The averaged Hebbian learning equation, the exponential-type geodesics of the finite discrete distributions, and their quantum statistical analogues

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1 Introduction

This article aims to provide a preview of integrable Hamiltonian forms of the averaged Hebbian learning equation (AHLE), the exponential-type geodesics (e-geodesics) of the finite discrete distributions, and their quantum analogues. The quantum analogues are two kinds of explicitly solvable matrix averaged Hebbian learning equations (MAHLEs) on the quantum statistical manifold (QSM), and their Hamiltonian forms. The entire contents including details on these equations and their Hamiltonian forms will be found in future papers [1, 2, 3] in preparation, which are motived directly by the papers [4] by Nakamura and [5, 6] by the author: In the paper [5], the extension of the AHLE referred to as the MAHLE-II in this article is constructed on the QSM, which admits the gradient equation form understood to be a natural quantum statistical analogue of Nakamura's gradient form of the AHLE [4]. Further, in [6], all the trajectories of the MAHLE-II are shown to be understood as the e-geodesics of the QSM.

The paper [1] deals with the Hamiltonian analysis of the AHLE on the cotangent bundle of the simplex. Although the Hamiltonian analysis on the cotangent bundle of the simplex looks merely an alternative version of Nakamura's Hamiltonian analysis on the tangent bundle of the simplex, the cotangent bundle version by the author [1] has a big advantage not only in connecting the AHLE with the e-geodesics of the finite discrete distributions but also in organizing quantum analogue of that connection together with discussing solvability and integrability of the systems dealt with. The Hamiltonian form of the MAHLE-I is found and studied in [2], which is shown to be integrable and explicitly solvable by quadrature. The Hamiltonian form of another MAHLE referred to as the MAHLE-II is found and studied in [3], which is shown to be explicitly solvable by quadrature. Further, in [3], any of the e-geodesics of the QSM is shown to be a trajectory of the Hamiltonian form of the MAHLE-II associated with one of the coefficient matrices.

The AHLE is the first order differential equation

$$\dot{\xi}_j = 2c_j\xi_j - 2\left(\sum_{k=1}^{n} c_k\xi_k\right)\xi_j \quad (j = 1, 2, \cdots, n) \quad \text{with} \quad \sum_{k=1}^{n} c_k = 0 \tag{1}$$

on the $n-1$ dimensional simplex

$$\mathcal{S}_{n-1} = \left\{\xi \in \mathbf{R}^n \;\middle|\; \sum_{k=1}^{n} \xi_k = 1,\, \xi_j > 0\,(j = 1, 2, \cdots, n)\right\} \tag{2}$$

without boundary, where c_js are constants. The overdot as attached to ξ_j on the lhs of (1) indicates the derivation in the time-variable, say t, throughout this article. Since Eq.(1) is invariant under the homogeneous translation, $c_j \mapsto c_j + a$ ($j = 1, 2, \cdots, n$, $a \in \mathbf{R}$), of the c_js, the requirement, $\sum_{k=1}^{n} c_k = 0$, in (1) is posed well. Note that the c_js in Nakamura [4] are set positive though.

To discuss quantum statistical analogues of both the AHLE and its Hamiltonian form, it is convenient to prepare the notation of three subspaces of the space, denoted by $M(n)$, of $n \times n$ complex matrices:

$$\begin{aligned} &H_n : \text{the space of } n \times n \text{ Hermitean matrices} \\ &H_n^+ : \text{the space of } n \times n \text{ positive definite Hermitean matrices} \\ &H_n^{tr0} : \text{the space of } n \times n \text{ traceless Hermitean matrices} \end{aligned} \tag{3}$$

The notation for the three spaces listed in (3) and $M(n)$ will be frequently used in this section and section 4 especially. Under the notation above, the quantum statistical manifold (QSM) is defined to be

$$Q_n = \{\rho \in H_n^+ \,|\, \mathrm{Tr}\,(\rho) = 1\}, \tag{4}$$

where Tr stands for the trace of matrices in this article. The MAHLEs are then described as the first order differential equations,

$$\text{MAHLE-I} \quad \dot{\rho} = (\rho\Gamma^\dagger + \Gamma\rho) - \mathrm{Tr}\,(\rho\Gamma^\dagger + \Gamma\rho)\,\rho \quad \text{with} \quad \Gamma \in (H_n^+ \cdot H_n)^{tr0}, \tag{5}$$

$$\text{MAHLE-II} \quad \dot{\rho} = (\rho C + C\rho) - 2\mathrm{Tr}\,(C\rho)\,\rho \quad \text{with} \quad C \in H_n^{tr0} \tag{6}$$

where Γ and C are constant matrices and $(H_n^+ \cdot H_n)^{tr0}$ denotes the set of traceless matrices in the form AB with $A \in H_n^+$ and $B \in H_n$. In the case that Γ and C in the MAHLE-I and MAHLE-II are diagonal respectively, the MAHLEs can be restricted to the submanifold,

$$\mathcal{D}_n = \{\rho \in Q_n \,|\, \rho : \text{diagonal}\}, \tag{7}$$

of Q_n, and both of the MAHLEs thus restricted on $\mathcal{D}_n$ are identical with the AHLE. Hence, it makes sense that we refer the ODEs (5) and (6) on Q_n as the matrix averaged Hebbian learning equations (MAHLEs).

The contents of this article is outlined in what follows. In section 2, the symplectic reduction is organized to characterize the cotangent bundle $T^*\mathcal{S}_{n-1}$ of $\mathcal{S}_{n-1}$ as the reduced phase space of the cotangent bundle T^*B_n of the positive 2^n-ant, B_n, of $\mathbf{R}^n$. In section 3, a Hamiltonian system on $T^*\mathcal{S}_{n-1}$ is constructed as the reduced Hamiltonian system of another Hamiltonian system on T^*B_n. The reduced system thus constructed is shown to be explicitly solvable and completely integrable. Furthermore, this reduced Hamiltonian system is a Hamiltonian form of both the family of the AHLEs and the exponential-type geodesics on the space of the finite discrete distributions. Section 4 is devoted to quantum analogues of sections 2 and 3. After a quantum analogue of the symplectic reduction made in section 2, a pair of Hamiltonian systems on the cotangent bundle T^*Q_n of the quantum statistical manifold Q_n is presented: One is an explicitly solvable and integrable Hamiltonian system that is shown to be a Hamiltonian form of the family of the MAHLE-Is. Another is an explicitly solvable parametric Hamiltonian system as a Hamiltonian form of the MAHLE-II: The family of those parametric Hamiltonian systems realizes all the exponential-type geodesics on Q_n. Section 5 is for conclusion.

