Nonintegrability and Chaos in Hamiltonian Systems with Saddle-Centers

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Abstract. In general, a Hamiltonian system is nonintegrabe if chaotic dynamics occurs. However, chaotic dynamics may not occur even if it is nonintegrable. Here we are interested in the following question: *Does chaotic dynamics occur in a Hamiltonian system when it is nonintegrable?* We review some previous results related to this question for two-degree-of-freedom Hamiltonian systems with saddle-centers and homoclinic orbits. We also state some extensions of the results to a higher-order approximation, heteroclinic orbits and more- or infinite-degree-of-freedom systems. In particular, the extended theory shows that Arnold diffusion type motions can occur in three- or more-degree-of-freedom systems.

Key words: Nonintegrability; chaos; Hamiltonian system; saddle-center; reversible system; homoclinic orbit; heteroclinic orbit; Melnikov method; Morales-Ramis theory; differential Galois theory.
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1 Introduction

1.1 Background

Consider n-degree-of-freedom Hamiltonian systems of the form

$$\dot{x} = J_n \mathrm{D}H(x), \quad x \in \mathbb{R}^{2n}, \tag{1.1}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is smooth or analytic, J_n is the $2n \times 2n$ symplectic matrix

$$J_n = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}$$

and id_n is the $n \times n$ identity matrix. Letting $x = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$, we rewrite (1.1) as

$$\dot{q} = \mathrm{D}_p H(p, q), \quad \dot{p} = -\mathrm{D}_q H(p, q), \tag{1.2}$$

which has a well-known form in mechanics when q and p are position and momentum, respectively. We begin with the definition of integrability for (1.1) (see, e.g., Section 3.2 of [24]).

Definition 1.1. *The Hamiltonian system* (1.1) *is* (Liouville) integrable *if there exist n scalar functions $F_1(x)\,(= H(x)), F_2(x), \dots, F_n(x)$ such that*

(i) $\mathrm{D}F_1(x), \dots, \mathrm{D}F_n(x)$ *are linearly independent a.e.*;

(ii) $[F_i, F_j](x) := \mathrm{D}F_i(x)J_n\mathrm{D}F_j(x) \equiv 0$.

It is a well-known fact the dynamics of integrable Hamiltonian systems are very simple as stated in the following theorem (see, e.g., Chapter 10 of [3] for the details).

Theorem 1.1 (Liouville-Arnold)**.** *Suppose that Eq.* (1.1) *is integrable and let* $F(x) = (F_1(x), \ldots, F_n(x))$*. If the level set* $F^{-1}(c)$ *is connected and compact for some* $c \in \mathbb{R}^n$*, then the dynamics of* (1.1) *on* $F^{-1}(c)$ *are diffeomorphic to a linear flow on* $\mathbb{T}^n$*.*

Let ϕ_t denote the flow of (1.1) and let $\gamma(t)$ be a periodic orbit in (1.1). The *stable* and *unstable manifolds* of $\gamma(t)$, $W^{\mathrm{s}}(\gamma(t))$ and $W^{\mathrm{u}}(\gamma(t))$, are defined as

$$W^{\mathrm{s}}(\gamma(t)) = \{x \in \mathbb{R}^n \mid \lim_{t\to+\infty} \inf_{s\in\mathbb{R}} |\phi_t(x) - \gamma(s)| = 0\},$$
$$W^{\mathrm{u}}(\gamma(t)) = \{x \in \mathbb{R}^n \mid \lim_{t\to-\infty} \inf_{s\in\mathbb{R}} |\phi_t(x) - \gamma(s)| = 0\}.$$

Assume that $W^{\mathrm{s}}(\gamma(t))$ and $W^{\mathrm{u}}(\gamma(t))$ are of dimension n. In this situation we state the well-known Smale-Birkhoff homoclinic theorem (see, e.g., Section 5.3 or [12] or Section 26 of [33]) as follows.

Theorem 1.2. *If* $W^{\mathrm{s}}(\gamma(t))$ *and* $W^{\mathrm{u}}(\gamma(t))$ *intersect transversely on the* $(2n-1)$*-dimensional level set* $H^{-1}(c)$ *for some* $c \in \mathbb{R}$*, then chaotic dynamics occurs: There exists a chaotic invariant set* Λ *containing*

(i) *countably many periodic orbits*;

(ii) *uncountably many bounded nonperiodic orbits*;

(iii) *a dense orbit.*

From Theorem 1.1 we see that if chaotic dynamics occurs, then Eq. (1.1) is nonintegrabe. However, chaotic dynamics may not occur even if it is nonintegrable. So we are interested in the following question: *Does chaotic dynamics occur when Eq.* (1.1) *is nonintegrable?*

As an example, we consider a two-degree-of-freedom system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -x_1 - cx_2^2 - dx_1^2, \quad \dot{x}_4 = -x_2 - 2cx_1x_2, \tag{1.3}$$

with the Hamiltonian $H = \frac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2)+cx_1x_2^2+\frac{1}{3}dx_1^3$. Eq. (1.3) is a generalization of the Hénon-Heiles system [13], which was discussed in many references (see [30] for an earlier list of such references). A numerically computed chaotic orbit in (1.3) for $c = 1$, $d = -1$ and $H = 1/6$, which was actually treated by Hénon and Heiles [13], is displayed in Fig. 1. It is a well-know fact that Eq. (1.3) is integrable for $c/d = 0, \frac{1}{6}, 1$ (see, e.g., [9]). The noninetgrability of (1.3) was shown for $c/d \neq 0, \frac{1}{6}, \frac{1}{2}, 1$ by Ito [17,18] earlier and for $c/d = \frac{1}{2}$ by Morales-Ruiz et al. [27] more than two decades later. The Ziglin theory [45] was used in the former work while the Morales-Ramis (or Morales-Ramis-Simó) theory [24,26,27], which was regarded as an extension of the Ziglin theory based on differential Galois theory [5,29], was used in the latter. See Section 3.2 for an outline of the Morales-Ramis theory. On the other hand, the occurrence of chaos in (1.3) was proved by Grotta Ragazzo [10] for $c/d \neq 0, \frac{1}{6}, \frac{1}{2}, \frac{3}{4}, 1$. He used asymptotic properties of special solutions to stationary Schrödinger equations and a theorem of Lerman [19] for

