Hamel's Formalism for Infinite-Dimensional Nonholonomic Systems

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Abstract. This paper reviews some of the recent results on Hamel's formalism for infinite-dimensional mechanical systems. Of particular interest are applications to the dynamics of systems with velocity constraints, which we illustrate with constrained planar motion of an inextensible string.

Key words: Geometric Mechanics, Constrained Dynamics, Velocity Constraints
MSC(2010): 70F25, 37J60, 70H33

1 Introduction

This paper highlights some of the contemporary developments [22] of the formalism introduced, independently, by Volterra [27], Maggi [18], Poincaré [21], Boltzmann [7], and Hamel [12]. We refer to this formalism as *Hamel's formalism* as Hamel gives the most comprehensive exposition in the finite-dimensional setting in his habilitation thesis [12].

Hamel's formalism is an evolution of Euler's approach to the dynamics of rigid bodies [9] and fluids [10, 11], in which the angular and spatial velocities are used instead of material velocity. The angular velocity of a rigid body and spatial velocity of a fluid are examples of *nonmaterial velocity*, which contains the information about system's velocity, but is not the rate of change of system's configuration with respect to time. For a finite degree of freedom system, nonmaterial velocity is usually introduced as a collection of velocity components relative to a set of vector fields that span system's velocity space.

One of the reasons for using nonmaterial velocity is that the Euler–Lagrange equations are not always effective for analyzing the dynamics of a mechanical system of interest. For example, it is difficult to study a rotating rigid body if the Euler–Lagrange equations, either intrinsically or in generalized coordinates, are used to represent the dynamics. On the other hand, the use of the angular velocity components relative to a body frame as initiated by Euler [9] results in a much simpler representation of dynamics. In a similar fashion, Euler [10, 11] uses *convective velocity* to represent the dynamics of ideal incompressible fluid. Euler's approach was further developed by Lagrange [17] for reasonably general Lagrangians on the rotation group and by Poincaré [21] for arbitrary Lie groups (see [19] for details and history).

The nonmaterial velocity used in [17] and [21] is associated with a group action. Hamel's formalism utilizes nonmaterial velocity that is unrelated to a group action on the configuration space. Hamel's equations include both the Euler–Lagrange and Euler–Poincaré equations as special cases.

As clearly seen from his paper, Hamel was particularly motivated by nonholonomic mechanics. Hamel's formalism features the simplicity of an analytic representation of constraints and the intrinsic absence of Lagrange multipliers in equations of motion and is exceptionally effective for studying finite-dimensional constrained and multibody systems and understanding their dynamics, both analytically and numerically; see e.g. [20], [13], [14], [15], [6], [3], [29] and references therein.

The paper reviews the recent development of Hamel's formalism for infinite-dimensional mechanical systems motivated by the importance of nonmaterial velocity in continuum mechanics, as demonstrated by Arnold [2] and Ebin and Marsden [8], and by recent development of infinite-dimensional nonholonomic mechanics (see e.g. [4] and [24, 25, 26]).

Being a survey, this paper leaves many technical details out. Interested readers are referred to [22] for these details, applications to systems with symmetry, etc. We concentrate on the formulation of the Hamilton and Lagrange–d'Alembert variational principles for Hamel's equations, which is carried out by constructing, in a coordinate-independent manner, the following key components of the formalism: (i) a bracket operation on the system's (infinite-dimensional) velocity space and (ii) the formula for variation of nonmaterial velocity. The illustrate the utility of the formalism, we study constrained planar motion of an inextensible string.

2 Infinite-Dimensional Mechanics

In this section we introduce a coordinate-free approach to Hamel's formalism. Thus, instead of frames, we use linear velocity substitutions that, in general, are not induced by a (local) configuration coordinate change. For functional-analytic technicalities, interested readers are referred to [22]. It is safe to assume that all infinite-dimensional configuration spaces are Banach manifolds; however, the results remain correct for much more general settings, such as convenient spaces.

