E})_\varepsilon$ is influenced by the penalization first, and then the penalized PDE is homogenized, and that in the case where $\gamma = \infty$ it is homogenized first, and then is penalized. In the case where $\gamma \in (0,\infty)$ we can interpret that $(\mathrm{E})_\varepsilon$ is homogenized and penalized at the same time.

2. Preliminaries

For any Borel subset $\Omega \subset \mathbf{R}^m$, $\mathcal{B}(\Omega)$ denotes the space of all Borel functions on Ω, and $\mathcal{B}^\infty(\Omega)$ denotes the Banach space of bounded Borel functions f on Ω with norm $\|f\|_\infty$, where we write $\|f\|_\infty = \sup_\Omega |f|$. I denotes a fixed finite interval, with length $|I| > 0$, and also the identity operator on a given space.

Throughout this paper we fix positive numbers κ_0, κ_1, with $\kappa_0 < \kappa_1$, and assume that k is a Borel function on $I \times I$ such that $\kappa_0 \le k(\xi,\eta) \le \kappa_1$ for all $\xi, \eta \in I$.

Next we define the continuous linear operator $K : \mathcal{B}^\infty(I) \to \mathcal{B}^\infty(I)$ by

$$Kf(\xi) = \int_I k(\xi,\eta) f(\eta)\,d\eta \qquad \text{for } \xi \in I.$$

We define $\overline{k}$ by

$$\overline{k}(\xi) = \int_I k(\xi,\eta)\,d\eta \qquad \text{for } \xi \in I$$

and define $C : \mathcal{B}^\infty(I) \to \mathcal{B}^\infty(I)$ and $L : \mathcal{B}^\infty(I) \to \mathcal{B}^\infty(I)$ by

$$Cf(\xi) = \overline{k}(\xi) f(\xi) \qquad \text{for } \xi \in I$$

and

$$Lf(\xi) = \int_I \frac{k(\xi,\eta)}{\overline{k}(\xi)} f(\eta)\,d\eta \qquad \text{for } \xi \in I.$$

We set

$$l(\xi,\eta) = \frac{k(\xi,\eta)}{\overline{k}(\xi)} \qquad \text{for } \xi,\ \eta \in I.$$

By the Fredholm-Riesz-Schauder theory, there exists a unique function $r \in \mathcal{B}^\infty(I)$ such that

$$\int_I r(\xi) l(\xi,\eta)\,d\xi = r(\eta) \qquad \text{for all } \eta \in I, \tag{2.1}$$

$$\int_I r(\xi)\,d\xi = 1. \tag{2.2}$$

Moreover, by the Perron-Frobenius theory, we see that $r(\xi) > 0$ for all $\xi \in I$. Then by (2.1) we see that

$$\frac{\kappa_0}{\kappa_1 |I|} \le r(\xi) \le \frac{\kappa_1}{\kappa_0 |I|} \qquad \text{for } \xi \in I. \tag{2.3}$$

We define $\overline{r}$ by

$$\overline{r}(\xi) = \frac{r(\xi)}{\overline{k}(\xi)} \Big/ \int_I \frac{r(\eta)}{\overline{k}(\eta)}\,d\eta \qquad \text{for } \xi \in I.$$

Then from (2.3) we have

$$\frac{\kappa_0^3}{\kappa_1{}^3|I|} \le \overline{r}(\xi) \le \frac{\kappa_1^3}{\kappa_0{}^3|I|} \qquad \text{for } \xi \in I. \tag{2.4}$$

For any integrable function $h : I \to \mathbf{R}$, we define

$$\{h\}^{\perp,\infty} = \{f \in \mathcal{B}^\infty(I) \mid \int_I h(\xi)f(\xi)d\xi = 0\}.$$

Since $\mathrm{Im}(K - C) \subset \{\overline{r}\}^{\perp,\infty}$, we may regard $K - C$ as an operator from $\{\overline{r}\}^{\perp,\infty}$ into $\{\overline{r}\}^{\perp,\infty}$. Observe that the bounded linear operator $L - I : \{r\}^{\perp,\infty} \to \{\mathbf{1}\}^{\perp,\infty}$ is invertible, where $\mathbf{1}(\xi) = 1$ for all $\xi \in I$. Consequently, $K - C$ is invertible. We denote this inverse operator by $(K - C)^{-1}$.