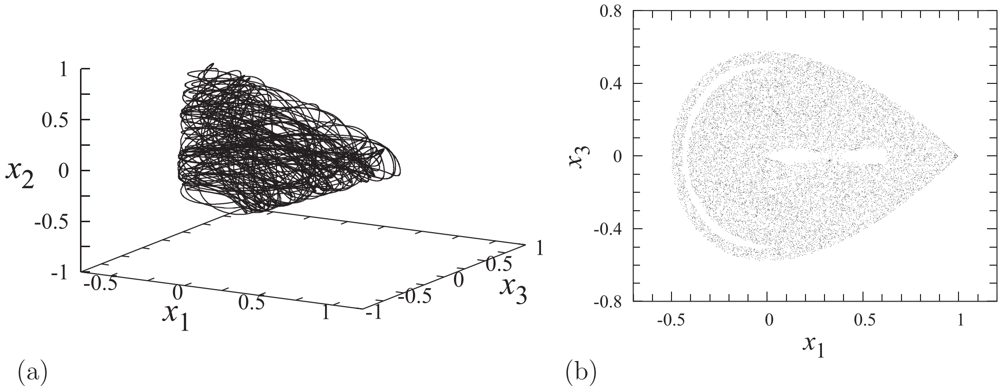

Figure 1: Numerically computed chaotic orbit in (1.3) for $c = 1$, $d = -1$ and $H = 1/6$: (a) Its projection onto the three-dimensional (x_1, x_2, x_3)-space; (b) its projection onto the two-dimensional (x_1, x_3)-space on the three-dimensional Poincaré section at $x_2 = 0$.

two-degree-of-freedom Hamiltonian systems with saddle-centers and homoclinic orbits. A different method based on a fundamental idea of Melnikov's method [12, 33], was also developed for such systems and used to prove the same result in [34]. See Section 2.2 for an outline of the method. Moreover, it was shown by an extension of the method in [41] that Eq. (1.3) exhibits chaotic dynamics for $c/d = \frac{3}{4}$. Thus, the nonintegrability is closely related to the occurrence of chaos in (1.3) although the occurrence of chaos for $c/d = \frac{1}{2}$ is still an open problem.

1.2 Object of this review

In this article, we review some previous results [34, 37] related to our question for two-degree-of-freedom Hamiltonian systems with saddle-centers,

$$\dot{x} = J_1 \mathrm{D}_x H(x, y), \quad \dot{y} = J_1 \mathrm{D}_y H(x, y), \quad (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2, \tag{1.4}$$

where $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is C^{r+1} $(r \geq 3)$ except that it is analytic when we discuss the nonintegrability of (1.4). Note that J_1 is the 2×2 symplectic matrix

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We assume the following on (1.4):

- **(A1)** For any $x \in \mathbb{R}^2$, $\mathrm{D}_x H(0, 0) = \mathrm{D}_y H(x, 0) = 0$;
- **(A2)** $J_1 \mathrm{D}_x^2 H(0, 0)$ has a pair of positive and negative eigenvalues, and there exists a homoclinic orbit $(x, y) = (x^{\mathrm{h}}(t), 0)$ such that $\lim_{t \to \pm\infty} x^{\mathrm{h}}(t) = 0$ (see Fig. 2);
- **(A3)** $J_1 \mathrm{D}_y^2 H(0, 0)$ has a pair of purely imaginary eigenvalues.

It follows from (A1)-(A3) that the x-plane, $\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid y = 0\}$, is invariant under the flow of (1.4); the origin $(x, y) = (0, 0)$ is a saddle-center; and by the Lyapunov center

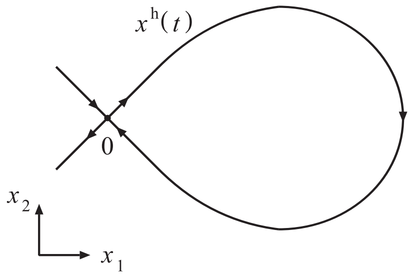

Figure 2: Homoclinic orbit $x^{\mathrm{h}}(t)$ on the x-plane

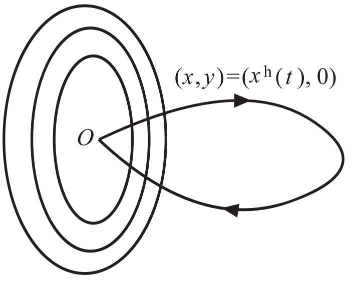

Figure 3: Periodic orbits near the origin

theorem (e.g., [1, 22]) there exists a family of periodic orbits near it (see Fig. 3). The periodic orbits have two-dimensional stable and unstable manifolds, which may intersect transversely on the level set $H^{-1}(c)$ for some $c \in \mathbb{R}$. Hence, by Theorem 1.2 chaotic dynamics may occur in (1.4).

We also state some extensions of the above results to a higher-order approximation in [41], heteroclinic orbits in [31], and more- or infinite-degree-of-freedom systems in [35,38]. In particular, the extended theory shows that Arnold diffusion type motions, which is not slow but very similar to well-know Arnold diffusion [2, 20], can occur in three- or more-degree-of-freedom systems. Moreover, a buckled beam, which was studied as an early example of infinite dimensional systems having chaotic motions by Holmes and Marsden [16] (see also [14, 23]) when it is subjected to damping and periodic external force, is shown to still exhibit chaotic motions even when it is not subjected to them.

The outline of this article is as follows: In Section 2 we describe the Melnikov-type method developed in [34] for detection of chaos in the two-degree-of-freedom Hamiltonian system (1.4). We begin with the standard Melnikov method [12, 33] and end with briefly illustrating the theory for an example including (1.3) as a special case. In Section 3 we describe the result of [37] on a relationship between nonintegrability and chaos for (1.4). Necessary parts of the differential Galois theory and Morales-Ramis theory are also briefly provided. We state several extensions of [31, 35, 38, 41] for the above results in Section 4 and finally give some comments on future work in Section 5.

2 Detection of Chaos

2.1 Standard Melnikov method

We begin with the standard Melnikov method. See Section 4.5 of [12] or Section 28 of [33] for the details. Consider two-dimensional time-periodic systems of the form

$$\dot{x} = J_1 \mathrm{D}H(x) + \varepsilon g(x,t), \quad x \in \mathbb{R}^2, \tag{2.1}$$

where $0 < \varepsilon \ll 1$, $H : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ are C^{r+1} and C^r $(r \leq 2)$, respectively, and $g(x,t)$ is T-periodic in t $(T > 0)$ for any $x \in \mathbb{R}^2$. We assume that the origin $x = 0$ is a hyperbolic saddle and has a homoclinic orbit $x^{\mathrm{h}}(t)$ in (2.1) with $\varepsilon = 0$. It follows that when $\varepsilon > 0$ is sufficiently small, near $x = 0$ there exists a hyperbolic periodic orbit $\gamma_\varepsilon(t)$ which has two-dimensional stable and unstable manifolds, $W^{\mathrm{s}}(\gamma_\varepsilon(t))$ and $W^{\mathrm{u}}(\gamma_\varepsilon(t))$.