2.1 Lagrangian Mechanics

Let M be an infinite-dimensional smooth manifold modeled on a vector space W and let TM be its kinematic tangent bundle with the projection $\pi_M : TM \to M$. Consider the ***initial inclusion map*** $i : Q \to M$ and the pullback vector bundle $P = i^*TM$. For convenience, we will think of Q as a subset of M. Note that Q is usually not a submanifold of M, see [16] for details.

A ***Lagrangian*** is a smooth function $L : P \to \mathbb{R}$. The dynamics for this Lagrangian is defined by ***Hamilton's principle***: The curve $\gamma : [a, b] \to Q$ is a trajectory if

$$\delta \int_a^b L\, dt = 0$$

along γ.

2.2 Hamel's Formalism and Hamilton's Principle

Let U be an open subset of M containing $q \in Q$ and let

$$U \times W \ni (q, \xi) \mapsto (q, \Psi_q \xi) \in \pi_M^{-1}(U) \subset TM \tag{2.1}$$

be a fiber-preserving diffeomorphism that is linear in the second input. Hence, for each $q \in U$, both $\Psi_q : W \to T_q M$ and $\Psi_q^{-1} : T_q M \to W$ are invertible bounded linear operators smoothly dependent on q in an open subset $i^{-1}(U) \subset Q$. As in general the Lagrangian fails to be defined on TM, it is necessary to consider various forms of equations of motion, such as weak and strong forms.

For each $\xi \in W$, the operator $\Psi_q : W \to T_q M$ defined in (2.1) outputs the vector $\Psi_q \xi \in T_q M$ for each $q \in U$. Thus, each $\xi \in W$ generates the vector field

$$\Psi\xi(q) := \Psi_q \xi$$

on U.

Given two vectors $\xi, \eta \in W$, define an antisymmetric bilinear operation on W by

$$[\xi, \eta]_q := \Psi_q^{-1}[\Psi\xi, \Psi\eta](q), \tag{2.2}$$

where the bracket on the right-hand side is the Jacobi–Lie bracket on the manifold M.

Next, for arbitrary $\xi, \eta, \zeta \in W$, we have

$$\begin{aligned}\Psi_q\big([[\xi, \eta]_q, \zeta]_q + [[\eta, \zeta]_q, \xi]_q + [[\zeta, \xi]_q, \eta]_q\big)\\ = [[\Psi\xi, \Psi\eta], \Psi\zeta](q) + [[\Psi\eta, \Psi\zeta], \Psi\xi](q) + [[\Psi\zeta, \Psi\xi], \Psi\eta](q) = 0,\end{aligned}$$

implying, in view of invertibility of Ψ_q, the Jacobi identity for the bracket $[\xi, \eta]_q$. Therefore, for each $q \in U$, the space W with the introduced bracket operation is a Lie algebra, denoted hereafter W_q.

The ***dual bracket*** $[\xi, \alpha]_q^*$ is a W^*-valued bilinear operation on $W \times W^*$ defined by

$$\big\langle [\xi, \alpha]_q^*, \eta \big\rangle_W := \big\langle \alpha, [\xi, \eta]_q \big\rangle_{W}, \quad \xi, \eta \in W, \quad \alpha \in W^*.$$

Let $\dot{q}$ and δq denote the velocity and the virtual displacement at $q \in Q$. From now on, the inverse images of $\dot{q}$ and δq are written as $\xi, \eta \in W$, that is, $\dot{q} = \Psi_q \xi$ and $\delta q = \Psi_q \eta$.

Interpreting ξ as an independent variable that replaces $\dot{q}$ (locally) defines the Lagrangian as a smooth function of (q, ξ) on $U \times W$:

$$l(q, \xi) := L(q, \Psi_q \xi). \tag{2.3}$$

The equations of motion written when (q, ξ) are selected as (local) coordinates on the velocity phase space are called ***Hamel's equations***.