Before we give the definition of viscosity solutions of

$$F(x, u(x,\xi), D_x u(x,\xi), \xi) = \int_I k(\xi,\eta)[u(x,\eta) - u(x,\xi)]d\eta \quad \text{for } (x,\xi) \in \mathbf{R} \times I, \tag{E}$$

where F is Borel measurable on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times I$ such that for each $\xi \in I$ the function $F(\cdot,\xi)$ is continuous on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$, we introduce the notation. We denote by $\mathcal{U}^+(\mathbf{R}^n \times I)$ the set of those functions u on $\mathbf{R}^n \times I$ such that for each $x \in \mathbf{R}^n$ the function $u(x,\cdot)$ is Borel measurable and integrable in I and for each $\xi \in I$ the function $u(\cdot,\xi)$ is upper semicontinuous in $\mathbf{R}^n$. We set $\mathcal{U}^-(\mathbf{R}^n \times I) = -\mathcal{U}^+(\mathbf{R}^n \times I)$. For any $\Omega \subset \mathbf{R}^m$, $C(\Omega) \otimes \mathcal{B}(I)$ denotes the set of functions f on $\Omega \times I$ such that for each $x \in \Omega$ the function $f(x,\cdot)$ is Borel measurable in I and for each $\xi \in I$ the function $f(\cdot,\xi)$ is continuous in Ω. We call a continuous function $\omega : [0,\infty) \to [0,\infty)$ a modulus if ω is non-decreasing in $[0,\infty)$ and $\omega(0) = 0$.

Definition. (i) *We call $u \in \mathcal{U}^+(\mathbf{R}^n \times I)$ a viscosity subsolution of* (E) *if whenever $\varphi \in C^1(\mathbf{R}^n)$, $\xi \in I$, and $u(\cdot,\xi) - \varphi$ attains its local maximum at $\hat{x}$, then*

$$F(\hat{x}, u(\hat{x},\xi), D\varphi(\hat{x}), \xi) \le \int_I k(\xi,\eta)[u(\hat{x},\eta) - u(\hat{x},\xi)]d\eta.$$

(ii) *We call $u \in \mathcal{U}^-(\mathbf{R}^n \times I)$ a viscosity supersolution of* (E) *if whenever $\varphi \in C^1(\mathbf{R}^n)$, $\xi \in I$, and $u(\cdot,\xi) - \varphi$ attains its local minimum at $\hat{x}$, then*

$$F(\hat{x}, u(\hat{x},\xi), D\varphi(\hat{x}), \xi) \ge \int_I k(\xi,\eta)[u(\hat{x},\eta) - u(\hat{x},\xi)]d\eta.$$

(iii) *We call $u \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ a viscosity solution of* (E) *if it is both a viscosity sub- and supersolution of* (E).

3. Three cell problems

We begin this section by giving our assumptions on H.

(A1) $H \in C(\mathbf{R}^{2n}) \otimes \mathcal{B}(I)$.

(A2) $\lim_{R\to\infty} \inf\{H(x,p,\xi) \mid x,p \in \mathbf{R}^n, \xi \in I, |p| \geq R\} = \infty$.

(A3) For each $R > 0$ the family $\{H(\cdot + z,\cdot,\cdot) \mid z \in \mathbf{R}^n\}$ of functions is relatively compact in $\mathcal{A}(\mathbf{R}^n \times B(0,R) \times I)$, where $\mathcal{A}(\mathbf{R}^n \times B(0,R) \times I)$ denotes the set of functions $f \in C(\mathbf{R}^n \times B(0,R)) \otimes \mathcal{B}(I)$, with norm $\|\cdot\|_{\mathcal{A}(R^n\times B(0,R)\times I)} := \sup_{\mathbf{R}^n\times B(0,R)\times I} |\cdot|$, which satisfy for a modulus μ_R and a positive constant M_R,

$$|f(x,p,\xi) - f(y,q,\xi)| \leq \mu_R(|x-y| + |p-q|), \quad |f(x,p,\xi)| \leq M_R \quad \text{for all } x,y \in \mathbf{R}^n, p,q \in B(0,R), \xi \in I, \tag{\#}$$

where $B(0,R)$ denotes the closed ball of $\mathbf{R}^n$ with radius R centered at the origin.

(A4) The family $\{H(\cdot + z,\cdot,\cdot) \mid z \in \mathbf{R}^n\}$ of functions is subset of $\mathcal{A}(\mathbf{R}^{2n} \times I)$, where $\mathcal{A}(\mathbf{R}^{2n} \times I)$ denotes the set of functions $f \in C(\mathbf{R}^{2n}) \otimes \mathcal{B}(I)$ such that for each $R > 0$ there exist a modulus μ_R and a positive constant M_R for which condition (#) is satisfied. Moreover, for every sequence $\{z_j\} \subset \mathbf{R}^n$ there are a subsequence $\{z_{j_k}\} \subset \{z_j\}$ and a function $\tilde{H} \in \mathcal{A}(\mathbf{R}^{2n} \times I)$ such that

$$\lim_{k\to\infty} \sup_{(x,p,\xi)\in\mathbf{R}^n\times\mathbf{R}^n\times I} |H(x + z_{j_k}, p, \xi) - \tilde{H}(x,p,\xi)| = 0.$$

Assumptions (A3) and (A4) relate to the almost periodic homogenization. Note that (A4) is a stronger condition than (A3).