2 Symplectic reduction of T^*B_n to $T^*\mathcal{S}_{n-1}$

We start with considering the cotangent bundle

$$T^*\mathcal{S}_{n-1} = \left\{(\xi,\eta) \in \mathcal{S}_{n-1} \times \mathbf{R}^n \,|\, \xi^T\eta = 0\right\} \tag{8}$$

of $\mathcal{S}_{n-1}$ as the phase space for the Hamiltonian form of the AHLE, where ^{T} indicates the transpose operation. Following Nakamura [4], we endow the Riemannian metric

$$ds^2 = \sum_{k=1}^{n} \frac{1}{\xi_k} d\xi_k \otimes d\xi_k \tag{9}$$

with $\mathcal{S}_{n-1}$. The $T^*\mathcal{S}_{n-1}$ is identified with the tangent bundle

$$T\mathcal{S}_{n-1} = \left\{(\xi,u) \in \mathcal{S}_{n-1} \times \mathbf{R}^n \,\middle|\, \sum_{k=1}^{n} u_k = 0\right\} = \mathcal{S}_{n-1} \times \mathbf{R}^{n-1} \tag{10}$$

of $\mathcal{S}_{n-1}$ through the diffeomorphism

$$\beta : (\xi,\eta) \in T^*\mathcal{S}_{n-1} \mapsto (\xi,u) \in T\mathcal{S}_{n-1} \quad \text{with} \quad u_j = \xi_j\eta_j \quad (j = 1,2,\cdots,n). \tag{11}$$

We endow the canonical symplectic form

$$d\sigma = \sum_{k=1}^{n} d\eta_k \wedge d\xi_k \quad \text{with} \quad \sigma = \sum_{k=1}^{n} \eta_k d\xi_k \tag{12}$$

with $T^*\mathcal{S}_{n-1}$. We note that the symplectic form on $T\mathcal{S}_{n-1}$ adopted in Nakamura [4] is understood to be the pull-back, $(\beta^{-1})^*(d\sigma)$, of $d\sigma$ by β^{-1}.

In connection with information geometry [7], we have the following lemma concerning the Riemannian manifold $(\mathcal{S}_{n-1}, ds^2)$ (see also subsection 3.4):

Lemma 2.1 ([1]) *The Riemannian manifold $(\mathcal{S}_{n-1}, ds^2)$ for the AHLE is the statistical manifold of the finite discrete distributions with n elementary events.*

Owing to this lemma, we can identify a Hamiltonian form of the family of AHLEs with that of the e-geodesics in section 3.

When we wish to make Hamiltonian analysis on $T^*\mathcal{S}_{n-1}$, we have to handle with the constraints, $\xi^T\eta = 0$ and $u^T\eta + \xi^T v = 0$, governing the tangent bundle,

$$T(T^*S_{n-1}) = \{(\xi,\eta,u,v) \in \mathcal{S}_{n-1} \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \,|\, \xi^T\eta = 0,\ u^T\eta + \xi^T v = 0\}, \tag{13}$$

of $T^*\mathcal{S}_{n-1}$. In order to avoid such constraints in our analysis, we introduce the cotangent bundle

$$T^*B_n = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n \,|\, x_j > 0 \ (j = 1,2,\cdots,n)\} \tag{14}$$

of the positive 2^n-ant

$$B_n = \{x \in \mathbf{R}^n \,|\, x_j > 0 \ (j = 1,2,\cdots,n)\} \tag{15}$$

as the 'extended phase space' of $(T^*\mathcal{S}_{n-1}, d\sigma)$, where $d\tilde{\sigma}$ is the canonical symplectic form

$$d\tilde{\sigma} = \sum_{k=1}^{n} dy_k \wedge dx_k \quad \text{with} \quad \tilde{\sigma} = \sum_{k=1}^{n} y_k dx_k. \tag{16}$$

The account for referring $(T^*B_n, d\tilde{\sigma})$ to as the 'extended phase space' is that $(T^*B_n, d\tilde{\sigma})$ is reduced symplectically to the phase space $(T^*\mathcal{S}_{n-1}, d\sigma)$. As easily seen from (14), we have no constraint on $T(T^*B_n)$ other than the positivity of x_j $(j = 1, 2, \cdots, n)$.

The base manifold B_n of T^*B_n is made into the principal $\mathbf{R}$-bundle over $\mathcal{S}_{n-1}$ with the $\mathbf{R}$-action

$$\psi_s : x \in B_n \mapsto e^s x \in B_n \quad (s \in \mathbf{R}) \tag{17}$$

and the projection

$$\mu : x \in B_n \mapsto \left(\sum_{k=1}^{n} x_k \right)^{-1} x \in \mathcal{S}_{n-1}. \tag{18}$$

With B_n, we endow the Riemannian metric

$$d\tilde{s}^2 = \left(\sum_{k=1}^{n} x_k \right)^{-1} \sum_{h=1}^{n} \frac{1}{x_h} dx_h \otimes dx_h, \tag{19}$$

so that the projection $\mu : B_n \to \mathcal{S}_{n-1}$ becomes a Riemannian submersion. Under this geometric circumstance, the reduction made in this section can be understood to afford a new example of Kummer's theorem [8] on the symplectic reduction.

Remark We note here that Nakamura endowed the metric, $(\sum_{k=1}^{n} x_k) \times d\tilde{s}^2$, with B_n in his Hamiltonian analysis on the tangent bundle TS^{n-1} [4].