Given a smooth curve $q(t) \in Q$, $t \in [a, b]$, its ***variation*** is a smooth one-parameter family of curves

$$[a, b] \times [-\varepsilon, \varepsilon] \ni (t, s) \mapsto \beta(t, s) \in Q \quad \text{such that} \quad \beta(t, 0) = q(t).$$

An ***infinitesimal variation*** δq is defined by

$$\delta q(t, s) := \frac{\partial}{\partial s}\beta(t, s).$$

When this field is evaluated along the curve $q(t)$, we write $\delta q(t)$, i.e.,

$$\delta q(t) := \delta q(t, 0) = \frac{\partial}{\partial s}\bigg|_{s=0}\beta(t, s). \tag{2.4}$$

Thus, a variation of a smooth curve $q(t) \in Q$ defines a curve $\eta(t) \in W$:

$$\delta q(t) = \Psi_{q(t)}\, \eta(t).$$

Theorem 2.1 (Hamilton's Principe for Hamel's Equations)**.** *Let $L : P \to \mathbb{R}$ be a Lagrangian and l be its representation in local coordinates (q, ξ). Then, the following statements are equivalent:*

(i) *The curve $q(t)$, where $a \le t \le b$, is a critical point of the action functional*

$$\int_a^b L(q, \dot q)\, dt \tag{2.5}$$

on the space of curves in Q connecting q_a to q_b on the interval $[a, b]$, where we choose variations of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$.

(ii) *The curve $q(t)$ satisfies the weak form of the Euler–Lagrange equations*

$$\int_a^b \left\langle \frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta \dot q}, \delta q \right\rangle dt = 0. \tag{2.6}$$

If, additionally, $i_* T_q Q$ is dense in $T_q M$ for every $q \in Q$, the curve $q(t)$ satisfies the strong form of the Euler–Lagrange equations,

$$\frac{d}{dt}\frac{\delta L}{\delta \dot q} - \frac{\delta L}{\delta q} = 0. \tag{2.7}$$

(iii) *The curve $(q(t), \xi(t))$ is a critical point of the functional*

$$\int_a^b l(q, \xi)\, dt \tag{2.8}$$

with respect to variations $\delta\xi$, induced by the variations

$$\delta q = \Psi_q\, \eta, \tag{2.9}$$

and given by

$$\delta \xi = \dot\eta + [\xi, \eta]_q. \tag{2.10}$$

(iv) *The curve $(q(t), \xi(t))$ satisfies the* ***weak form of the Hamel equations***

$$\int_a^b \left\langle \Psi_q^* \frac{\delta l}{\delta q} + \left[\xi, \frac{\delta l}{\delta \xi}\right]_q^* - \frac{d}{dt}\frac{\delta l}{\delta \xi}, \eta \right\rangle dt = 0, \quad \eta \in \Psi_q^{-1}(T_q Q) \tag{2.11}$$

coupled with the equations $\dot q = \Psi_q \xi$. If $i_* T_q Q$ is dense in $T_q M$ for every $q \in Q$, the curve $(q(t), \xi(t))$ satisfies the ***strong form of the Hamel equations***

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \left[\xi, \frac{\delta l}{\delta \xi}\right]_q^* + \Psi_q^* \frac{\delta l}{\delta q} \tag{2.12}$$

coupled with the equation $\dot q = \Psi_q \xi$.

Proof. The equivalence of (i) and (ii) is well-documented in the literature.

To prove the equivalence of (i) and (iii), start with evaluating the quantities $\delta \dot q$ and $d(\delta q)/dt$. Recall that

$$\delta q(t) = \frac{\partial}{\partial s}\bigg|_{s=0} \beta(t,s) = \Psi_{q(t)}\, \eta(t), \quad \text{where} \quad \eta(t) \in W.$$

Using the definition (2.4) of the field δq,

$$\delta \Psi_{q(t)} = \frac{\partial}{\partial s}\bigg|_{s=0} \Psi_{\beta(t,s)} = \delta q(t)[\Psi_{q(t)}] = \big(\Psi_{q(t)}\, \eta(t)\big)[\Psi_{q(t)}]. \tag{2.13}$$

Hereafter, $v[f]$ denotes the derivative of the function f along the vector filed v; in particular, in (2.13) an operator-valued function is differentiated.

