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**Homogenization and penalization of Hamilton-Jacobi equations with integral terms**

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1. Introduction

We consider the functional partial differential equation

\[ u^\varepsilon(x, \xi) + H\left(\frac{x}{\varepsilon}, Du^\varepsilon(x, \xi), \xi\right) = \frac{1}{\delta(\varepsilon)} \int_I k(\xi, \eta) [u^\varepsilon(x, \eta) - u^\varepsilon(x, \xi)] d\eta \quad (E)_\varepsilon \]

for \((x, \xi) \in \mathbb{R}^n \times I,\)

where \(\varepsilon\) and \(\delta(\varepsilon)\) are a positive parameter and a positive parameter satisfying \(\delta(\varepsilon) \to 0\) as \(\varepsilon \searrow 0\) respectively, \(I\) is a finite interval of \(\mathbb{R}\), \(H\) is a Borel measurable function on \(\mathbb{R}^{2n} \times I\) such that for each \(\xi \in I\) the function \(H(\cdot, \xi)\) is continuous on \(\mathbb{R}^{2n}\), and \(k\) is a bounded, positive, Borel measurable function on \(I \times I\).

Equation \((E)_\varepsilon\) appears as a fundamental equation in optimal control of the system whose states are described by ordinary differential equations, subject to random changes of states in \(I\) and to control which induce the integral term in \((E)_\varepsilon\) and the nonlinearity of \(H\), respectively.

An evolution equation similar to \((E)_\varepsilon\) was considered in Ishii-Shimano[11]. They proved a convergence theorem in which the limit equation is identified with a nonlinear parabolic PDE. The second and third terms of \((E)_\varepsilon\) indicate the effects of homogenization and penalization, respectively. Our motivation is to study the interaction in the asymptotics between the effects of the almost periodic homogenization and penalization in \((E)_\varepsilon\).

In this paper we deal with the almost periodic homogenization. In [8], Ishii studied the almost periodic homogenization of Hamilton-Jacobi equations. There are many references concerning the homogenization of Hamilton-Jacobi equations. However most of these deal with the periodic homogenization. See e.g., [1,4,5,6,7,10]. Except for the periodic and almost periodic cases, Souganidis studied stochastic homogenization for the Cauchy problem for first-order PDE in [12], and Arisawa dealt with the quasi-periodic homogenization for second-order Hamilton-Jacobi-Bellman equations in [3].

Our plan is the following. In Section 2 we explain some properties for the integral operator of \((E)_\varepsilon\) and give our definition of viscosity solutions. In Section 3 we consider three cell problems. These cell problems play important parts in proofs of our main theorems. In Section 4 we state convergence theorems which are our main theorems. Our main theorems, Theorems 4.2, 4.3 and 4.4, say that the equation, which the limit
function of the viscosity solution \( u^\epsilon \) of \((E)_\epsilon \), as \( \epsilon \to 0 \), varies according to the ranges of 
\( \gamma := \lim_{\epsilon \to 0} \delta(\epsilon)/\epsilon \), \( 0 < \gamma < \infty \), or \( \gamma = \infty \). In Section 5 we deal with functional first-order PDE including two positive parameters. Theorem 5.2 says that in the case where \( \gamma = 0 \) \((E)_\epsilon \) is influenced by the penalization first, and then the penalized PDE is homogenized, and that in the case where \( \gamma = \infty \) it is homogenized first, and then is penalized. In the case where \( \gamma \in (0, \infty) \) we can interpret that \((E)_\epsilon \) is homogenized and penalized at the same time.

2. Preliminaries

For any Borel subset \( \Omega \subset \mathbb{R}^m \), \( B(\Omega) \) denotes the space of all Borel functions on \( \Omega \), and \( B^\infty(\Omega) \) denotes the Banach space of bounded Borel functions \( f \) on \( \Omega \) with norm \( ||f||_\infty = \sup_\Omega |f| \). \( I \) denotes a fixed finite interval, with length \( |I| > 0 \), and also the identity operator on a given space.

Throughout this paper we fix positive numbers \( \kappa_0, \kappa_1 \), with \( \kappa_0 < \kappa_1 \), and assume that \( k \) is a Borel function on \( I \times I \) such that \( \kappa_0 \leq k(\xi, \eta) \leq \kappa_1 \) for all \( \xi, \eta \in I \).