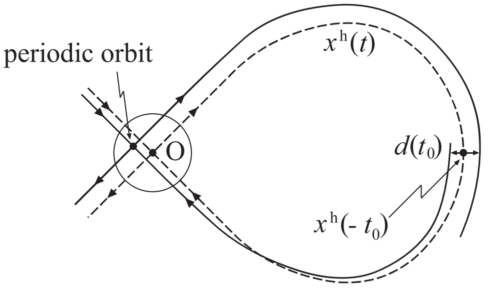

Figure 4: Perturbed phase space for (2.1)

We can approximate orbits on $W^{\mathrm{s}}(\gamma_\varepsilon(t))$ and $W^{\mathrm{u}}(\gamma_\varepsilon(t))$ as $x = x^{\mathrm{h}}(t - t_0) + \varepsilon\xi(t)$, where $\xi = \xi(t)$ is a solution to the variational equation

$$\dot{\xi} = J_1 \mathrm{D}_x^2 H(x^{\mathrm{h}}(t - t_0))\xi + g(x^{\mathrm{h}}(t - t_0), t).$$

See Fig. 4. The signed distance $d(t_0)$ between $W^{\mathrm{s}}(\gamma_\varepsilon(t))$ and $W^{\mathrm{u}}(\gamma_\varepsilon(t))$ near $x = x^{\mathrm{h}}(-t_0)$ and $t = 0 \mod T$ can be estimated as

$$d(t_0) = \frac{\varepsilon M(t_0)}{\mathrm{D}H(x^{\mathrm{h}}(-t_0))} + O(\varepsilon^2),$$

where

$$M(t_0) = \int_{-\infty}^{\infty} \mathrm{D}_x H(x^{\mathrm{h}}(t), 0) \cdot g(x^{\mathrm{h}}(t), t + t_0)\mathrm{d}t$$

and the dot represents the inner product. We call $M(t_0)$ the *Melnikov function*.

Theorem 2.1. *If $M(t_0)$ has a simple zero, then for $\varepsilon > 0$ sufficiently small $W^{\mathrm{s}}(\gamma_\varepsilon(t))$ and $W^{\mathrm{u}}(\gamma_\varepsilon(t))$ intersect transversely.*

Using the Smale-Birkhoff homoclinic theorem [12,33] (see also Theorem 1.1) and Theorem 2.1, we see that if $M(t_0)$ has a simple zero, then chaotic dynamics occurs in (2.1).

2.2 Melnikov-type method

We now consider the two-degree-of-freedom Hamiltonian system (1.4) under assumptions (A1)-(A3), and describe the Melnikov-type method developed in [34]. See [34] for the details. The Hamiltonian $H(x, y)$ is assumed to be C^{r+1} ($r \geq 3$).

As stated in Section 1.2, there exists a family of periodic orbits near the saddle-center at $(x, y) = (0, 0)$. Let $\gamma_\varepsilon(t)$ denote a periodic orbit of the family such that $\max_{t\in\mathbb{R}} |\gamma_\varepsilon(t)| = O(\varepsilon)$, where $0 < \varepsilon \ll 1$. We can approximate orbits on the two-dimensional stable and unstable manifolds of $\gamma_\varepsilon(t)$, $W^{\mathrm{s}}(\gamma_\varepsilon(t))$ and $W^{\mathrm{u}}(\gamma_\varepsilon(t))$, as $x = x^{\mathrm{h}}(t-t_0)+\varepsilon^2\xi(t)$, $y = \varepsilon\eta(t)$, where $(\xi, \eta) = (\xi(t), \eta(t))$ is a solution to the variational equation

$$\begin{aligned}
\dot{\xi} &= J_1 \mathrm{D}_x^2 H(x^{\mathrm{h}}(t - t_0), 0)\xi + g(x^{\mathrm{h}}(t - t_0), \eta) + O(\varepsilon),\\
\dot{\eta} &= J_1 \mathrm{D}_y^2 H(x^{\mathrm{h}}(t - t_0), 0)\eta + O(\varepsilon),
\end{aligned}$$

where $g(x,\eta)=\frac{1}{2}J\mathrm{D}_x\mathrm{D}_y^2H(x,0)(\eta,\eta)$. Applying an argument used in the standard Melnikov method, we see that the signed distance $d(t_0)$ between $W^{\mathrm{s}}(\gamma_\varepsilon(t))$ and $W^{\mathrm{u}}(\gamma_\varepsilon(t))$ near $(x,y)=(x^{\mathrm{h}}(-t_0),0)$ is estimated as

$$d(t_0)=\frac{\varepsilon M(t_0)}{\mathrm{D}_xH(x^{\mathrm{h}}(-t_0),0)}+O(\varepsilon^2),$$

where $M(t_0)$ is the Melnikov function given by

$$M(t_0)=\int_{-\infty}^{\infty}\mathrm{D}_xH(x^{\mathrm{h}}(t),0)\cdot g(x^{\mathrm{h}}(t),\eta(t+t_0))\mathrm{d}t.$$

We can also see that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{D}_y^2H(x^{\mathrm{h}}(t),0)(\eta(t+t_0),\eta(t+t_0))]\\&=-2\mathrm{D}_xH(x^{\mathrm{h}}(t),0)(\eta(t+t_0),\eta(t+t_0))\cdot g(x^{\mathrm{h}}(t),\eta(t+t_0)),\end{aligned}$$

so that

$$M(t_0)=-\tfrac{1}{2}\mathrm{D}_y^2H(x^{\mathrm{h}}(t),0)(\eta(t+t_0),\eta(t+t_0))\Big|_{-\infty}^{\infty}.$$

See [34].

Let $\Psi(t)$ and $\Phi(t)$ be, respectively, fundamental matrices of

$$\dot\eta=J_1\mathrm{D}_y^2H(x^{\mathrm{h}}(t),0)\eta \tag{2.2}$$

and

$$\dot\eta=J_1\mathrm{D}_y^2H(0,0)\eta \tag{2.3}$$

such that $\Phi(0)=\mathrm{id}_2$. Let $B_\pm=\lim_{t\to\pm\infty}\Phi(-t)\Psi(t)$ and let $B_0=B_+B_-^{-1}$. We can write $\eta(t)=\Psi(t-t_0)B_-^{-1}\Phi(t_0)\eta_0+O(\varepsilon)$ with $\eta_0\ (\neq 0)\in\mathbb{R}^2$, which satisfies

$$\eta(t)\to\begin{cases}\Phi(t)\eta_0 & \text{as } t\to-\infty;\\ \Phi(t-t_0)B_0\Phi(t_0)\eta_0 & \text{as } t\to\infty.\end{cases}$$

Thus, the Melnikov function is rewritten as

$$M(t_0)=q_0(\eta_0)-q_0(B_0\Phi(t_0)\eta_0),\quad \eta_0(\neq 0)\in\mathbb{R}^2,$$

where $q_0(\eta)=\frac{1}{2}\mathrm{D}_y^2H(0,0)(\eta,\eta)$ for $\eta\in\mathbb{R}^2$. We obtain the following theorem.