Similarly,

$$\tfrac{d}{dt}\Psi_{q(t)} = \dot q(t)[\Psi_{q(t)}] = \big(\Psi_{q(t)}\, \xi(t)\big)[\Psi_{q(t)}],$$

and therefore

$$\delta \dot q = \big(\Psi_q\, \eta\big)[\Psi_q]\xi + \Psi_q\, \delta\xi, \quad \tfrac{d}{dt}\delta q = \big(\Psi_q\, \xi\big)[\Psi_q]\eta + \Psi_q\, \dot\eta.$$

From $\delta \dot q = \frac{d}{dt}\delta q$, one obtains

$$\Psi_q\left(\delta\xi - \dot\eta\right) = \big(\Psi\xi\big)[\Psi\eta](q) - \big(\Psi\eta\big)[\Psi\xi](q) = [\Psi\xi, \Psi\eta](q) = \Psi_q\, [\xi, \eta]_q,$$

which implies formula (2.10).

To prove the equivalence of (iii) and the weak form of Hamel's equations (2.11), we use the above formula and compute the variation of the action (2.8):

$$\begin{aligned} \delta \int_a^b l(q,\xi)\, dt &= \int_a^b \left(\left\langle \frac{\delta l}{\delta q}, \delta q \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \delta \xi \right\rangle \right) dt \\ &= \int_a^b \left(\left\langle \frac{\delta l}{\delta q}, \Psi_q\, \eta \right\rangle + \left\langle \frac{\delta l}{\delta \xi}, \dot\eta + [\xi, \eta]_q \right\rangle \right) dt \\ &= \int_a^b \left\langle \Psi_q^* \frac{\delta l}{\delta q} + \left[\xi, \frac{\delta l}{\delta \xi}\right]_q^* - \frac{d}{dt}\frac{\delta l}{\delta \xi}, \eta \right\rangle dt. \end{aligned}$$

If $i_* T_q Q$ is dense in $T_q M$ for every $q \in Q$, then the subspace $\Psi_{q(t)}^{-1}(i_* T_{q(t)} Q)$ is dense in W for each t and the variational derivative vanishes if and only if the strong form of the Hamel equations (2.12) is satisfied. $\square$

Example 2.1. The configuration manifold for an inextensible string moving in the plane is the space of smooth embeddings $\mathrm{Emb}([0,1],\mathbb{R}^2)$. We will view $\mathbb{R}^2$ as a complex plane. Given $z \in \mathrm{Emb}([0,1],\mathbb{C})$, the inextensibility condition reads $|z_s| = 1$, $0 \le s \le 1$. For simplicity, we assume no resistance to bending. Therefore, the Lagrangian reads

$$L(z) = \int_0^1 \tfrac{1}{2}\big(|\dot z|^2 - \lambda(|z_s|^2 - 1)\big)\, ds,$$

where $\lambda : [0,1] \to \mathbb{R}$ is the Lagrange multiplier (tension) associated with the inextensibility constraint. The boundary conditions for the Lagrange multiplier are a part of the requirement $\delta L = 0$. For a free motion of a string, these conditions read

$$\lambda|_{s=0} = \lambda|_{s=1} = 0. \tag{2.14}$$

Let

$$\dot z = \Psi_z\, \xi := z_s \xi, \tag{2.15}$$

so the velocity components to be used to construct Hamel's equations are represented by a complex-valued function $\xi = \xi(s,t)$. Geometrically, the real and imaginary parts of ξ are the tangent and normal velocity components of the points of the string.

The Lagrangian becomes

$$l = \int_0^1 \tfrac{1}{2}\big(\bar z_s z_s \bar\xi \xi - \lambda(\bar z_s z_s - 1)\big)\, ds,$$

in which the density should be understood as a function of $(z_s, \bar z_s, \xi, \bar\xi\,)$ and the Lagrange multiplier λ.