Next we define the continuous linear operator \( K : B^\infty(I) \to B^\infty(I) \) by 
\[
Kf(\xi) = \int_I k(\xi, \eta)f(\eta)d\eta \quad \text{for} \quad \xi \in I.
\]

We define \( \overline{k} \) by 
\[
\overline{k}(\xi) = \int_I k(\xi, \eta)d\eta \quad \text{for} \quad \xi \in I
\]
and define \( C : B^\infty(I) \to B^\infty(I) \) and \( L : B^\infty(I) \to B^\infty(I) \) by 
\[
 Cf(\xi) = \overline{k}(\xi)f(\xi) \quad \text{for} \quad \xi \in I
\]
and 
\[
 Lf(\xi) = \int_I \frac{k(\xi, \eta)}{\overline{k}(\xi)} f(\eta)d\eta \quad \text{for} \quad \xi \in I.
\]

We set 
\[
l(\xi, \eta) = \frac{k(\xi, \eta)}{\overline{k}(\xi)} \quad \text{for} \quad \xi, \eta \in I.
\]

By the Fredholm-Riesz-Schauder theory, there exists a unique function \( r \in B^\infty(I) \) such that 
\[
\int_I r(\xi)l(\xi, \eta)d\xi = r(\eta) \quad \text{for all} \quad \eta \in I, \quad (2.1)
\]
\[
\int_I r(\xi)d\xi = 1. \quad (2.2)
\]
Moreover, by the Perron-Frobenius theory, we see that \( r(\xi) > 0 \) for all \( \xi \in I \). Then by (2.1) we see that 
\[
\frac{\kappa_0}{\kappa_1|I|} \leq r(\xi) \leq \frac{\kappa_1}{\kappa_0|I|} \quad \text{for} \quad \xi \in I. \quad (2.3)
\]

We define \( \overline{r} \) by 
\[
\overline{r}(\xi) = \frac{r(\xi)}{k(\xi)} / \int_I r(\eta)d\eta \quad \text{for} \quad \xi \in I.
\]
Then from (2.3) we have
\[ \frac{\kappa_0^3}{\kappa_1^3 |I|} \leq \overline{\tau}(\xi) \leq \frac{\kappa_1^3}{\kappa_0^3 |I|} \quad \text{for } \xi \in I. \tag{2.4} \]

For any integrable function \( h : I \to \mathbb{R} \), we define
\[ \{h\}^{-\infty} = \{ f \in \mathcal{B}^\infty(I) \mid \int_I h(\xi)f(\xi)d\xi = 0 \}. \]

Since \( \text{Im}(K - C) \subset \{\overline{\tau}\}^{-\infty} \), we may regard \( K - C \) as an operator from \( \{\overline{\tau}\}^{-\infty} \) into \( \{\overline{\tau}\}^{-\infty} \). Observe that the bounded linear operator \( L - I : \{\overline{\tau}\}^{-\infty} \to \{1\}^{-\infty} \) is invertible, where \( 1(\xi) = 1 \) for all \( \xi \in I \). Consequently, \( K - C \) is invertible. We denote this inverse operator by \( (K - C)^{-1} \).

Before we give the definition of viscosity solutions of
\[ F(x, u(x, \xi), D_x u(x, \xi), \xi) = \int_I k(\xi, \eta)[u(x, \eta) - u(x, \xi)]d\eta \quad \text{for } (x, \xi) \in \mathbb{R} \times I, \tag{E} \]
where \( F \) is Borel measurable on \( \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times I \) such that for each \( \xi \in I \) the function \( F(\cdot, \xi) \) is continuous on \( \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \), we introduce the notation. We denote by \( \mathcal{U}^+(\mathbb{R}^n \times I) \) the set of those functions \( u \) on \( \mathbb{R}^n \times I \) such that for each \( x \in \mathbb{R}^n \) the function \( u(\cdot, \cdot) \) is Borel measurable and integrable in \( I \) and for each \( \xi \in I \) the function \( u(\cdot, \xi) \) is upper semicontinuous in \( \mathbb{R}^n \). We set \( \mathcal{U}^-(\mathbb{R}^n \times I) = -\mathcal{U}^+(\mathbb{R}^n \times I) \). For any \( \Omega \subset \mathbb{R}^n \), \( C(\Omega) \otimes \mathcal{B}(I) \) denotes the set of functions \( f \) on \( \Omega \times I \) such that for each \( x \in \Omega \) the function \( f(\cdot, \cdot) \) is Borel measurable in \( I \) and for each \( \xi \in I \) the function \( f(\cdot, \xi) \) is continuous in \( \Omega \). We call a continuous function \( \omega : [0, \infty) \to [0, \infty) \) a modulus if \( \omega \) is non-decreasing in \( [0, \infty) \) and \( \omega(0) = 0 \).