Theorem 2.2. *If $M(t_0)$ has a simple zero, then for $\varepsilon>0$ sufficiently small $W^{\mathrm{s}}(\gamma_\varepsilon(t))$ and $W^{\mathrm{u}}(\gamma_\varepsilon(t))$ intersect transversely on the level set $H^{-1}(c)$ with $c=H(\gamma_\varepsilon(t))$.*

Using Theorem 1.1 and Theorem 2.2, we see that if $M(t_0)$ has a simple zero, then chaotic dynamics occurs in (1.4).

2.3 Example

We apply the method of Section 2.2 to a two-degree-of-freedom system with the Hamiltonian

$$H(x,y)=\tfrac{1}{2}(-x_1^2+\omega^2 y_1^2)+\frac{\beta_1}{n+1}x_1^{n+1}+\tfrac{1}{2}\beta_2 x_1^{n-1}y_1^2+\tfrac{1}{2}(x_2^2+y_2^2)+O(y_1^3). \qquad (2.4)$$

It includes the Hénon-Heiles system (1.3) as a special case of $n=2$ in which

$$\beta_1=\frac{2c}{\sqrt{1+\mu^2}},\quad \beta_2=\frac{2(d-c)}{\sqrt{1+\mu^2}},\quad \beta_3=\frac{\mu(c+d)}{\sqrt{1+\mu^2}},$$
$$\omega=\frac{\mu}{\sqrt{2}},\quad x_{10}=\frac{\sqrt{1+\mu^2}}{2c},\quad \mu=\sqrt{2-\frac{d}{c}}$$

if $c/d>1/2$ and

$$\beta_1=d,\quad \beta_2=2c,\quad \beta_3=0,\quad \omega=\sqrt{1-\frac{2c}{d}}$$

if $c/d<1/2$. We easily see that assumptions (A1)-(A3) hold and

$$\begin{aligned}x^{\mathrm{h}}(t)=&\left(\left(\frac{n+1}{2\beta_1}\right)^{1/(n-1)}\operatorname{sech}^{2/(n-1)}\left(\frac{n-1}{2}t\right),\right.\\ &\left.\left(\frac{n+1}{2\beta_1}\right)^{1/(n-1)}\operatorname{sech}^{2/(n-1)}\left(\frac{n-1}{2}t\right)\tanh\left(\frac{n-1}{2}t\right)\right).\end{aligned} \qquad (2.5)$$

We compute

$$M(t_0)=\omega^2 b(\sqrt{1+b^2}\cos(2\omega t_0+\phi_0)+b),$$

where

$$b=\begin{cases}\dfrac{\cos^2\pi\sqrt{\sigma}/2}{\sinh^2 2\pi\omega/(n-1)} & \text{if } \sigma>0;\\[2mm] \dfrac{\cosh^2\pi\sqrt{-\sigma}/2}{\sinh^2 2\pi\omega/(n-1)} & \text{if } \sigma<0,\end{cases}$$

and

$$\sigma=\frac{8\beta_2(n+1)}{\beta_1(n-1)^2}+1.$$

We see that if

$$\frac{\beta_2}{\beta_1}\neq\frac{(n-1)^2}{2(n+1)}\ell(\ell+1),\quad \ell\in\mathbb{N}, \qquad (2.6)$$

then $M(t_0)$ has a simple zero so that by Theorem 2.2 chaotic dynamics occurs. The same condition was also obtained by Grotta Ragazzo [10] although his approach was valid only for a restricted class of two-degree-of-freedom Hamiltonians of the form $H(q,p)=\frac{1}{2}|p|^2+V(q)$ with $q,p\in\mathbb{R}^2$ (cf. Eq. (1.2)), where $V(q)$ is a C^{r+1} function. Note that condition (2.6) holds for (1.3) when $c/d\neq 0,1,\frac{1}{2},\frac{3}{4},1$.

3 Nonintegrability and Chaos

3.1 Differential Galois theory

We present such an introductory material of differential Galois theory as needed below. See [5, 29] for thorough explanations of the theory.

Let $\mathbb{K}$ be a *differential field* endowed with a derivation ∂, and consider linear systems of the form

$$\partial y = Ay, \quad A \in \mathrm{gl}(n, \mathbb{K}). \tag{3.1}$$

Let $\mathrm{C}_{\mathbb{K}} := \{a \in \mathbb{K} \mid \partial a = 0\}$ be the *field of constants* of $\mathbb{K}$. For instance, when $\mathbb{K} = \mathbb{C}(t)$, then $\mathrm{C}_{\mathbb{K}} = \mathbb{C}$. A *differential field extension* $\mathbb{L} \supset \mathbb{K}$ is a field extension such that $\mathbb{L}$ is a differential field and the derivations on $\mathbb{L}$ and $\mathbb{K}$ coincide on $\mathbb{K}$. Let Ξ be a fundamental matrix of (3.1). A differential field extension $\mathbb{L} \supset \mathbb{K}$ is called a *Picard-Vessiot extension* if

(i) $\mathbb{L}$ is generated by $\mathbb{K}$ and entries of Ξ;

(ii) $\mathrm{C}_{\mathbb{L}} = \mathrm{C}_{\mathbb{K}}$.

We now fix a Picard-Vessiot extension $\mathbb{L} \supset \mathbb{K}$ and a fundamental matrix Ξ with entries in $\mathbb{L}$. Let σ be a $\mathbb{K}$-*automorphism* of $\mathbb{L}$, i.e., a field automorphism of $\mathbb{L}$ such that $\partial(\sigma(a)) = \sigma(\partial a)$ for any $a \in \mathbb{L}$ and $\sigma(a) = a$ for any $a \in \mathbb{K}$. Since $\partial\sigma(\Xi) = \sigma(\partial\Xi) = \sigma(A\Xi) = A\sigma(\Xi)$, we see that $\sigma(\Xi)$ is another fundamental matrix of (3.1). Hence, by a fundamental result of linear differential equations, we have $\sigma(\Xi) = \Xi M_\sigma$ for some $M_\sigma \in \mathrm{GL}(n, \mathrm{C}_{\mathbb{L}})$. A group of $\mathbb{K}$-automorphisms of $\mathbb{L}$ is called the *differential Galois group* $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ of (3.1). An algebraic group $G \subset \mathrm{GL}(n, \mathrm{C}_{\mathbb{L}})$ generally has a unique irreducible component of G containing the identity element, which is called the *identity component* G^0. We denote the identity component of $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ by $\mathrm{Gal}(\mathbb{L}/\mathbb{K})^0$.