Next, formula (2.2) for the string becomes

$$[\Psi\xi, \Psi\eta](z) = \frac{d}{d\tau}\bigg|_{\tau=0} \big((z + \tau z_s \xi)_s \eta - (z + \tau z_s \eta)_s \xi\big) = z_s(\xi_s \eta - \eta_s \xi) = \Psi_z\,[\xi,\eta]_z.$$

That is,

$$[\xi,\eta]_z = \xi_s \eta - \xi \eta_s. \tag{2.16}$$

Instead of establishing the formulae for the dual bracket and dual operator Ψ^*, it is more efficient in this example to directly work with the variational principle. We have:

$$\frac{\delta l}{\delta z}\,\delta z + \frac{\delta l}{\delta \xi}\,\delta\xi + \frac{\delta l}{\delta \bar z}\,\delta \bar z + \frac{\delta l}{\delta \bar\xi}\,\delta\bar\xi, \tag{2.17}$$

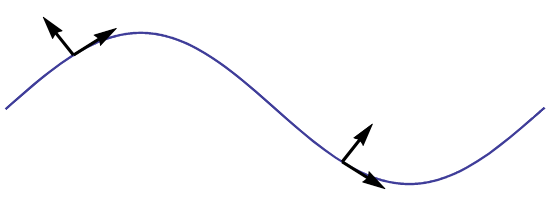

Figure 1: An inextensible planar string.

and since l is real-valued, the two last terms are obtained from the first two by conjugation. Thus, it is sufficient to evaluate the last two terms in (2.17):

$$\begin{aligned}
\frac{\delta l}{\delta \bar z}\,\delta \bar z + \frac{\delta l}{\delta \bar\xi}\,\delta\bar\xi &= \frac{\delta l}{\delta \bar z}\,\delta \bar z + \frac{\delta l}{\delta \bar\xi}\big(\bar\xi_s\bar\eta - \bar\xi\bar\eta_s\big) - \frac{d}{dt}\frac{\delta l}{\delta\bar\xi}\,\bar\eta \\
&= \int_0^1 \tfrac12\Big(\big(\lambda z_s - z_s\bar\xi\xi\big)_s\delta\bar z + \bar z_s z_s\xi\big(\bar\xi_s\bar\eta - \bar\xi\bar\eta_s\big) - \tfrac{d}{dt}\big(\bar z_s z_s\xi\big)\bar\eta\Big)\,ds - \bar z_s z_s\lambda\bar\eta\Big|_{s=0}^{s=1} \\
&= \int_0^1 \tfrac12\Big(z_s\bar z_{ss}\xi\bar\xi + z_s\bar z_s\xi\bar\xi_s + \lambda_s\bar z_s z_s + \lambda z_{ss}\bar z_s - \tfrac{d}{dt}\big(\bar z_s z_s\xi\big)\Big)\bar\eta\,ds - \tfrac12\bar z_s z_s\lambda\bar\eta\Big|_{s=0}^{s=1},
\end{aligned}$$

which, after imposing the constraint $\bar z_s z_s = 1$, implies

$$\dot\xi = \xi\bar\xi_s + \lambda_s + i\varkappa\big(\lambda - \bar\xi\xi\big) \tag{2.18}$$

as well as the tension conditions (2.14). Here, $\varkappa$ is the (signed) curvature of the curve $[0,1] \ni s \mapsto z(s) \in \mathbb{C}$. The right-hand side of (2.18) gives an explicit representation of the terms on the right-hand side of Hamel's equations (2.12) for the string.

3 Mechanics with Constraints

Here we discuss infinite-dimensional dynamics with velocity constraints.

3.1 The Lagrange–d'Alembert Principle

We confine our attention to constraints that are linear and homogeneous in the velocity. Accordingly, we consider a configuration space Q and a distribution $\mathcal{D}$ on Q that describes these constraints. Recall that a distribution $\mathcal{D}$ is a collection of linear subspaces of the tangent spaces of Q; we denote these spaces by $\mathcal{D}_q \subset T_qQ$, one for each $q \in Q$.

A curve $q(t) \in Q$ is said to ***satisfy the constraints*** if $\dot q(t) \in \mathcal{D}_{q(t)}$ for all t. This distribution will, in general, be nonintegrable; i.e., the dynamics will be, in general, nonholonomic.

The condition for a curve to satisfy the constraints is, by itself, insufficient for the development of constrained mechanics. One needs a mechanism that relates the trajectories of the unconstrained and constrained systems. For the ideal constraints in the finite-dimensional setting, this is accomplished by a projection. Thus, constraints define a submanifold of the velocity phase space and a projection onto this submanifold.