**Definition.** (i) We call \( u \in \mathcal{U}^+(\mathbb{R}^n \times I) \) a viscosity subsolution of (E) if whenever \( \varphi \in C^1(\mathbb{R}^n), \xi \in I, \) and \( u(\cdot, \xi) - \varphi \) attains its local maximum at \( \hat{x} \), then
\[ F(\hat{x}, u(\hat{x}, \xi), D\varphi(\hat{x}), \xi) \leq \int_I k(\xi, \eta)[u(\hat{x}, \eta) - u(\hat{x}, \xi)]d\eta. \]

(ii) We call \( u \in \mathcal{U}^-(\mathbb{R}^n \times I) \) a viscosity supersolution of (E) if whenever \( \varphi \in C^1(\mathbb{R}^n), \xi \in I, \) and \( u(\cdot, \xi) - \varphi \) attains its local minimum at \( \hat{x} \), then
\[ F(\hat{x}, u(\hat{x}, \xi), D\varphi(\hat{x}), \xi) \geq \int_I k(\xi, \eta)[u(\hat{x}, \eta) - u(\hat{x}, \xi)]d\eta. \]

(iii) We call \( u \in C(\mathbb{R}^n) \otimes \mathcal{B}(I) \) a viscosity solution of (E) if it is both a viscosity subsolution and supersolution of (E).

### 3. Three cell problems

We begin this section by giving our assumptions on \( H \).

(A1) \( H \in C(\mathbb{R}^{2n}) \otimes \mathcal{B}(I) \).
\((A2)\) \(\lim_{R \to \infty} \inf \{H(x, p, \xi) | x, p \in \mathbb{R}^n, \xi \in I, |p| \geq R \} = \infty.\)

\((A3)\) For each \(R > 0\) the family \(\{H(\cdot + z, \cdot, \cdot) | z \in \mathbb{R}^n\}\) of functions is relatively compact in \(\mathcal{A}(\mathbb{R}^n \times B(0, R) \times I)\), where \(\mathcal{A}(\mathbb{R}^n \times B(0, R) \times I)\) denotes the set of functions \(f \in C(\mathbb{R}^n \times B(0, R)) \otimes \mathcal{B}(I)\), with norm \(\| \cdot \|_{\mathcal{A}(\mathbb{R}^n \times B(0, R) \times I)} := \sup_{\mathbb{R}^n \times B(0, R) \times I} |\cdot|\), which satisfy for a modulus \(\mu_R\) and a positive constant \(M_R\),

\[
|f(x, p, \xi) - f(y, q, \xi)| \leq \mu_R(|x - y| + |p - q|), \quad |f(x, p, \xi)| \leq M_R
\]

for all \(x, y \in \mathbb{R}^n, p, q \in B(0, R), \xi \in I,\) (#)

where \(B(0, R)\) denotes the closed ball of \(\mathbb{R}^n\) with radius \(R\) centered at the origin.

\((A4)\) The family \(\{H(\cdot + z, \cdot, \cdot) | z \in \mathbb{R}^n\}\) of functions is subset of \(\mathcal{A}(\mathbb{R}^{2n} \times I)\), where \(\mathcal{A}(\mathbb{R}^{2n} \times I)\) denotes the set of functions \(f \in C(\mathbb{R}^{2n}) \otimes \mathcal{B}(I)\) such that for each \(R > 0\) there exist a modulus \(\mu_R\) and a positive constant \(M_R\) for which condition (#) is satisfied. Moreover, for every sequence \(\{z_j\} \subset \mathbb{R}^n\) there are a subsequence \(\{z_{j_k}\} \subset \{z_j\}\) and a function \(\tilde{H} \in \mathcal{A}(\mathbb{R}^{2n} \times I)\) such that

\[
\lim_{k \to \infty} \sup_{(x, p, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \times I} |H(x + z_{j_k}, p, \xi) - \tilde{H}(x, p, \xi)| = 0.
\]

Assumptions (A3) and (A4) relate to the almost periodic homogenization. Note that (A4) is a stronger condition than (A3).

Example. We consider the function \(H(x, p, \xi) = b(\xi)|p|^m + f(x),\) where \(m > 0, b \in B^\infty(I)\) is positive, and \(f \in C(\mathbb{R}^n)\) is almost periodic. Then the function \(H\) satisfies (A1), (A2) and (A4).