Example. We consider the function $H(x,p,\xi) = b(\xi)|p|^m + f(x)$, where $m > 0$, $b \in \mathcal{B}^\infty(I)$ is positive, and $f \in C(\mathbf{R}^n)$ is almost periodic. Then the function H satisfies (A1), (A2) and (A4).

Theorem 3.1. *Assume that* (A1)-(A3) *hold. Let* $\hat{p} \in \mathbf{R}^n$. *There is a unique constant* $\lambda \in \mathbf{R}$ *such that for each* $\theta > 0$ *there is a bounded and Lipschitz continuous viscosity solution* v *of*

$$\begin{cases} \displaystyle\int_I \bar{r}(\eta) H(x, \hat{p} + Dv(x), \eta)\, d\eta & \leq \lambda + \theta \text{ for } x \in \mathbf{R}^n, \\ \displaystyle\int_I \bar{r}(\eta) H(x, \hat{p} + Dv(x), \eta)\, d\eta & \geq \lambda - \theta \text{ for } x \in \mathbf{R}^n. \end{cases}$$

The problem of finding a constant λ described in the above theorem is a type of the so-called ergodic problem. We adapted here the formulation of Arisawa[2].

We can define the effective function $\bar{H}_0 : \mathbf{R}^n \to \mathbf{R}$ by setting $\bar{H}_0(\hat{p}) = \lambda$, where λ is the constant given by Theorem 3.1.

Proposition 3.2. $\bar{H}_0$ *is continuous on* $\mathbf{R}^n$.

We refer to [8] for a proof of Theorem 3.1 and Proposition 3.2.

Theorem 3.3. *Assume that* (A1), (A2) *and* (A4) *hold. Let* $\hat{p} \in \mathbf{R}^n$ *and* $\gamma > 0$. *There is a unique constant* $\lambda_\gamma \in \mathbf{R}$ *such that for each* $\theta > 0$ *there is a bounded viscosity solution*

$v \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ *of*

$$\begin{cases} H(x,\hat{p}+D_x v(x,\xi),\xi) & \leq \lambda_\gamma + \theta + \dfrac{1}{\gamma}\displaystyle\int_I k(\xi,\eta)\,[v(x,\eta)-v(x,\xi)]\,d\eta \\ & \qquad \text{for } (x,\xi) \in \mathbf{R}^n \times I, \\ H(x,\hat{p}+D_x v(x,\xi),\xi) & \geq \lambda_\gamma - \theta + \dfrac{1}{\gamma}\displaystyle\int_I k(\xi,\eta)\,[v(x,\eta)-v(x,\xi)]\,d\eta \\ & \qquad \text{for } (x,\xi) \in \mathbf{R}^n \times I. \end{cases}$$

Here we define $\bar{H}_\gamma : \mathbf{R}^n \to \mathbf{R}$ by setting $\bar{H}_\gamma(\hat{p}) = \lambda_\gamma$, where λ_γ is the constant given by Theorem 3.3.

Proposition 3.4. *$\bar{H}_\gamma$ is continuous on $\mathbf{R}^n$.*

Theorem 3.5. *Assume that* (A1)-(A3) *hold. Let $p \in \mathbf{R}^n$. There is a unique function $\lambda \in \mathcal{B}^\infty(I)$ such that for each $\theta > 0$ there is a bounded viscosity solution $v \in C(\mathbf{R}^n)\otimes\mathcal{B}(I)$ of*

$$\begin{cases} H(x,\hat{p}+D_x v(x,\xi),\xi) \leq \lambda(\xi) + h(\xi) & \text{for } (x,\xi) \in \mathbf{R}^n \times I, \\ H(x,\hat{p}+D_x v(x,\xi),\xi) \geq \lambda(\xi) - h(\xi) & \text{for } (x,\xi) \in \mathbf{R}^n \times I, \end{cases}$$

for all $(x,\xi) \in \mathbf{R}^n \times I$, where $h \in \mathcal{B}^\infty(I)$ and h satisfies $\int_I |h(\eta)|d\eta \leq \theta$.

Here we define $\bar{H}_\infty : \mathbf{R}^n \times I \to \mathbf{R}$ by setting $\bar{H}_\infty(\hat{p},\xi) = \lambda(\xi)$, where λ is the function given by Theorem 3.5.

Proposition 3.6. *$\bar{H}_\infty \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$. Moreover, for each $R > 0$ there is a modulus ω_R such that*

$$|\bar{H}_\infty(p,\xi) - \bar{H}_\infty(q,\xi)| \leq \omega_R(|p-q|) \qquad \text{for all } p,q \in B(0,R), \xi \in I.$$

4. Convergence theorems

We state uniqueness and existence results for $(\mathrm{E})_\varepsilon$.