For a projection to be meaningful in the infinite-dimensional case, we require that $\mathcal{D}$ be a locally ***splitting subbundle*** of TQ. That is, for each q there exists a chart (U,h) of Q such that $Th(\pi_Q^{-1}(U)\cap\mathcal{D}) = h(U)\times W^{\mathcal{D}}$, where the closed subspace $W^{\mathcal{D}}$ of the model space W is ***splitting, or complemented***, i.e., there is a closed subspace $W^{\mathcal{U}}$ of W such that $W^{\mathcal{D}}\oplus W^{\mathcal{U}} = W$ and the projection $\pi^{\mathcal{D}}$ uniquely determined by setting $(\operatorname{Ker}\pi^{\mathcal{D}}, \operatorname{Im}\pi^{\mathcal{D}}) = (W^{\mathcal{U}}, W^{\mathcal{D}})$ is *continuous.*

To simplify the exposition, in the rest of the section we assume that Lagrangians are defined on TQ and state the results for strong equations of motion. Similar statements for weak equations are straightforward to obtain. The following Lagrange–d'Alembert principle is known to be equivalent to the dynamics of systems with ideal constraints:

Definition 3.1. *The* ***Lagrange–d'Alembert equations of motion*** *for the system are those determined by*

$$\delta \int_a^b L(q, \dot{q})\, dt = 0,$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$ and $\delta q(t) \in \mathcal{D}_{q(t)}$ for each t where $a \leq t \leq b$.

This principle is supplemented by the condition that the curve $q(t)$ itself satisfies the constraints. Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation. This is well known to be important to obtain the correct mechanical equations (see [5] for a discussion and references).

The Lagrange–d'Alembert principle is equivalent to the equations

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{q}} - \frac{\delta L}{\delta q} \in \mathcal{D}_q^{\circ}, \quad \dot{q} \in \mathcal{D}_q. \tag{3.1}$$

Here,

$$\mathcal{D}_q^{\circ} = \left\{ a \in T_q^* Q \mid \langle a, v \rangle = 0, v \in \mathcal{D}_q \right\}.$$

3.2 The Constrained Hamel Equations

Given a nonholonomic system, that is, a Lagrangian $L : TQ \to \mathbb{R}$ and constraint distribution $\mathcal{D}$, select the operators $\Psi_q : W \to T_q Q$ on $U \subset Q$ such that there exist closed subspaces $W^{\mathcal{D}}$, $W^{\mathcal{U}} \subset W$, $W = W^{\mathcal{D}} \oplus W^{\mathcal{U}}$, and $\Psi_q = \Psi_q^{\mathcal{D}} \oplus \Psi_q^{\mathcal{U}}$, where $\Psi_q^{\mathcal{D}} : W^{\mathcal{D}} \to \mathcal{D}_q$ and $\Psi_q^{\mathcal{U}} : W^{\mathcal{U}} \to \mathcal{U}_q$ and their inverses are bounded linear operators smoothly dependent on $q \in U$.[1]

Each $\dot{q} \in TQ$ is then uniquely decomposed as

$$\dot{q} = \Psi_q \xi^{\mathcal{D}} + \Psi_q \xi^{\mathcal{U}}, \quad \text{where} \quad \Psi_q \xi^{\mathcal{D}} \in \mathcal{D}_q, \tag{3.2}$$

i.e., $\Psi_q \xi^{\mathcal{D}}$ is the component of $\dot{q}$ along $\mathcal{D}_q$. Similarly, each $\alpha \in W^*$ uniquely decomposes as

$$\alpha = \alpha_{\mathcal{D}} + \alpha_{\mathcal{U}},$$

where $\alpha_{\mathcal{D}}$ and $\alpha_{\mathcal{U}}$ denote the components of α along the duals of $W^{\mathcal{D}}$ and $W^{\mathcal{U}}$, respectively:

$$\alpha_{\mathcal{D}} = \left(\pi^{\mathcal{D}}\right)^* \circ \alpha|_{W^{\mathcal{D}}} \quad \text{and} \quad \alpha_{\mathcal{U}} = \left(\mathrm{id} - \left(\pi^{\mathcal{D}}\right)^*\right) \circ \alpha|_{W^{\mathcal{U}}},$$