Theorem 3.1. Assume that (A1)-(A3) hold. Let \(\hat{p} \in \mathbb{R}^n.\) There is a unique constant \(\lambda \in \mathbb{R}\) such that for each \(\theta > 0\) there is a bounded and Lipschitz continuous viscosity solution \(v\) of

\[
\left\{ \begin{array}{ll}
\int_I \nabla(\eta)H(x, \hat{p} + Dv(x), \eta) d\eta & \leq \lambda + \theta \quad \text{for } x \in \mathbb{R}^n, \\
\int_I \nabla(\eta)H(x, \hat{p} + Dv(x), \eta) d\eta & \geq \lambda - \theta \quad \text{for } x \in \mathbb{R}^n.
\end{array} \right.
\]

The problem of finding a constant \(\lambda\) described in the above theorem is a type of the so-called ergodic problem. We adapted here the formulation of Arisawa[2].

We can define the effective function \(\tilde{H}_0 : \mathbb{R}^n \to \mathbb{R}\) by setting \(\tilde{H}_0(\hat{p}) = \lambda,\) where \(\lambda\) is the constant given by Theorem 3.1.

Proposition 3.2. \(\tilde{H}_0\) is continuous on \(\mathbb{R}^n.\)

We refer to [8] for a proof of Theorem 3.1 and Proposition 3.2.

Theorem 3.3. Assume that (A1), (A2) and (A4) hold. Let \(\hat{p} \in \mathbb{R}^n\) and \(\gamma > 0.\) There is a unique constant \(\lambda_{\gamma} \in \mathbb{R}\) such that for each \(\theta > 0\) there is a bounded viscosity solution
$v \in C(\mathbb{R}^n) \otimes B(I)$ of

$$
\begin{align*}
H(x, \hat{p} + D_x v(x, \xi), \xi) &\leq \lambda_{\gamma} + \theta + \frac{1}{\gamma} \int_I k(\xi, \eta) [v(x, \eta) - v(x, \xi)] d\eta \\
H(x, \hat{p} + D_x v(x, \xi), \xi) &\geq \lambda_{\gamma} - \theta + \frac{1}{\gamma} \int_I k(\xi, \eta) [v(x, \eta) - v(x, \xi)] d\eta
\end{align*}
$$

for $(x, \xi) \in \mathbb{R}^n \times I$.

Here we define $\bar{H} : \mathbb{R}^n \rightarrow \mathbb{R}$ by setting $\bar{H}(\hat{p}) = \lambda(\xi)$, where $\lambda(\xi)$ is the constant given by Theorem 3.3.

**Proposition 3.4.** $\bar{H}$ is continuous on $\mathbb{R}^n$.

**Theorem 3.5.** Assume that (A1)-(A3) hold. Let $p \in \mathbb{R}^n$. There is a unique function $\lambda \in B^\infty(I)$ such that for each $\theta > 0$ there is a bounded viscosity solution $v \in C(\mathbb{R}^n) \otimes B(I)$ of

$$
\begin{align*}
H(x, \hat{p} + D_x v(x, \xi), \xi) &\leq \lambda(\xi) + h(\xi) &\text{for } (x, \xi) \in \mathbb{R}^n \times I, \\
H(x, \hat{p} + D_x v(x, \xi), \xi) &\geq \lambda(\xi) - h(\xi) &\text{for } (x, \xi) \in \mathbb{R}^n \times I,
\end{align*}
$$

for all $(x, \xi) \in \mathbb{R}^n \times I$, where $h \in B^\infty(I)$ and $h$ satisfies $\int_I |h(\eta)| d\eta \leq \theta$.

Here we define $\bar{H}_\infty : \mathbb{R}^n \times I \rightarrow \mathbb{R}$ by setting $\bar{H}_\infty(\hat{p}, \xi) = \lambda(\xi)$, where $\lambda$ is the function given by Theorem 3.5.

**Proposition 3.6.** $\bar{H}_\infty \in C(\mathbb{R}^n) \otimes B(I)$. Moreover, for each $R > 0$ there is a modulus $\omega_R$ such that

$$
|\bar{H}_\infty(p, \xi) - \bar{H}_\infty(q, \xi)| \leq \omega_R(|p - q|) \quad \text{for all } p, q \in B(0, R), \xi \in I.
$$

## 4. Convergence theorems

We state uniqueness and existence results for $(E)_\epsilon$.