Theorem 4.1. *Assume that* (A1)-(A3) *hold. Let $\varepsilon > 0$. There is a unique bounded viscosity solution $u^\varepsilon \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ of* $(\mathrm{E})_\varepsilon$.

Consult sections 3 and 4 of [9] for the proof of Theorem 4.1. However, note that the equations considered in [9] are slightly different from $(\mathrm{E})_\varepsilon$.

Theorem 4.2. *Assume that* (A1)-(A3) *hold and that $\lim_{\varepsilon\to 0} \delta(\varepsilon)/\varepsilon = 0$. Let u^ε be the bounded viscosity solution of* $(\mathrm{E})_\varepsilon$ *and u be the (unique) bounded viscosity solution of*

$$u(x) + \bar{H}_0(Du(x)) = 0 \qquad \text{for } x \in \mathbf{R}^n. \tag{LE$_0$}$$

Then

$$\lim_{\varepsilon\searrow 0} \sup\{|u^\varepsilon(x,\xi) - u(x)| \mid x \in \mathbf{R}^n, \xi \in I\} = 0.$$

Theorem 4.3. *Assume that* (A1), (A2) *and* (A4) *hold and that*

$$\lim_{\varepsilon \to 0} \frac{\delta(\varepsilon)}{\varepsilon} = \gamma \in (0, \infty).$$

Let u^ε be the bounded viscosity solution of $(\mathrm{E})_\varepsilon$ *and u be the bounded viscosity solution of*

$$u(x) + \bar{H}_\gamma(Du(x)) = 0 \qquad \text{for } x \in \mathbf{R}^n. \qquad (\mathrm{LE})_\gamma$$

Then

$$\lim_{\varepsilon \searrow 0} \sup\{|u^\varepsilon(x,\xi) - u(x)| \mid x \in \mathbf{R}^n, \xi \in I\} = 0.$$

Theorem 4.4. *Assume that* (A1)-(A3) *hold and that* $\lim_{\varepsilon \to 0} \delta(\varepsilon)/\varepsilon = \infty$. *Let u^ε be the bounded viscosity solution of* $(\mathrm{E})_\varepsilon$ *and u be the bounded viscosity solution of*

$$u(x) + \int_I \bar{r}(\eta)\bar{H}_\infty(Du(x),\eta)d\eta = 0 \qquad \text{for } x \in \mathbf{R}^n. \qquad (\mathrm{LE})_\infty$$

Then

$$\lim_{\varepsilon \searrow 0} \sup\{|u^\varepsilon(x,\xi) - u(x)| \mid x \in \mathbf{R}^n, \xi \in I\} = 0.$$

5. Functional first-order PDE with two parameters

In this section we consider the functional PDE with two parameters:

$$u^{\varepsilon,\delta}(x,\xi) + H\left(\frac{x}{\varepsilon}, Du^{\varepsilon,\delta}(x,\xi), \xi\right) = \frac{1}{\delta}\int_I k(\xi,\eta)\left[u^{\varepsilon,\delta}(x,\eta) - u^{\varepsilon,\delta}(x,\xi)\right] d\eta \qquad (\mathrm{E})_{\varepsilon,\delta}$$
$$\text{for } (x,\xi) \in \mathbf{R}^n \times I,$$

where ε and δ are positive parameters.

We give a result for the existence and uniqueness of viscosity solution of $(\mathrm{E})_{\varepsilon,\delta}$ without proving it. (See Theorem 4.1.)

Theorem 5.1. *Assume that* (A1)-(A3) *hold. Let* ε, $\delta > 0$. *There is a unique bounded viscosity solution* $u^{\varepsilon,\delta} \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ *of* $(\mathrm{E})_{\varepsilon,\delta}$.

We consider the asymptotic behavior of the viscosity solution of $(\mathrm{E})_{\varepsilon,\delta}$, as $\delta \searrow 0$, and then $\varepsilon \searrow 0$ or $\varepsilon \searrow 0$, and then $\delta \searrow 0$. We state a main theorem of this section.

Theorem 5.2. *Assume that* (A1)-(A3) *hold.*

(i) *If u is a bounded viscosity solution of* $(\mathrm{LE})_0$, *then*

$$u(x) = \lim_{\varepsilon \searrow 0} \lim_{\delta \searrow 0} u^{\varepsilon,\delta}(x,\xi) \qquad \text{for } (x,\xi) \in \mathbf{R}^n \times I.$$

(ii) *If u is a bounded viscosity solution of* $(\mathrm{LE})_\infty$, *then*

$$u(x) = \lim_{\varepsilon \searrow 0} \lim_{\delta \searrow 0} u^{\varepsilon,\delta}(x,\xi) \qquad \text{for } (x,\xi) \in \mathbf{R}^n \times I.$$

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