where $\left(\pi^{\mathcal{D}}\right)^*$ is the dual of $\pi^{\mathcal{D}}$. Using (3.2), the constraints read

$$\xi = \xi^{\mathcal{D}} \quad \text{or} \quad \xi^{\mathcal{U}} = 0, \tag{3.3}$$

which implies

$$\eta = \eta^{\mathcal{D}} \quad \text{or} \quad \eta^{\mathcal{U}} = 0. \tag{3.4}$$

The Lagrange–d'Alembert principle then implies the following theorem:

[1] In general, $U \neq Q$, as numerous finite-dimensional examples demonstrate.

Theorem 3.1. *The dynamics of a nonholonomic system is represented by the* ***strong form of constrained Hamel equations***

$$\left(\frac{d}{dt}\frac{\delta l}{\delta \xi} - \left[\xi^{\mathcal{D}}, \frac{\delta l}{\delta \xi}\right]_q^* - \Psi_q^* \frac{\delta l}{\delta q}\right)_{\mathcal{D}} = 0, \quad \xi^{\mathcal{U}} = 0, \quad \dot{q} = \Psi_q \, \xi^{\mathcal{D}}. \tag{3.5}$$

Example 3.1. Consider an inextensible string moving in the plane subject to the vanishing normal velocity constraint. See Figure 1. One may think of a motion of a sharp string on the horizontal ice. Using the notations introduced in Example 2.1, the constraint reads $\xi = \bar{\xi}$, i.e., $\xi \in \mathbb{R}$. Equations (3.5) for the constrained string thus become

$$\dot{\xi} = \xi \xi_s + \lambda_s, \tag{3.6}$$
$$\dot{z} = z_s \xi, \quad \xi = \bar{\xi}. \tag{3.7}$$

along with the inextensibility condition.

It is geometrically evident (or can be confirmed with a simple calculation) that the inextensibility condition in the presence of constraint $\xi = \bar{\xi}$ implies $\xi_s = 0$. That is, all points of the string have the same speed, and (3.6) becomes

$$\dot{\xi} = \lambda_s. \tag{3.8}$$

As λ_s can be evaluated at any s in (3.8), we conclude that

$$\dot{\xi} = 0,$$

i.e., $\xi = \text{const}$ throughout the motion. This is in agreement with the motion of the Chaplygin sleigh for which the velocity of the contact point relative to the body frame is constant. Unlike the sleigh, the constrained string motion is not completely determined by its initial state. Indeed, any solution of (3.7) is of the form

$$z = \phi(s + \xi t),$$

where ϕ is an arbitrary twice-differentiable complex-valued function. The initial conditions define ϕ on the segment $[0, 1]$. Outside this segment, the function ϕ is unknown, unless, for example, the motion of the front end of the string has been prescribed. The motion of the constrained string is therefore *purely kinematic*: The string follows its front end, which moves at a constant speed.

This behavior is similar to that of the degenerate Chaplygin sleigh specified by the Lagrangian $l = \frac{1}{2}\zeta\bar{\zeta}$ and constraint $\zeta = \bar{\zeta}$, where $\zeta = e^{-i\theta}\dot{z}$. For the degenerate sleigh, the dynamics reads

$$\dot{\zeta} = 0, \quad \dot{z} = e^{i\theta}\zeta,$$

where $\theta(t)$ is an arbitrary function. Thus, the motions are not identified by the initial conditions.

Example 3.2. Consider the Chaplygin sleigh with an inextensible string attached. Assume that the string is constrained as in Example 3.1, i.e., the normal velocity of each point of the string is zero. This system is SE(2)-invariant. The string position z is measured relative to the sleigh, so that ω and ζ are the angular and linear velocity components of the sleigh.