We proceed to the symplectic reduction of $(T^*B_n, d\tilde{\sigma})$ associated with the symplectically lifted $\mathbf{R}$-action

$$\tilde{\psi}_s : (x, y) \in T^*B_n \mapsto (e^s x, e^{-s} y) \in T^*B_n \tag{20}$$

of ψ_s. The moment map associated with $\tilde{\psi}_s$ takes the form

$$J : (x, y) \in T^*B_n \mapsto y^T x \in \mathbf{R}. \tag{21}$$

On the inverse image $J^{-1}(0)$ of J, the $\mathbf{R}$-action $\tilde{\psi}_s$ is free and proper. Hence due to the reduction theorem by Marsden and Weinstein [9], the quotient set $J^{-1}(0)/\mathbf{R}$ is allowed to have a differentiable structure as the reduced phase space. Indeed, by the map

$$\nu : (x, y) \in J^{-1}(0) \mapsto \left(\left(\sum_{k=1}^{n} x_k \right)^{-1} x, \left(\sum_{k=1}^{n} x_k \right) y \right) \in T^*\mathcal{S}_{n-1}, \tag{22}$$

the quotient manifold $J^{-1}(0)/\mathbf{R}$ is realized as $T^*\mathcal{S}_{n-1}$. Then it follows from Kummer's theorem [8] that the reduced symplectic form is identical with the canonical symplectic form $d\sigma$ on $T^*\mathcal{S}_{n-1}$. Namely, $d\tilde{\sigma}$ and $d\sigma$ satisfy the relation

$$\iota^* d\tilde{\sigma} = \nu^* d\sigma, \tag{23}$$

where $\iota : J^{-1}(0) \to T^*B_n$ is the inclusion map. In a summary, we have the following theorem.

Theorem 2.2 ([1]) *The extended phase space* $(T^*B_n, d\tilde{\sigma})$ *is reduced to the phase space* $(T^*\mathcal{S}_{n-1}, d\sigma)$ *by the symplectic* $\mathbf{R}$*-action* $\tilde{\psi}_s$ *defined by (20).*

Remark We do not have to go to an explicit expression of $d\sigma$ into details since we can reduce any Hamiltonian equations invariant under $\tilde{\psi}_s$ on T^*B_n to Hamiltonian equations on $T^*\mathcal{S}_{n-1}$ by using not $d\sigma$ but the projection ν.

3 Hamiltonian form of both the family of the AHLEs and the e-geodesics

We organize a Hamiltonian system on $(T^*\mathcal{S}_{n-1}, d\sigma)$ as a Hamiltonian form of both the family of the AHLEs and the e-geodesics on $\mathcal{S}_{n-1}$, that is given along with the symplectic reduction from another integrable Hamiltonian system on the extended phase space $(T^*B_n, d\tilde{\sigma})$.

3.1 Completely integrable Hamiltonian system $(T^*B_n, d\tilde{\sigma}, \tilde{K})$

Examining closely the Hamiltonian form of the AHLE found by Nakamura [4], we consider the Hamiltonian system $(T^*B_n, d\tilde{\sigma}, \tilde{K})$ with the Hamiltonian

$$\tilde{K}(x, y) = \sum_{k=1}^{n} (x_k y_k)^2 \quad ((x, y) \in T^*B_n), \tag{24}$$

whose Hamiltonian equation takes the form

$$\dot{x}_j = 2x_j^2 y_j, \quad \dot{y}_j = -2x_j y_j^2 \quad (j = 1, 2, \cdots, n). \tag{25}$$

Since Eq.(25) admits $\{x_j y_j\}_{j=1,2,\cdots,n}$ as first integrals, it is easily integrated to be

$$\begin{aligned} x_j(t) &= \exp\big(2t x_j(0) y_j(0)\big)\, x_j(0), \\ y_j(t) &= \exp\big(-2t x_j(0) y_j(0)\big)\, y_j(0) \end{aligned} \qquad (j = 1, 2, \cdots, n). \tag{26}$$

Theorem 3.1 ([1]) *The Hamiltonian system* $(T^*B_n, d\tilde{\sigma}, \tilde{K})$ *is explicitly solvable by quadrature with the solution (26) and completely integrable in the sense that it admits* n *functionally independent and commutative first integrals* $\{x_j y_j\}_{j=1,2,\cdots,n}$.

Remark The defining equation (24) of the Hamiltonian $\tilde{K}$ can be read as the relation among the first integrals $\{x_j y_j\}_{j=1,2,\cdots,n}$ and the Hamiltonian $\tilde{K}$.

3.2 The reduced Hamiltonian system $(T^*\mathcal{S}_{n-1}, d\sigma, K)$

Since $\tilde{K}(x, y)$ is invariant under the $\mathbf{R}$-action $\tilde{\psi}_s$, we can apply the symplectic reduction organized in section 2 to $(T^*B_n, d\tilde{\sigma}, \tilde{K})$, too. According to Marsden and Weinstein [9], the reduced Hamiltonian $K(\xi, \eta)$ is defined to satisfy $\tilde{K} \circ \iota = K \circ \nu$. By calculation with $(\xi, \eta) = \nu(x, y)$ and $x^T y = \xi^T \eta = 0$, we obtain the reduced Hamiltonian in the form,

$$K(\xi, \eta) = \sum_{k=1}^{n} (\xi_k \eta_k)^2 \quad ((\xi, \eta) \in T^*\mathcal{S}_{n-1}). \tag{27}$$

Lemma 3.2 ([1]) *The Hamiltonian system* $(T^*B_n, d\tilde{\sigma}, \tilde{K})$ *is reduced by the* $\mathbf{R}$*-action* $\tilde{\psi}_s$ *to the Hamiltonian system* $(T^*\mathcal{S}_{n-1}, d\sigma, K)$.

In turn, we derive the Hamiltonian equation for the reduced Hamiltonian system $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ together with its solution. Since any trajectories of $(T^*B_n, d\tilde{\sigma}, \tilde{K})$ subject to $(x(0), y(0)) \in J^{-1}(0)$ are placed on $J^{-1}(0)$, we can apply the projection ν and its differential $\nu_* : T(J^{-1}(0)) \to T(T^*\mathcal{S}_{n-1})$ to the solution (26) with $(x(0), y(0)) \in J^{-1}(0)$ and the differential equation (25) restricted on $J^{-1}(0)$, respectively. By calculation, we have

$$\begin{aligned} \dot{\xi}_j &= 2(\xi_j\eta_j)\xi_j - 2\left(\sum_{k=1}^{n}(\xi_k\eta_k)\xi_k\right)\xi_j \\ \dot{\eta}_j &= -2(\xi_j\eta_j)\eta_j + 2\left(\sum_{k=1}^{n}(\xi_k\eta_k)\xi_k\right)\eta_j \end{aligned} \qquad (j = 1, 2, \cdots, n) \tag{28}$$

as the Hamiltonian equation for the reduced Hamiltonian system $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ together with its solution

$$\begin{aligned} \xi_j(t) &= S(t)^{-1}\exp\left(2t\xi_j(0)\eta_j(0)\right)\xi_j(0) \\ \eta_j(t) &= S(t)\,\exp(-2t\xi_j(0)\eta_j(0))\,\eta_j(0) \end{aligned} \qquad (j = 1, 2, \cdots, n) \tag{29}$$