**Theorem 4.1.** Assume that (A1)-(A3) hold. Let $\epsilon > 0$. There is a unique bounded viscosity solution $u^\epsilon \in C(\mathbb{R}^n) \otimes B(I)$ of $(E)_\epsilon$.

Consult sections 3 and 4 of [9] for the proof of Theorem 4.1. However, note that the equations considered in [9] are slightly different from $(E)_\epsilon$.

**Theorem 4.2.** Assume that (A1)-(A3) hold and that $\lim_{\epsilon \rightarrow 0} \delta(\epsilon)/\epsilon = 0$. Let $u^\epsilon$ be the bounded viscosity solution of $(E)_\epsilon$ and $u$ be the (unique) bounded viscosity solution of

$$
u(x) + \tilde{H}_0(Du(x)) = 0 \quad \text{for } x \in \mathbb{R}^n. \quad (LE)_0
$$

Then

$$
\lim_{\delta \rightarrow 0} \sup \{|u^\epsilon(x, \xi) - u(x)| \mid x \in \mathbb{R}^n, \xi \in I\} = 0.
$$
Theorem 4.3. Assume that (A1), (A2) and (A4) hold and that
\[ \lim_{\varepsilon \to 0} \frac{\delta(\varepsilon)}{\varepsilon} = \gamma \in (0, \infty). \]
Let \( u^\varepsilon \) be the bounded viscosity solution of \((E)_\varepsilon\) and \( u \) be the bounded viscosity solution of \n
\[ u(x) + \bar{H}_\gamma(Du(x)) = 0 \quad \text{for } x \in \mathbb{R}^n. \]  

(LE)_\gamma

Then
\[ \lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^n, \xi \in I} |u^\varepsilon(x, \xi) - u(x)| = 0. \]

Theorem 4.4. Assume that (A1)-(A3) hold and that \( \lim_{\varepsilon \to 0} \delta(\varepsilon)/\varepsilon = \infty \). Let \( u^\varepsilon \) be the bounded viscosity solution of \((E)_\varepsilon\) and \( u \) be the bounded viscosity solution of
\n
\[ u(x) + \int_I \overline{r}(\eta)H\left(Du(x), \eta \right)d\eta = 0 \quad \text{for } x \in \mathbb{R}^n. \]  

(LE)_\infty

Then
\[ \lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^n, \xi \in I} |u^\varepsilon(x, \xi) - u(x)| = 0. \]

5. Functional first-order PDE with two parameters

In this section we consider the functional PDE with two parameters:

\[ u^{\varepsilon,\delta}(x, \xi) + H\left(\frac{x}{\varepsilon}, Du^{\varepsilon,\delta}(x, \xi)\right) = \frac{1}{\delta} \int_I k(\xi, \eta) \left[u^{\varepsilon,\delta}(x, \eta) - u^{\varepsilon,\delta}(x, \xi)\right]d\eta \quad (E)_{\varepsilon,\delta} \]

where \( \varepsilon \) and \( \delta \) are positive parameters.

We give a result for the existence and uniqueness of viscosity solution of \((E)_{\varepsilon,\delta}\) without proving it. (See Theorem 4.1.)

Theorem 5.1. Assume that (A1)-(A3) hold. Let \( \varepsilon, \delta > 0 \). There is a unique bounded viscosity solution \( u^{\varepsilon,\delta} \in C(\mathbb{R}^n) \otimes B(I) \) of \((E)_{\varepsilon,\delta}\).

We consider the asymptotic behavior of the viscosity solution of \((E)_{\varepsilon,\delta}\), as \( \delta \searrow 0 \), and then \( \varepsilon \searrow 0 \) or \( \varepsilon \searrow 0 \), and then \( \delta \searrow 0 \). We state a main theorem of this section.

Theorem 5.2. Assume that (A1)-(A3) hold.

(i) If \( u \) is a bounded viscosity solution of \((LE)_0\), then
\[ u(x) = \lim_{\varepsilon \searrow 0} \lim_{\delta \searrow 0} u^{\varepsilon,\delta}(x, \xi) \quad \text{for } (x, \xi) \in \mathbb{R}^n \times I. \]
(ii) If $u$ is a bounded viscosity solution of $(LE)_{\infty}$, then
\[ u(x) = \lim_{\epsilon \searrow 0} \lim_{\delta \searrow 0} u^{\epsilon, \delta}(x, \xi) \quad \text{for } (x, \xi) \in \mathbb{R}^n \times I. \]

References