3.2 Morales-Ramis theory

We consider (1.1) as a complex Hamiltonian system. Let $\bar{x}(t)$ be a nonconstant particular solution to (1.1). We write solutions to (1.1) near $\bar{x}(t)$ as $x = \bar{x}(t) + \delta\xi$ with $\xi \in \mathbb{C}^{2n}$, where $0 < \delta \ll 1$. Substituting this expression into (1.1) and keeping the resulting equation up to $O(\delta)$, we obtain the variational equation of (1.1) around $\bar{x}(t)$,

$$\dot{\xi} = J_n \mathrm{D}^2 H(\bar{x}(t))\xi. \tag{3.2}$$

Let G^0 be the identity component of the differential Galois group for (3.2). The following theorem was proved in [26] (see also [24]).

Theorem 3.1 (Morales-Ruiz and Ramis)**.** *If Eq.* (1.1) *is meromorphically integrable near $\bar{x}(t)$, then G^0 is abelian.*

It follows from Theorem 3.1 that if G^0 is not abelian, then Eq. (1.1) is meromorphically nonintegrable. We write solutions to (1.1) as $x = \bar{x}(t) + \delta\xi^{(1)} + \frac{1}{2}\delta^2\xi^{(2)} + \cdots$ to obtain higher-order variational equations like (3.2). Morales-Ruiz et al. [27] extended Theorem 3.1 and obtained a stronger necessary condition for meromorphic integrability of (1.1) with the differential Galois group for the linearization of the higher-order variational equations. See [27] for the details.

3.3 Relationship between nonintegrability and chaos

We return to the two-degree-of-freedom system (1.4). The Hamiltonian $H(x, y)$ is assumed to be analytic. The variational equation of (1.4) around $(x^{\mathrm{h}}(t), 0)$ is given by

$$\dot{\xi} = J_1 \mathrm{D}_x^2 H(x^{\mathrm{h}}(t), 0)\xi, \quad \dot{\eta} = J_1 \mathrm{D}_y^2 H(x^{\mathrm{h}}(t), 0)\eta, \quad \xi, \eta \in \mathbb{C}^2. \tag{3.3}$$

We call the first and second equations of (3.3) the *tangential* and *normal variational equations*, respectively. Let G^0 be the identity component of the differential Galois group for the normal variational equation when the domain of the independent variable t is restricted to a neighborhood of $\mathbb{R} \cup \{\pm\infty\}$ in a Riemann surface (see [25, 37, 42, 43] for the details). It follows from Theorem 3.1 that if G^0 is not abelian then Eq. (1.4) is meromorphically nonintegrable near $(x^{\mathrm{h}}(t), 0)$. Furthermore, we can prove the following theorem [37].

Theorem 3.2. *If G^0 is not abelian, then for $\varepsilon > 0$ sufficiently small $W^{\mathrm{s}}(\gamma_\varepsilon(t))$ and $W^{\mathrm{u}}(\gamma_\varepsilon(t))$ intersect transversely on the level set $H^{-1}(c)$ with $c = H(\gamma_\varepsilon(t))$.*

A similar result for a restricted class of two-degree-of-freedom natural Hamiltonian systems was obtained by Morales-Ruiz and Peris [25] earlier based on the result of Grotta Ragazzo [10]. Thus, if G^0 is not abelian, then not only the system (1.4) is nonintegrable but also chaotic dynamics occurs.

We now apply Theorem 3.2 to the Hamiltonian (2.4). The normal variational equation around the homoclinic orbit (2.5) becomes

$$\dot{\eta}_1 = \eta_2, \quad \dot{\eta}_2 = -\left(\omega^2 + \frac{\beta_2(n+1)}{2\beta_1}\operatorname{sech}^2\left(\frac{n-1}{2}t\right)\right)\eta_1, \tag{3.4}$$

which is transformed to the Gauss hypergeometric equation

$$s(1-s)\frac{\mathrm{d}^2\zeta}{\mathrm{d}s^2} + \left(1 + \frac{2i\omega}{n-1}\right)(1-2s)\frac{\mathrm{d}\zeta}{\mathrm{d}s} - \left(\frac{2i\omega}{n-1} - \rho\right)\left(\frac{2i\omega}{n-1} + \rho + 1\right) = 0 \tag{3.5}$$

under the change of variables

$$s = \tfrac{1}{2}\left(1 - \tanh\left(\frac{n-1}{2}t\right)\right), \quad \eta_1 = (4s(1-s))^{i\omega/(n-1)}\left(\frac{n-1}{2}t\right)\zeta,$$

where

$$\rho = \tfrac{1}{2}(\sqrt{\sigma} - 1), \quad \sigma = \frac{8\beta_2(n+1)}{\beta_1(n-1)^2} + 1.$$

Note that the singular points $s = 0$ and 1 in (3.5), respectively, correspond to $t = \infty$ and $-\infty$ in (3.4). Using an argument in Section 5 of [37], we see that when $\omega > 0$, if condition (2.6) holds, then the identity component of the differential Galois group of (3.5) is not abelian, so that G^0 is not abelian. Using Theorem 3.2, we reobtain the result of Section 2.3.

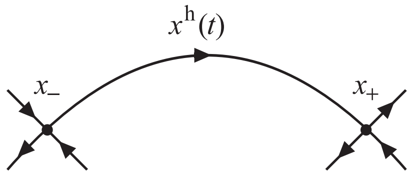

Figure 5: Heteroclinic orbit $x^{\mathrm{h}}(t)$ on the x-plane.

4 Several Extensions

4.1 Higher-order Melnikov method

We consider the two-degree-freedom system (1.4). Let $\Psi(t)$ and $\Phi(t)$ be, respectively, fundamental matrices of (2.2) and (2.3) with $\Phi(0) = \mathrm{id}_2$, as in Section 2.2. Recall that $B_\pm = \lim_{t\to\pm\infty} \Phi(-t)\Psi(t)$ and $B_0 = B_+B_-^{-1}$. Let

$$\begin{aligned} q_j(x,\eta) =& \frac{1}{(j+2)!}\mathrm{D}_y^{j+2}H(x,0)(\underbrace{\eta,\dots,\eta}_{j+2}), \quad j = 0,1,\dots, \\ K(v) =& \int_0^\infty [q_1(x^{\mathrm{h}}(t),\Psi(t)v) - q_1(0,\Phi(t)B_+v)]\mathrm{d}t \\ &+ \int_{-\infty}^0 [q_1(x^{\mathrm{h}}(t),\Psi(t)v) - q_1(0,\Phi(t)B_-v)]\mathrm{d}t. \end{aligned}$$

We define the *first-* and *second-order Melnikov functions* as

$$M_1(t_0) = q_0(0,\eta_0) - q_0(0,B_0\Phi(t_0)\eta_0)$$

and

$$\begin{aligned} M_2(t_0) =& -\Phi(t_0)\eta_0 \cdot \mathrm{D}_y^2H(0,0)B_-J_1\mathrm{D}_vK(B_-^{-1}\Phi(t_0)\eta_0) \\ &+ q_1(0,\Phi(t_0)\eta_0) - q_1(0,B_0\Phi(t_0)\eta_0), \end{aligned}$$

respectively, where η_0 $(\neq 0) \in \mathbb{R}^2$. In this situation, we can prove the following [41].