with

$$S(t) = \sum_{k=1}^{n}\exp\left(2t\xi_k(0)\eta_k(0)\right)\xi_k(0). \tag{30}$$

The solution (29) is understood to be obtained by quadrature since the projection process, $(\xi(t), \eta(t)) = \nu(x(t), y(t))$ with $(x(0), y(0)) \in J^{-1}(0)$, is of algebraic manipulation and since the solution process for (26) is of quadrature. Further, as seen easily from (28), the reduced Hamiltonian system admits $\{\xi_j\eta_j\}_{j=1,2,\cdots,n}$ as first integrals. Since $\{\xi_j\eta_j\}_{j=1,2,\cdots,n}$ are the reduction of $\{x_jy_j\}_{j=1,2,\cdots n}$, $n-1$ integrals, $\{\xi_j\eta_j\}_{j=1,2,\cdots,n-1}$ of them are functionally independent and commutative with relation $\sum_{k=1}^{n-1}\xi_k\eta_k = -\xi_n\eta_n$. Hence, we have the following theorem.

Theorem 3.3 ([1]) *The Hamiltonian system* $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ *is explicitly solvable by quadrature with the solution (29) and completely integrable in the sense it admits* $n-1$ *functionally independent and commutative first integrals* $\{\xi_j\eta_j\}_{j=1,2,\cdots,n-1}$ *subject to* $\xi_n\eta_n = -\sum_{k=1}^{n-1}\xi_k\eta_k$.

Remark The defining equation (27) of the Hamiltonian K and $\sum_{k=1}^{n-1}\xi_k\eta_k = -\xi_n\eta_n$ can be read as the relation among the first integrals $\{\xi_j\eta_j\}_{j=1,2,\cdots,n}$ and the Hamiltonian K.

3.3 The family of the ALHEs and $(T^*\mathcal{S}_{n-1}, d\sigma, K)$

Fixing the values of the first integrals $\xi_j(t)\eta_j(t)$ to be $c_j = \xi_j(0)\eta_j(0)$ $(j = 1, 2, \cdots, n)$ in Eqs. (29) and (30), we see that $\xi_j(t)$ $(j = 1, 2, \cdots, n)$ of (29) satisfy the AHLE given by (1). On referring to the family of differential equations in the form (1) associated with all the sets of coefficients $\{\{c_j\}_{j=1,2,\cdots,n}\}$ with $\sum_{k=1}^{n}c_k = 0$ as the family of AHLEs, we understand that the solution (29) of the Hamiltonian system $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ exhausts all the trajectories of the family of AHLEs and vice versa. Therefore, we have the following theorem.

Theorem 3.4 ([1]) *The Hamiltonian system* $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ *is a Hamiltonian form of the family of the AHLEs.*

3.4 $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ and the e-geodesics

We start with considering $\mathcal{S}_{n-1}$ as the statistical manifold of finite discrete distributions. Let $\mathcal{E} = \{1, 2, \cdots, n\}$ be the set of n elementary events, and let $p(\ell, \xi)$ be the probability of a finite discrete distribution defined by

$$p(\ell; \xi) = \xi_\ell \quad (\ell \in \mathcal{E},\ \xi \in \mathcal{S}_{n-1}). \tag{31}$$

Then the family of all the finite discrete distributions $\{\{p(\ell, \xi)\}_{\ell \in \mathcal{E}}\}_{\xi \in \mathcal{S}_{n-1}}$ is naturally identified with $\mathcal{S}_{n-1}$. The intrinsic coordinates of $\mathcal{S}_{n-1}$ defined by

$$\zeta_j = \log \frac{\xi_j}{\xi_n} \quad (j = 1, 2, \cdots, n-1) \tag{32}$$

are called the exponential coordinates, which are used very often for information geometry of $\mathcal{S}_{n-1}$ (see Amari and Nagaoka [7] as the standard literature). The exponential coordinates ζ are, however, not so convenient for global analysis of mechanics on $\mathcal{S}_{n-1}$ like the objective of this article. Using (32), we can draw explicit expressions of the Fisher metric and the exponential-type geodesics in terms of our ξ from those known in terms of the conventional exponential coordinates ζ (see [7]) as follows.

By a straightforward calculation, we have the identity

$$ds^2 = \sum_{k=1}^{n} \frac{1}{\xi_k} d\xi_k \otimes d\xi_k = \sum_{h,\ell=1}^{n-1} \left(\frac{\delta_{h\ell}}{\zeta_h} + \frac{1}{1 - \sum_{m=1}^{n-1} \zeta_m} \right) d\zeta_h \otimes d\zeta_\ell, \tag{33}$$

on the Fisher metric between the expressions in our coordinates ξ and in the conventional exponential coordinates ζ (see [7]), where δ_{hm} denotes the Kronecker's delta. The exponential-type geodesics are described in the straight-line form

$$\zeta_j(t) = 2c_j t + \zeta_j(0) \quad (t \in \mathbf{R},\ j = 1, 2, \cdots, n-1) \tag{34}$$

with arbitrary constants $c_j \in \mathbf{R}$ $(j = 1, 2, \cdots, n-1)$ in terms of the exponential coordinates ζ according to [7]. Through the relation (32), Eq. (34) is shown, by calculatiion, to be equivalent to Eq. (29) under the first integral constraint

$$\xi_j(t)\eta_j(t) \equiv \xi_j(0)\eta_j(0) = c_j \quad (j = 1, 2, \cdots, n) \quad \text{with} \quad c_n = -\sum_{k=1}^{n-1} c_k. \tag{35}$$

In a summary, we have the following theorem.

Theorem 3.5 ([1]) *The Hamiltonian system* $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ *is the Hamiltonian form of the exponential-type geodesics on the statistical manifold* $\mathcal{S}_{n-1}$ *of the finite discrete distributions.*

Combining Theorem 3.5 with Theorem 3.3, we have the following theorem.

Theorem 3.6 ([1]) *The Hamiltonian system* $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ *for the exponential-type geodesics on* $\mathcal{S}_{n-1}$ *is explicitly solvable by quadrature with the solution (29) and completely integrable in the sense it admits* $n-1$ *functionally independent and commutative first integrals* $\xi_j\eta_j$ $(j = 1, 2, \cdots, n-1)$ *with* $\xi_n\eta_n = -\sum_{k=1}^{n-1} \xi_k\eta_k$.