The *absolute* velocity of the string, ξ, is computed to be

$$\xi = z_s^{-1}\big(\dot z + \zeta + i\omega z\big),$$

This effectively defines the operator Ψ. The Lagrangian, which is system's kinetic energy, reads

$$l = \tfrac{1}{2}\big(J\omega^2 + m\bar\zeta\zeta\big) + \tfrac{1}{2}\int_0^1 \big(\bar\xi\xi - \lambda(\bar z_s z_s - 1)\big)\, ds.$$

The constraint are given by $\bar\zeta = \zeta$ and $\bar\xi = \xi$.

Hamel's equations for this system become

$$\dot\omega = 0, \tag{3.9}$$
$$m\dot\zeta = \lambda_0, \tag{3.10}$$
$$\dot\xi = \xi\xi_s + \lambda_s, \tag{3.11}$$

where ζ and ξ are real-valued. These equations should be amended with the coupling conditions

$$z|_{s=0} = 0, \quad z_s|_{s=0} = 1, \quad \xi|_{s=0} = \zeta. \tag{3.12}$$

These simply state that the string is attached to the blade at the contact point of the blade and ice and the velocity of the attached string end equals the velocity of the blade.

Arguing as in Example 3.1, one concludes that ξ is independent of s. Thus, equation (3.11) becomes

$$\dot\xi = \lambda_s.$$

Equation (3.9) implies $\omega = \text{const}$.

The tension λ is obtained by solving the equation

$$\lambda_{ss} = 0,$$

and since $\lambda|_{s=1} = 0$, we conclude that

$$\lambda = (s-1)\dot\xi. \tag{3.13}$$

Therefore, $\lambda|_{s=0} = -\dot\xi$, which, in combination with (3.10) and (3.12), yields $\zeta = \text{const}$. The velocity coupling condition then implies that the blade moves at a constant speed ξ. Using (3.13), we conclude that $\lambda = 0$.

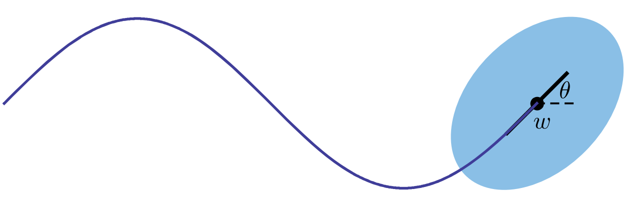

Figure 2: The Chaplygin sleigh coupled to a constrained string.

Summarizing, the Chaplygin sleigh with the constrained string attached generically undergoes uniform circular motion. Nongeneric trajectories are straight lines. The string (possibly after some period of time) follows the trajectory of the contact point of the sleigh.

It is interesting to point out that in this example the shape dynamics (string's motion) is modulated by the group dynamics (skate's motion). This is the opposite of typical reconstruction in finite-dimensional constrained systems discussed in [5].

We note also that the qualitative dynamics of this system—uniform circular or straight line motion—is consistent with the behavior of integrable Hamiltonian systems. One may raise the question of whether it is integrable in a more precise sense—with infinitely-many conserved quantities. We intend to investigate that in a forthcoming publication.

4 Concluding Remarks

Many interesting features of the formalism, such as symmetry reduction, have not been included in this short survey. It should be noted, however, that performing symmetry reduction may not always be a good idea. For instance, while the string motion in Examples 2.1 and 3.2 is SE(2)-invariant, not carrying out symmetry reduction results in simpler analysis. Interested readers are referred to our paper [22] for more details and for important functional-analytic technicalities. See also [23], [28] and [1] for the the extension of the formalism to field-theoretic setting with applications to analysis and numerical simulations.

Acknowledgments

We would like to thank Professors Yongxin Guo, Francois Gay-Balmaz, Vakhtang Putkaradze, and Tudor Ratiu for valuable discussions. The research of AMB was partially supported by NSF grant DMS-1613819 and AFOSR grant FA 0550-18-0028. The research of DS was partially supported by the China Scholarship Council. DS wishes to thank support and hospitality of North Carolina State University during his visit. The research of YBK was partially supported by NDSEG Fellowship. The research of DVZ was partially supported by NSF grants DMS-0908995 and DMS-1211454. DVZ would like to acknowledge support and hospitality of the Beijing Institute of Technology, where a part of this work was carried out.

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