Theorem 4.1. *If $M_1(t_0) \equiv 0$ and $M_2(t_0)$ has a simple zero, then for $\varepsilon > 0$ sufficiently small $W^{\mathrm{s}}(\gamma_\varepsilon(t))$ and $W^{\mathrm{u}}(\gamma_\varepsilon(t))$ intersect transversely on the level set $H^{-1}(c)$ with $c = H(\gamma_\varepsilon(t))$.*

Using Theorem 4.1, we can show that chaotic dynamics occurs in (1.3) when $c/d = \frac{3}{4}$. See [41] for the details.

4.2 Heteroclinic orbits

We still consider the two-degree-freedom system (1.4) but assume the following instead of (A1)-(A3):

(A1') For any $x \in \mathbb{R}^2$, $\mathrm{D}_yH(x,0) = 0$ and for some $x_\pm \in \mathbb{R}^2$, $\mathrm{D}_xH(x_\pm,0) = 0$;

(A2') $J_1 \mathrm{D}_x^2 H(x_\pm, 0)$ has a pair of positive and negative eigenvalues, and there exists a heteroclinic orbit $(x, y) = (x^{\mathrm{h}}(t), 0)$ such that $\lim_{t\to\pm\infty} x^{\mathrm{h}}(t) = x_\pm$ (see Fig. 5);

(A3') $J_1 \mathrm{D}_y^2 H(x_\pm, 0)$ has a pair of purely imaginary eigenvalues, $i\omega_\pm$ and $-i\omega_\pm$.

It follows from (A1')-(A3') that the x-plane is invariant under the flow of (1.4); there are two saddle-centers at $(x, y) = (x_\pm, 0)$; and by the Lyapunov center theorem (e.g., [1, 22]) there exists a family of periodic orbits near each saddle-center (cf. Fig. 3). Let $\gamma_{\pm,\varepsilon}(t)$ denote periodic orbits of the families such that $\max_{t\in\mathbb{R}} |\gamma_{\pm,\varepsilon(t)} - (x_\pm, 0)| = O(\varepsilon)$. The periodic orbits $\gamma_{\pm,\varepsilon}(t)$ have two-dimensional stable and unstable manifolds, $W^{\mathrm{s}}(\gamma_{\pm,\varepsilon}(t))$ and $W^{\mathrm{u}}(\gamma_{\pm,\varepsilon}(t))$. So $W^{\mathrm{u}}(\gamma_{-,\varepsilon}(t))$ may intersect $W^{\mathrm{s}}(\gamma_{+,\varepsilon}(t))$ transversely on the level set $H^{-1}(c)$ for $c \in \mathbb{R}$ when $H(\gamma_{+,\varepsilon}(t)) = H(\gamma_{-,\varepsilon}(t)) = c$ for some $c \in \mathbb{R}$.

Let $\Psi(t)$ and $\Phi_\pm(t)$ be, respectively, fundamental matrices of

$$\dot\eta = J_1 \mathrm{D}_y^2 H(x^{\mathrm{h}}(t), 0)\eta \quad \text{and} \quad \dot\eta = J_1 \mathrm{D}_y^2 H(x_\pm, 0)\eta$$

such that $\Phi_\pm(0) = \mathrm{id}_2$. Let $B_\pm = \lim_{t\to\pm\infty} \Phi_\pm(-t)\Psi(t)$ and $B_0 = B_+ B_-^{-1}$. We define the *Melnikov function* as

$$M(t_0) = q_-(\eta_0) - q_+(B_0 \Phi(t_0)\eta_0), \quad \eta_0 (\neq 0) \in \mathbb{R}^2,$$

where $q_\pm(\eta) = \frac{1}{2}\mathrm{D}_y^2 H(x_\pm, 0)(\eta, \eta)$ for $\eta \in \mathbb{R}^2$. In this situation, we can prove the following [31].

Theorem 4.2. *If $M(t_0)$ has a simple zero and $H(\gamma_{+,\varepsilon}(t)) = H(\gamma_{-,\varepsilon}(t)) = c$ for some $c \in \mathbb{R}$, then for $\varepsilon > 0$ sufficiently small $W^{\mathrm{u}}(\gamma_{-.\varepsilon}(t))$ intersects $W^{\mathrm{s}}(\gamma_{+,\varepsilon}(t))$ transversely on the level set $H^{-1}(c)$.*

Suppose that there also exists a heteroclinic orbit from $(x_+, 0)$ to $(x_-, 0)$ and the corresponding Melnikov function has a simple zero. Then, as in Theorem 4.2, $W^{\mathrm{u}}(\gamma_{+.\varepsilon}(t))$ intersects $W^{\mathrm{s}}(\gamma_{+,\varepsilon}(t))$ transversely. Thus, there exist transverse heteroclinic cycles yielding transverse homoclinic orbits to $\gamma_{\pm,\varepsilon}(t)$ (see, e.g., Section 26.1 of [33]). Hence, by Theorem 1.2 chaotic dynamics may occur in (1.4).

Let G^0 be the identity component of the differential Galois group for the variational equation of (1.4) around $(x, y) = (x^{\mathrm{h}}(t), 0)$ (cf. Eq. (3.3)). We have the following result similar to Theorem 3.2.

Theorem 4.3. *Suppose that $H(x, y)$ is analytic and $\omega_+ = \omega_-$. If G^0 is not abelian and $H(\gamma_{+,\varepsilon}(t)) = H(\gamma_{-,\varepsilon}(t)) = c$ for some $c \in \mathbb{R}$, then for $\varepsilon > 0$ sufficiently small $W^{\mathrm{u}}(\gamma_{-,\varepsilon}(t))$ intersects $W^{\mathrm{s}}(\gamma_{+,\varepsilon}(t))$ transversely on the level set $H^{-1}(c)$.*

See [43] for the proof. A result of [42] on Bogoyavlenskij nonintegrability [4] of general systems (which are not necessarily Hamiltonian) near homo- and heteroclinic orbits, along with Theorem 4.2, were used there. The statement of Theorem 4.3 does not necessarily hold for $\omega_+ \neq \omega_-$: $W^{\mathrm{u}}(\gamma_{-,\varepsilon}(t))$ may not intersect $W^{\mathrm{s}}(\gamma_{+,\varepsilon}(t))$ even if Eq. (1.4) is nonintegrable. See [43] for the details.