3.5 Conclusion of section 3

On closing this section, we show the following theorem as the net result of this section by putting Theorems 3.3, 3.4, and 3.6 together.

Theorem 3.7 ([1]) *The explicitly solvable and completely integrable Hamiltonian system* $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ *is the Hamiltonian form of both the family of AHLEs and the exponential-type geodesics of the finite discrete distributions.*

As the closing remark of section 3, we would like to mention of a distinction between Hamiltonian mechanics in this article and the previous studies on Hamiltonian mechanics on statistical manifolds (Fujiwara and Amari [10], Boumuki and Noda [11]). The phase spaces taken in [10] and [11] are even-dimensional classical statistical manifolds while the cotangent bundle of the statistical manifold of the finite discrete distributions are taken as the phase space in this article. Since we are interested in the exponential-type geodesics here, our cotangent-bundle setting looks more natural than the phase-space setting made in [10, 11] since geodesics on a given manifold are known very well to be governed by a second order differential equation on that manifold.

4 Quantum statistical analogues

In this section, we briefly give a pair of quantum analogues of the reduction of Hamiltonian system $(T^*B_n, d\tilde{\sigma}, \tilde{K})$ to $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ in order to find Hamiltonian forms of the MAHLE-I, the MAHLE-II, and the exponential-type geodesics on the quantum statistical manifold Q_n.

4.1 Symplectic reduction of $T^*H_n^+$ to T^*Q_n

The organization of this subsection is almost on a parallel with that of section 2. Let us start with considering the space of $n \times n$ positive definite Hermitean matrices H_n^+ and its submanifold Q_n defined by (4) as quantum statistical counterparts of B_n and $\mathcal{S}_{n-1}$, respectively. We endow the Riemannian metrics

$$\langle \Xi, \Xi' \rangle_\rho = \mathrm{Tr}\,(\Xi\, L_\rho(\Xi')) \quad (\rho \in Q_n,\ \Xi, \Xi' \in H_n^{tr0} \cong T_\rho Q_n) \tag{36}$$

with Q_n, and

$$\langle\!\langle X, X' \rangle\!\rangle_r = \frac{1}{\mathrm{Tr}\,(r)}\,\mathrm{Tr}\,(X \tilde{L}_r(X')) \quad (r \in H_n^+,\ X, X' \in H_n \cong T_\rho H_n^+) \tag{37}$$

with H_n^+, where $L_\rho(\Xi)$ is the symmetric logarithmic derivative (SLD) of $\Xi \in T_\rho Q_n$ defined by

$$\Xi = \frac{1}{2}\Big(\rho L_\rho(\Xi) + L_\rho(\Xi)\rho\Big) \quad (\rho \in Q_n,\ \Xi \in H_n^{tr0} \cong T_\rho Q_n) \tag{38}$$

(see [12]), and $\tilde{L}_r(X)$ the extended one of $X \in T_r H_n^+$ by

$$X = \frac{1}{2}\Big(r\tilde{L}_r(X) + \tilde{L}_r(X) r\Big) \quad (r \in H_n^+,\ X \in H_n \cong T_r H_n^+). \tag{39}$$

The metric $\langle\cdot,\cdot\rangle$ defined by (36) is the quantum SLD-Fisher metric (cf. Hayashi [12]), so that the Riemannian manifold $(Q_n, \langle\cdot,\cdot\rangle)$ is called the quantum statistical manifold (QSM). We have the following lemma on a relation between $(\mathcal{S}_{n-1}, ds^2)$ and $(Q_n, \langle\cdot,\cdot\rangle)$.

Lemma 4.1 ([2, 3]) *Let the submanifold $\mathcal{D}_n$ of Q_n be defined by (7). Then the Riemannian submanifold $(\mathcal{D}_n, \langle\cdot,\cdot\rangle|_{\mathcal{D}_n})$ of the quantum statistical manifold $(Q_n, \langle\cdot,\cdot\rangle)$ is the isometric embedding of the Riemannian manifold $(\mathcal{S}_{n-1}, ds^2)$ of finite discrete distributions.*

In view of Lemma 4.1, a differential equation on Q_n deserves for being referred to as a matrix averaged Hebbian learning equation (MAHLE) if it can be restricted on $\mathcal{D}_n$ and if the restriction becomes the AHLE: A pair of MAHLEs to be dealt with in what follows are introduced already by (5) and (6).

As a quantum statistical counterpart of the principal $\mathbf{R}$-bundle $\mu : B_n \to \mathcal{S}_{n-1}$, H_n^+ is made into the principal $\mathbf{R}$-bundle over Q_n with the $\mathbf{R}$-action

$$\Psi_s : r \in H_n^+ \mapsto e^s r \in H_n^+ \quad (s \in \mathbf{R}) \tag{40}$$

and the projection

$$\mu_Q : r \in H_n^+ \mapsto (\mathrm{Tr}\,(r))^{-1} r \in Q_n. \tag{41}$$

Due to this $\mathbf{R}$-bundle structure of H_n^+ over Q_n, the symplectic reduction applied to $T^*H_n^+$ in what follows can be understood to afford another new example of Kummer's theorem [8] in addition to the reduction made in section 2.

We consider the cotangent bundles, $T^*H_n^+$ of H_n^+ and T^*Q_n of Q_n, which take the forms

$$T^*H_n^+ = H_n^+ \times H_n \tag{42}$$

and

$$T^*Q_n = \{(\rho, \Pi) \in Q_n \times H_n \,|\, \mathrm{Tr}\,(\rho\Pi) = 0\}, \tag{43}$$

respectively. Note that we have the diffeomorphism

$$\beta_Q : (\rho, \Pi) \in T^*Q_n \mapsto (\rho, L_\rho^{-1}(\Pi)) \in TQ_n \simeq Q_n \times H_n^{tr0} \tag{44}$$

as the quantum statistical analogue of (11), where L_ρ^{-1} denotes the inverse of the SLD L_ρ (see (38)).

With $T^*H_n^+$, the canonical symplectic form $d\tilde{\Lambda}$ is endowed, which is defined to be

$$\begin{gathered} d\tilde{\Lambda}_{(r,P)}((X,Y), (X',Y')) = \mathrm{Tr}\,(YX') - \mathrm{Tr}\,(XY') \\ ((r,P) \in T^*H_n^+, (X,Y), (X',Y') \in T_{(r,P)}(T^*H_n^+) = H_n \times H_n) \end{gathered} \tag{45}$$

with

$$\tilde{\Lambda}_{(r,P)}(X,Y) = \mathrm{Tr}\,(PX). \quad ((r,P) \in T^*H_n^+, (X,Y) \in T_{(r,P)}(T^*H_n^+)). \tag{46}$$

We proceed to the symplectic reduction of the phase space $(T^*H_n^+, d\tilde{\Lambda})$ by the symplectic $\mathbf{R}$-action

$$\tilde{\Psi}_s : (r,\, P) \in T^*H_n^+ \mapsto (e^s r,\, e^{-s}P) \in T^*H_n^+ \quad (s \in \mathbf{R}). \tag{47}$$

On a parallel with the reduction made in section 2, we find the moment map

$$J_Q : (r, P) \in T^*H_n^+ \mapsto \mathrm{Tr}\,(rP) \in \mathbf{R} \tag{48}$$

associated with the $\mathbf{R}$-action $\tilde{\Psi}_s$. The quotient set, $J_Q^{-1}(0)/\mathbf{R}$, of the level set $J_Q^{-1}(0)$ by the $\mathbf{R}$-action $\tilde{\Psi}_s$ is the reduced phase space: By the smooth map,

$$\nu_Q : (r, P) \in J_Q^{-1}(0) \mapsto (\rho, \Pi) = \Big((\mathrm{Tr}\,(r))^{-1} r, \mathrm{Tr}\,(r)P\Big) \in T^*Q_n \cong J_Q^{-1}(0)/\mathbf{R}, \tag{49}$$

$J^{-1}(0)/\mathbf{R}$ is realized as T^*Q_n.

Following the same procedure as used in section 2, we obtain the reduced symplectic form on T^*Q_n: According to Kummer's theorem [8], the reduced symplectic form turns out to be the canonical one, which is denoted by $d\Lambda$ henceforce. The $d\Lambda$ is of course determined by the formula $\iota_Q^* d\tilde{\Lambda} = \nu_Q^* d\Lambda$ similar to Eq. (23), where $\iota_Q : J_Q^{-1}(0) \to T^*Q_n$ is the inclusion. On the same account as given in the ending remark of section 2, we will not need an explicit expression of $d\Lambda$ in what follows. At conclusion of this subsection, we have the following theorem.

Theorem 4.2 ([2]) *The phase space $(T^*H_n^+, d\tilde{\Lambda})$ is reduced to the phase space $(T^*Q_n, d\Lambda)$ by the symplectic $\mathbf{R}$-action $\tilde{\Psi}_s$ defined by (47).*

4.2 The exponential-type geodesics on Q_n

We move to the exponential-type (e-) geodesics on Q_n in turn. To discuss the e-geodesics in this article and other papers [1, 2, 3, 6] by the author, the SLD defined by (38) is taken as the logarithmic derivative. Then, according to Hayashi [12], the e-geodesic denoted by $\rho_e(t)$ on Q_n with the initial condition

$$\rho_e(0) = \rho_0, \quad \dot{\rho}_e(0) = \Xi_0 \tag{50}$$

is given explicitly in the form

$$\rho_e(t) = \Big\{\mathrm{Tr}\,\Big(e^{\frac{t}{2}L_{\rho_0}(\Xi_0)}\,\rho_0\,e^{\frac{t}{2}L_{\rho_0}(\Xi_0)}\Big)\Big\}^{-1} e^{\frac{t}{2}L_{\rho_0}(\Xi_0)}\,\rho_0\,e^{\frac{t}{2}L_{\rho_0}(\Xi_0)}, \tag{51}$$

where $L_{\rho_0}(\Xi_0)$ is the SLD of Ξ_0 at ρ_0. The differential of $\rho_e(t)$ in t satisfies the relation

$$\dot{\rho}_e(t) = \frac{1}{2}\{\rho_e(t)\,L_{\rho_0}(\Xi_0) + L_{\rho_0}(\Xi_0)\,\rho_e(t)\} - \mathrm{Tr}\,(L_{\rho_0}(\Xi_0)\rho_e(t))\,\rho_e(t) \tag{52}$$

(see [12]). Equation (52) has the following alternative form

$$\begin{aligned}\dot{\rho}_e(t) =& \rho_e(t)\frac{1}{2}\left(L_{\rho_0}(\Xi_0) - \frac{\mathrm{Tr}\,(L_{\rho_0}(\Xi_0))}{n}I_n\right) + \frac{1}{2}\left(L_{\rho_0}(\Xi_0) - \frac{\mathrm{Tr}\,(L_{\rho_0}(\Xi_0))}{n}I_n\right)\rho_e(t)\\ &- \mathrm{Tr}\,\left(\left(L_{\rho_0}(\Xi_0) - \frac{\mathrm{Tr}\,(L_{\rho_0}(\Xi_0))}{n}I_n\right)\rho_e(t)\right)\rho_e(t)\end{aligned} \tag{53}$$

with

$$\mathrm{Tr}\left(L_{\rho_0}(\Xi_0) - \frac{\mathrm{Tr}\,(L_{\rho_0}(\Xi_0))}{n} I_n\right) = 0, \tag{54}$$

where I_n denotes the identity matrix of degree n. Equation (53) is in a very convenient form for connecting the e-geodesics with the MAHLE-II.

4.3 The MAHLE-I and its Hamiltonian form

As a natural quantum statistical analogue of the Hamiltonian system $(T^*B_n, d\tilde{\sigma}, \tilde{K})$, we consider the Hamiltonian system $(T^*H_n^+, d\tilde{\Lambda}, \tilde{F}_I)$ with the Hamiltonian

$$\tilde{F}_I(r,P) = \mathrm{Tr}\,(rPrP) \quad ((r,P) \in T^*H_n^+), \tag{55}$$

whose Hamiltonian equation is calculated to be

$$\dot{r} = 2rPr, \quad \dot{P} = -2PrP. \tag{56}$$

Since rP $(= (Pr)^\dagger)$ is allowed as a matrix-valued first integral of (56), we easily solve (56) to be

$$\begin{aligned} r(t) &= \exp\big(t\,(r(0)P(0))\big)\, r(0)\, \exp\big(t\,(r(0)P(0))\big)^\dagger, \\ P(t) &= \big(\exp(-t\,r(0)P(0))\big)^\dagger\, P(0)\, \exp(-t\,r(0)P(0)) \end{aligned} \tag{57}$$

by quadrature. The matrix-valued first integral rP consists of n^2, a half of the dimension of $T^*H_n^+$, functionally independent real-valued first integrals which form a non-commutative Lie algebra. We note here that the defining equation (55) of the Hamiltonian $\tilde{F}_I$ can be read as the relation among the entries of rP and the Hamiltonian $\tilde{F}_I$. Then, we have the following theorem.