4.3 Three- or more-degree-of-freedom Hamiltonian systems

We consider $(n+1)$-degree-of-freedom Hamiltonian systems of the form

$$\dot{x} = J_1 \mathrm{D}_x H(x,y), \quad \dot{y} = J_n \mathrm{D}_y H(x,y), \quad (x,y) \in \mathbb{R}^2 \times \mathbb{R}^{2n}, \tag{4.1}$$

where $n \geq 2$, $H : \mathbb{R}^2 \times \mathbb{R}^{2n} \to \mathbb{R}$ is C^{r+1} $(r \geq 2n+4)$. We make the following assumptions:

(A1") For any $x \in \mathbb{R}^2$, $\mathrm{D}_x H(0,0) = \mathrm{D}_y H(x,0) = 0$;

(A2") $J_1 \mathrm{D}_x^2 H(0,0)$ has a pair of positive and negative eigenvalues, and there exists a homoclinic orbit $(x,y) = (x^{\mathrm{h}}(t), 0)$.

(A3") $J_n \mathrm{D}_y^2 H(0,0)$ has n pairs of purely imaginary eigenvalues $\pm i\omega_j$, $j = 1, \dots, n$, satisfying the nonresonant condition

$$k \cdot \omega = k_1\omega_1 + \cdots + k_n\omega_n \neq 0$$

for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ such that $1 \leq |k| = \sum_{j=1}^n |k_j| \leq 4$.

It follows from (A1")-(A3") that the x-plane is invariant under the flow of (4.1) and the origin $(x,y) = (0,0)$ is a saddle-center. We can also show that there is a symplectic transformation $(x,y) \mapsto (s,u,I,\psi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{T}^n$ such that the Hamiltonian H is expressed as

$$H(s,u,I,\psi) = \lambda\, su + \omega \cdot I + \tfrac{1}{2}(AI \cdot I) + g(s,u,I,\psi)$$

near the origin, where A is an $n \times n$ matrix, and $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R}$ is C^{r+1} for $I \neq 0$ and of higher order than 2 in s, u and I. Moreover, we assume the following:

(A4") A is nonsingular.

Using the invariant manifold theory [7,8] (see also [32]) and a version of the KAM theorem [28], we see from (A4") that a Cantor set of n-dimensional invariant tori which have $(2n+1)$-dimensional stable and unstable manifolds exists near the origin. See [38] for more details.

Let $\Psi(t)$ and $\Phi(\omega t)$ be, respectively, fundamental matrices of

$$\dot{\eta} = J_n \mathrm{D}_y^2 H(x^{\mathrm{h}}(t), 0)\eta \quad \text{and} \quad \dot{\eta} = J_n \mathrm{D}_y^2 H(0,0)\eta$$

such that $\Phi(0) = \mathrm{id}_n$. Let $B_\pm = \lim_{t\to\pm\infty} \Phi(-\omega t)\Psi(t)$ and $B_0 = B_+ B_-^{-1}$. We define the *Melnikov function* as

$$M(\theta; r) = q_0(r) - q_0(B_0\Phi(\theta)r), \quad r \in \mathbb{R}_+^n = \prod_{j=1}^n (0,\infty),$$

where $q_0(\eta) = \frac{1}{2}\mathrm{D}_y^2 H(0,0)(\eta,\eta)$. In this situation we have the following theorem [38].

Theorem 4.4. *Suppose that $M(\theta; r)$ has a simple zero at $\theta = \theta_0$ for some $r \in \mathbb{R}_+^n$ fixed. Then the unstable manifold of an invariant torus near the origin intersects the stable manifold of another invariant torus near the origin transversely on the level set $H^{-1}(c)$ for some $c \in \mathbb{R}$. Here the projections of the two invariant manifolds onto the y-space are close to $\{\varepsilon\Phi(\theta)r \mid \theta \in \mathbb{T}^n\}$ and $\{\varepsilon\Phi(\theta)B_0\Phi(\theta_0)r \mid \theta \in \mathbb{T}^n\}$, where $\varepsilon > 0$ is sufficiently small..*

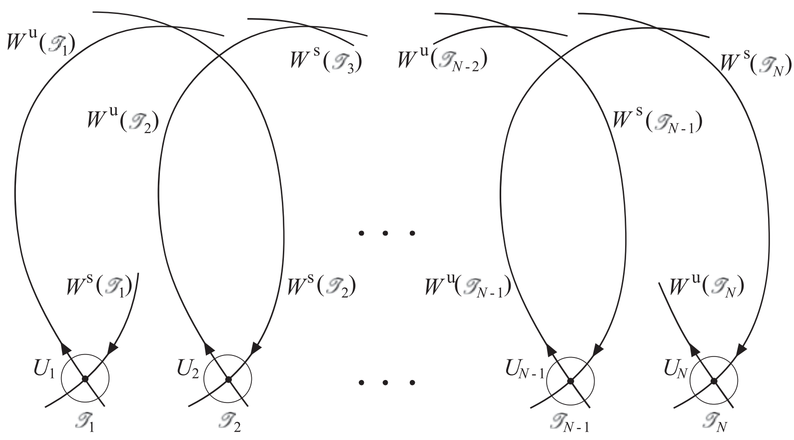

Figure 6: Transition chain

Let $N > 1$ be an integer and suppose that $M(\theta; r)$ has a simple zero at $\theta = \theta^j \in \mathbb{T}^n$ for $r = r^j \in \mathbb{R}^n_+$, $j = 1, \ldots, N-1$, such that $\Phi(\tilde{\theta}^j)r^{j+1} = B_0\Phi(\theta^j)r^j$ for some $\tilde{\theta}^j \in \mathbb{T}^n$. Let $W^{\mathrm{s}}(\mathscr{T}_j)$ and $W^{\mathrm{u}}(\mathscr{T}_j)$, respectively, denote the $(2n+1)$-dimensional stable and unstable manifolds of $\mathscr{T}_j$ for $j = 1, \ldots, N$. Using Theorem 4.4, we can find N invariant tori $\mathscr{T}_1, \ldots, \mathscr{T}_N$ such that $W^{\mathrm{u}}(\mathscr{T}_j)$ intersects $W^{\mathrm{s}}(\mathscr{T}_{j+1})$ transversely for $j = 1, \ldots, N-1$, on the level set $H^{-1}(c)$ for some $c \in \mathbb{R}$ such that $H(\mathscr{T}_j) = c$, $j = 1, \ldots, N$. We refer to the sequence of invariant tori $\mathscr{T}_1, \ldots, \mathscr{T}_N$ as a *transition chain*. See Fig. 6. So we see that there exists an open set of points arbitrarily close to $\mathscr{T}_1$, connected by trajectories with points arbitrary close to $\mathscr{T}_N$ through points near $\mathscr{T}_j$, $j = 2, \ldots, N-1$. This is very similar to Arnold diffusion, which occurs in nearly integrable systems [2, 20], although the drift speed is not slow. See [38] for more details. Numerical evidence for Arnold diffusion type motions in a three-degree-of-freedom system was also provided in [36].