Theorem 4.3 ([2]) *The Hamiltonian system $(T^*H_n^+, d\tilde{\Lambda}, \tilde{F}_1)$ is explicitly solvable by quadrature and integrable in the sense that it admits the matrix-valued first integral rP consisting of n^2 functionally independent first integrals forming a non-commutative Lie algebra.*

Since the Hamiltonian $\tilde{F}_I$ of the Hamiltonian system $(T^*H_n^+, d\tilde{\Lambda}, \tilde{F}_I)$ is invariant under the symplectic $\mathbf{R}$-action $\tilde{\Psi}_s$ given by (47), we can apply the symplectic reduction by the $\mathbf{R}$-action $\tilde{\Psi}_s$ to $(T^*H_n^+, d\tilde{\Lambda}, \tilde{F}_1)$, too. According to the reduction formula $\tilde{F}_I \circ \iota_Q = F_I \circ \nu_Q$, for $\tilde{F}_I$, we obtain the function

$$F_I(\rho,\Pi) = \mathrm{Tr}\,(\rho\Pi\rho\Pi) \quad ((\rho,\Pi) \in T^*Q_n \subset Q_n \times H_n) \tag{58}$$

as the Hamiltonian of the reduced Hamiltonian system of $(T^*H_n^+, d\tilde{\Lambda}, \tilde{F}_1)$. The Hamiltonian equation of the reduced system $(T^*Q_n, d\Lambda, F_I)$ is available by applying the differential map of the projection $\nu_Q : J_Q^{-1}(0) \to T^*Q_n$ to the Hamiltonian equation (56) restricted on $J^{-1}(0)$, which turns out take the form

$$\begin{aligned} \dot{\rho} &= 2\rho\Pi\rho - 2\mathrm{Tr}\,(\rho\Pi\rho)\,\rho, \\ \dot{\Pi} &= -2\Pi\rho\Pi + 2\mathrm{Tr}\,(\rho\Pi\rho)\,\Pi. \end{aligned} \tag{59}$$

The solution of (59) is available directly by applying the projection ν_Q to the solution (57) with $(r(0), P(0)) \in J_Q^{-1}(0)$, which is written in the form

$$\begin{aligned} \rho_I(t) &= (S_I(t))^{-1} \exp\big(t\,\rho(0)\Pi(0)\big)\,\rho(0)\,\exp\big(t\,\rho(0)\Pi(0)\big)^{\dagger}, \\ \Pi_I(t) &= S_I(t)\,\exp\big(-t\,\rho(0)\Pi(0)\big)^{\dagger}\,\Pi(0)\,\exp\big(-t\,\rho(0)\Pi(0)\big) \end{aligned} \tag{60}$$

with

$$S_I(t) = \mathrm{Tr}\Big(\exp\big(t\,\rho(0)\Pi(0)\big)\,\rho(0)\,\exp\big(t\,\rho(0)\Pi(0)\big)^{\dagger}\Big). \tag{61}$$

Taking the defining constraint $\mathrm{Tr}\,(\rho\Pi) = 0$ into account, we see that $\rho\Pi$ is a matrix-valued first integral consisting of $n^2 - 1$, a half of $\dim T^*Q_n$, functionally independent real-valued first integrals forming a non-commutative Lie algebra. Hence, we can say that $(T^*Q_n, d\Lambda, F_I)$ is integrable. Further, as seen above, the process to reach (60) is the combination of quadrature to have (57) and the algebraic manipulations in the projection process by ν_Q, so that the solution (60) is understood to be obtained by quadrature. Hence, we have the following theorem.

Theorem 4.4 ([2]) *The Hamiltonian system $(T^*Q_n, d\Lambda, F_1)$ is explicitly solvable by quadrature and integrable in the sense that it admits the matrix-valued first integral $\rho\Pi$ consisting of n^2-1 functionally independent first integrals forming a non-commutative Lie algebra.*

We give an account for referring to $(T^*Q_n, d\Lambda, F_I)$ as a Hamiltonian form of the family of MAHLE-I with all the coefficient matrices $\{\Gamma \in (H_n^+ \cdot H_n)^{tr0}\}$. On denoting $\rho(0)\Pi(0)$ by Γ, we see that $\Gamma \in (H_n^+ \cdot H_n)^{tr0}$ and that the solution (60) with $\rho_I(0)\Pi_I(0) = \Gamma \in (H_n^+ \cdot H_n)^{tr0}$ satisfies the first order differential equation (5). Conversely, for any trajectory denoted by $\rho_\Gamma(t)$ of the MAHLE-I (5) with $\rho_\Gamma(0) = \rho_0$, we find the trajectory $(\rho_I(t), \Pi_I(t))$ of the Hamiltonian system $(T^*Q_n, d\Lambda, F_I)$ with $(\rho_I(0), \Pi_I(0)) = (\rho_0, \rho_0^{-1}\Gamma)$, whose $\rho_I(t)$ coincides with the trajectory $\rho_\Gamma(t)$ of the MAHLE-I. In a summary, we have the following theorem.

Theorem 4.5 ([2]) *The Hamiltonian system $(T^*Q_n, d\Lambda, F_1)$ is a Hamiltonian form of the family of the MAHLE-Is.*

At the end of this subsection, we make a mention of a relation between the trajectories of the family of the MAHLE-Is and the e-geodesics on Q_n: We can only find the relation which is shown in Theorem 3.7 under the isometry mentioned in Lemma 4.1.

4.4 The MAHLE-II and its Hamiltonian form

At the end of the previous subsection, we see that the Hamiltonian form, $(T^*Q_n, d\Lambda, F_I)$ of the MAHLE-I is not a Hamiltonian form of the e-geodesics on Q_n. On recalling that the coefficient matrix Γ in (5) originates in the matrix-valued first integral rP of the Hamiltonian system $(T^*H_n^+, d\tilde{\Lambda}, \tilde{F}_I)$, we start with considering another Hamiltonian

$$\tilde{F}_{II}^{C}(r, P) = \mathrm{Tr}\big(CrP + PrC\big), \tag{62}$$

on $(T^*H_n^+, d\tilde{\Lambda})$ which includes already a fixed traceless Hermitean matrix C having a role of coefficient matrix appearing in the MAHLE-II given by (6). The Hamiltonian equation of $(T^*H_n^+, d\tilde{\Lambda}, \tilde{F}_{II}^C)$ is calculated to be in a separation of variables form,

$$\dot{r} = rC + Cr, \quad \dot{P} = -PC - CP, \tag{63}$$

which is easily solved to be

$$r(t) = \exp(tC)\, r(0)\, \exp(tC), \quad P(t) = \exp(-tC)\, P(0)\, \exp(-tC) \tag{64}$$

by quadrature.