4.4 Undamped, buckled beam: An infinite-degree-of-freedom Hamiltonian system

We now consider an undamped, buckled beam with hinged ends, shown in Fig. 7. We adopt the following mathematical model of the beam as in [16]:

$$\ddot{u} + u'''' + \left[\Gamma - \kappa \int_0^1 (u')^2 \mathrm{d}\zeta\right] u'' = 0, \tag{4.2}$$

where u represents the transverse deflection, the prime and overdot represent partial differentiation with respect to z and t, respectively, and Γ and κ represent the compressive force and stiffness due to "membrane" effects, respectively. The boundary condition is given by $u(0) = u(1) = 0$ and $u''(0) = u''(1) = 0$. Especially, the distance ℓ between the hinged ends is non-dimensionalized such that $\ell = 1$. Eq. (4.2) is an infinite-dimensional Hamiltonian system with the symplectic form

$$\Omega((u_1, \dot{u}_1), (u_2, \dot{u}_2)) = \int_0^1 (\dot{u}_2 u_1 - \dot{u}_1 u_2)\mathrm{d}z$$

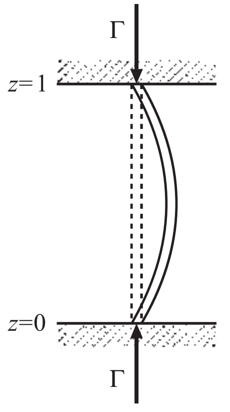

Figure 7: Buckled beam

and the Hamiltonian

$$H(u,\dot{u}) = \int_0^1 \left[\frac{1}{2}(\dot{u})^2 - \frac{\Gamma}{2}(u')^2 + \frac{1}{2}(u'')^2\right] \mathrm{d}z + \frac{\kappa}{4}\left[\int_0^1 (u')^2 \mathrm{d}z\right]^2.$$

We assume that $\pi^2 < \Gamma < 4\pi^2$, so that only the first mode $u = \sin \pi z$ is unstable.

Let

$$u = \sum_{l=1}^{n+1} a_{j_l} \sin j_l \pi z, \tag{4.3}$$

where $j_1 = 1$ and $j_l \in \mathbb{N}$, $l = 2, \ldots, n+1$, with $j_1 < \cdots < j_{n+1}$. Note that Eq. (4.3) satisfies the boundary condition. Substituting (4.3) into (4.2), we obtain

$$\begin{aligned} &\dot{x}_1 = x_2,\ \dot{x}_2 = x_1 - \left(x_1^2 + \sum_{m=1}^{n} j_{m+1}^2 y_m^2\right) x_1, \\ &\dot{y}_l = y_{n+l},\ \dot{y}_{n+l} = -\omega_l^2 y_l - j_{l+1}^2 \left(x_1^2 + \sum_{m=1}^{n} j_{m+1}^2 y_m^2\right) y_l, \quad l = 1, \ldots, n, \end{aligned} \tag{4.4}$$

which is an $(n+1)$-degree-of-freedom Hamiltonian system of the form (1.4) and (4.1) for $n = 1$ and $n > 1$, respectively, where

$$\omega_l = j_{l+1}\sqrt{\frac{(j_{l+1}\pi)^2 - \Gamma}{\Gamma - \pi^2}}, \quad l = 1, \ldots, n.$$

We also show that assumptions (A1)-(A3) or (A1")-(A3") without the nonresonant condition hold. Assuming the nonresonant condition for $n > 1$ and applying Theorems 2.2 and 4.4 to (4.4) for $n = 1$ and $n > 1$, respectively, we prove that chaotic vibrations occur in (4.2). See [35] for the details. We remark that assumption (A4") holds if

$$j_l^2 \neq \frac{m(m+1)}{2}, \quad m \in \mathbb{N}, \quad \text{i.e., } j_l \neq 6, 204, 6930, 235416, \ldots.$$

The case of $n = 1$ was also studied in [11] earlier.

5 Future Work

Finally, we give some comments on future work. First of all, one may raise the open problem of determining whether the Hénon-Heiles system (1.3) exhibits chaos for $c/d = \frac{1}{2}$. Especially, it makes the problem difficult that there exists only a degenerate saddle-center at which the Jacobian matrix has a double zero eigenvalue. To overcome this difficulty, the Lyapunov center theorem has to be extended. Since Eq. (1.3) is nonintegrable as stated in Section 1.1, it seems natural to expect that chaotic dynamics occurs then. A numerical simulation presented in [27] also supports this conjecture. Some preliminary result was obtained in [39].

Second, extensions of the results stated in Sections 2 and 3 to reversible systems with saddle-centers:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n},$$

where there exists a linear involution $R : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ ($R^2 = \mathrm{id}_{2n}$) such that $Rf(x) + f(Rx) = 0$. Reversible systems have some similar properties as Hamiltonian systems, e.g., the statement of the Lyapunov center theorem holds (see e.g., [6]), but may not have a first integral, so that their trajectories may not be restricted to a lower-dimensional space than the phase space and more complicated behavior may occur. Some preliminary result was obtained in [40].

As described in this article, for Hamiltonian systems with saddle-centers and homo- or heteroclinic orbits, especially in the two-degree-of-freedom case, we now understand a relationship between nonintegrability and chaos to some extent. However, when a Hamiltonian system with homo- or heteroclinic orbits has only hyperbolic saddles at which all eigenvalues of the Jacobian matrix are real, it is not so clear whether it can exhibit chaotic dynamics even for the two-degree-of-freedom case. No essential progress in this direction has been made since the work of Holmes [15] in 1980. For instance, for the heavy top, which has been one of important dynamical systems since the time of Euler and Lagrange, the problem of nonintegrability was completely solved by Ziglin [44, 46] (see also [21,47]) but the occurrence of chaos in a special case with only such a hyperbolic saddle is still an open problem. So a new theory for detecting chaos and discussing a relationship between nonintegarbility and chaos in Hamiltonian and non-Hamiltonian systems (e.g., reversible systems) with such hyperbolic saddles is expected.

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