Like in the case of MAHLE-I, we apply the symplectic reduction by the $\mathbf{R}$-action $\tilde{\Psi}_s$ (see (47)) to $(T^*H_n^+, d\tilde{\Lambda}, \tilde{F}_{II}^C)$, since the Hamiltonian $\tilde{F}_{II}^C$ is invariant under the symplectic action $\tilde{\Psi}_s$, too. Indeed, through the reduction formula $\tilde{F}_{II}^C \circ \iota_Q = F_{II}^C \circ \nu_Q$, we obtain the reduced Hamiltonian

$$F_{II}^C(\rho, \Pi) = \mathrm{Tr}\,(C\rho\Pi + \Pi\rho C) \quad ((\rho, \Pi) \in T^*Q_n \subset Q_n \times H_n). \tag{65}$$

By applying the differential map of the projection $\nu_Q : J_Q^{-1}(0) \to T^*Q_n$ (see (49)) to the Hamiltonian equation (63) restricted on $J_Q^{-1}(0)$, the Hamiltonian equation of the reduced system $(T^*Q_n, d\Lambda, F_{II}^C)$ is written in the form

$$\dot{\rho} = \rho C + C\rho - 2\mathrm{Tr}\,(C\rho)\,\rho, \quad \dot{\Pi} = -\Pi C - C\Pi + 2\mathrm{Tr}\,(C\rho)\,\Pi \quad (C \in H_n^{tr0}). \tag{66}$$

As seen immediately, the first equation in (66) for ρ is the very MAHLE-II given by (6). The solution of (66) is obtained directly from (64) with $(r(0), P(0)) \in J_Q^{-1}(0)$ through the projection $\nu : J_Q^{-1}(0) \to T^*Q_n$. which turns out, by calculation, to take the form

$$\begin{aligned} \rho_{II}(t) &= (S_{II}(t))^{-1} \exp\big(tC\big)\, \rho(0)\, \exp\big(tC\big), \\ \Pi_{II}(t) &= S_{II}(t)\, \exp\big(-tC\big)\, \Pi(0)\, \exp\big(-tC\big) \end{aligned} \tag{67}$$

with

$$S_{II}(t) = \mathrm{Tr}\Big(\exp\big(tC\big)\, \rho(0)\, \exp\big(tC\big)\Big). \tag{68}$$

From the expression, (67) and (68), of the solution of $(T^*Q_n, d\Lambda, F_{II})$, any first integrals seems to be hardly found, so that we say the explicit solvability of $(T^*Q_n, d\Lambda, F_{II}^C)$ only, here. In a summary, we have the following.

Theorem 4.6 ([3]) *The Hamiltonian system $(T^*Q_n, d\Lambda, F_{II}^C)$ is a Hamiltonian form of the MAHLE-II, which is explicitly solvable to be (67) with (68) by quadrature.*

We proceed to study a relation between the MAHLE-II and the exponential-type geodesics of Q_n. A comparison between the governing equation (53) with (54) for the e-geodesic with the initial condition (50) and the MAHLE-II given by (6) yields the following theorem as an extension of [6].

Theorem 4.7 ([3]) *All the exponential-type geodesics $\rho_e(t)$ with the initial condition (50) are the trajectories $\rho_{II}(t)$ of the MAHLE-II with the coefficient matrix*

$$C = \frac{1}{2}\left\{L_{\rho_0}(\Xi_0) - \frac{\mathrm{Tr}\left(L_{\rho_0}(\Xi_0)\right)}{n} I_n\right\}. \tag{69}$$

Conversely, all the trajectories $\rho_{II}(t)$ of the MAHLE-II with arbitrary $C \in H_n^{tr0}$ are the exponential-type geodesics $\rho_e(t)$ with the initial condition

$$\rho_e(0) = \rho_{II}(0), \quad \dot{\rho}_e(0) = \rho_{II}(0)C + C\rho_{II}(0) - 2\mathrm{Tr}\left(C\rho_{II}(0)\right)\rho_{II}(0). \tag{70}$$

In view of Theorem 4.7, we reach to the concluding theorem of this subsection.

Theorem 4.8 ([3]) *The family of Hamiltonian systems $\{(T^*Q_n, d\Lambda, F_{II}^C)\}_{C\in H_n^{tr0}}$ realizes all the exponential-type geodesics on Q_n. Conversely, any exponential-type geodesic on Q_n is a trajectory of a Hamiltonian system belonging to $\{(T^*Q_n, d\Lambda, F_{II}^C)\}_{C\in H_n^{tr0}}$.*

5 Conclusion

In this article, we have shown very briefly three kinds of Hamiltonian systems:

(1) The explicitly solvable and completely integrable Hamiltonian system $(T^*\mathcal{S}_{n-1}, d\sigma, K)$ is found as a Hamiltonian form of both the family of the AHLEs and the exponential-type geodesics on the finite discrete distributions.

(2) As one of the pair of quantum analogues of the Hamiltonian form $(T^*\mathcal{S}_{n-1}, d\sigma, K)$, the explicitly solvable and integrable Hamiltonian system $(T^*Q_n, d\Lambda, F_I)$ is found as a Hamiltonian form of the family of the MAHLE-Is.

(3) As another of the pair of quantum analogues of the Hamiltonian form $(T^*\mathcal{S}_{n-1}, d\sigma, K)$, the family of explicitly solvable Hamiltonian systems $\{(T^*Q_n, d\Lambda, F_{II}^C)\}_{C\in H_n^{tr0}}$ is found. This family is understood to be a family of Hamiltonian forms of the exponential-type geodesics on Q_n in the sense that the family of Hamiltonian systems realizes all the e-geodesics and vice versa.

All through the process of finding and studying these three kinds of systems, the symplectic reduction technique within the framework of cotangent bundles works very well. Many of details and proofs are consigned in three future papers [1, 2, 3].

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