Dirac structures and Lagrangian systems on tangent bundles

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Abstract

In mechanics, a Dirac structure, which is the unified notion of symplectic and Poisson structures, has been widely used to formulate mechanical systems with nonholonomic constraints, electric circuits as well as thermodynamic systems. In particular, the induced Dirac structure on the cotangent bundle from a given constraint distribution plays an essential role in the context of implicit Lagrangian and Hamiltonian systems. However, there has been almost no research on the Dirac geometry associated to the tangent bundle TQ, although it may be relevant with regular Lagrangian systems. In this paper, we introduce an induced Dirac structure on TQ, called a *Lagrangian Dirac structure*. For the regular case, we finally show that one can define a Lagrange-Dirac system on TQ.

1 Lagrangian systems

In this section, we shall make a short review on Lagrangian systems in the context of conventional Hamilton's principle as well as the induced symplectic structure called the Lagrangian two-form (see Marsden and Ratiu [1999]).

1.1 Hamilton's principle

Let us first recall Hamilton's principle in mechanics. Consider a mechanical system with an n-dimensional configuration manifold Q and let L be a Lagrangian on the tangent bundle TQ. Consider the following action functional

$$S(q) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt,$$

where $q(t),\ t \in [t_1, t_2] \subset \mathbb{R}$, denotes a curve joining $q_1 = q(t_1)$ and $q_2 = q(t_2)$ on Q and where $\dot{q}(t) = \frac{d}{dt}q(t)$ denotes the time derivative of $q(t)$. Hamilton's variational principle states that the motion $q(t)$ of the mechanical system is given by a solution curve of the critical condition of the action functional

$$\delta S(q) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{t_1}^{t_2} L\big(q_\epsilon(t), \dot{q}_\epsilon(t)\big)\mathrm{d}t = 0, \tag{1}$$

where $q_\epsilon(t) := q(t, \epsilon)$ is an arbitrary variation of $q(t)$ with fixed endpoints, $q_0(t) = q(t)$ and $q_0(t_1) = q(t_1)$, $q_0(t_2) = q(t_2)$, for all $\epsilon \in [-a, a]$. The infinitesimal variations associated with the variations $q_\epsilon(t)$ are given by

$$\delta q(t) := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q_\epsilon(t).$$

It follows from equation (1) that the solution curve $q(t)$ satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{2}$$

1.2 Lagrangian symplectic structures on tangent bundles

Legendre transform. Recall the Legendre transform associated to L is given by the fiber derivative $\mathbb{F}L : TQ \to T^*Q$ as

$$\mathbb{F}L(v) \cdot w = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(v + \epsilon\, w),$$

where $v, w \in T_qQ$ and $\mathbb{F}L(v)\cdot w$ indicates the derivative of L at v along the fiber T_qQ in the direction w. Notice that the map $\mathbb{F}L : TQ \to T^*Q$ is fiber-preserving over Q and it maps the fiber T_qQ to the fiber T_q^*Q. Thus, the *Legendre transform* $\mathbb{F}L : TQ \to T^*Q$ is locally represented by

$$\mathbb{F}L(q, v) = \left(q, \frac{\partial L}{\partial v}\right),$$

where $p := \frac{\partial L}{\partial v} \in T_q^*Q$ denotes the momentum variable in mechanics. When a given Lagrangian $L : TQ \to \mathbb{R}$ is *hyperregular*, the map $\mathbb{F}L : TQ \to T^*Q$ is to be a diffeomorphism.

Lagrangian forms on tangent bundles. Let $L : TQ \to \mathbb{R}$ be a hyperregular Lagrangian. Let $\pi_Q : T^*Q \to Q$ be the cotangent bundle projection. The cotangent bundle T^*Q naturally has the *canonical one-form* Θ defined by, for each $\alpha_q \in T_q^*Q$,

$$\Theta(\alpha_q) \cdot w_{\alpha_q} = \left\langle \alpha_q, T_{\alpha_q}\pi_Q(w_{\alpha_q}) \right\rangle,$$

where $w_{\alpha_q} \in T_{\alpha_q}T^*Q$ and the *canonical symplectic structure* is given by

$$\Omega = -\mathbf{d}\Theta.$$

In local coordinates $(q^1, ..., q^n)$ for $q \in Q$ and $(q^1, ..., q^n, p_1, ..., p_n)$ for $(q, p) \in T^*Q$, one has $\Theta = p_i\, dq^i$ and the *canonical symplectic structure* is represented by $\Omega = dq^i \wedge dp_i$.

By using the Legendre transform $\mathbb{F}L : TQ \to T^*Q$, we can define an induced one-form Θ_L on TQ, called the *Lagrangian one-form*, by

$$\Theta_L = (\mathbb{F}L)^*\Theta,$$

and also define the induced symplectic structure Ω_L on TQ, called the *Lagrangian two-form*, by

$$\Omega_L = (\mathbb{F}L)^*\Omega.$$

Since $\Omega = -\mathbf{d}\Theta$ holds and the exterior derivative $\mathbf{d}$ commutes with the pull-back, it reads

$$\Omega_L = -\mathbf{d}\Theta_L.$$

Note that the Lagrangian one-form Θ_L on TQ holds, for $v_q \in TQ$,

$$\Theta_L(v_q) \cdot w_{v_q} = \left\langle \mathbb{F}L(v_q), T_{v_q}\tau_Q(w_{v_q}) \right\rangle,$$

where $w_{v_q} \in T_{v_q}TQ$, and $\tau_Q : TQ \to Q$ is the tangent bundle projection.

Local expressions. Using local coordinates $(q^1, ..., q^n, v^1, ..., v^n)$ for $(q, v) \in TQ$, the coordinate expression of Θ_L may be represented by

$$\Theta_L = \frac{\partial L}{\partial v^i} dq^i,$$

and hence $\Omega_L = -\mathbf{d}\Theta_L$ may be locally denoted by

$$\Omega_L = \frac{\partial^2 L}{\partial v^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dq^i \wedge dv^j.$$

The induced bundle map $\Omega_L^\flat : TTQ \to T^*TQ$ associated with the Lagrangian two-form Ω_L is represented by the skew-symmetric matrix as

$$\Omega_L^\flat = \begin{pmatrix} \dfrac{\partial^2 L}{\partial v^j \partial q^i} - \dfrac{\partial^2 L}{\partial v^i \partial q^j} & -\dfrac{\partial^2 L}{\partial v^i \partial v^j} \\ \dfrac{\partial^2 L}{\partial v^i \partial v^j} & 0 \end{pmatrix}.$$

In the above, since we assume that L is hyperregular, the Hessian is nonsingular, i.e.,

$$\det\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) \neq 0$$

and therefore Ω_L is nondegenerate.

Lagrangian systems. In this paragraph, we shall make a brief review on the *Lagrangian system* in the context of the Lagrangian symplectic structure. Let us see how the intrinsic Euler-Lagrange equations can be formulated in the context of the Lagrangian two-form on the tangent bundle.

Let us define an energy E_L on TQ by, for $u = (q, v) \in TQ$,

$$E_L(u) := \langle \mathbb{F}L(u), u \rangle - L(u).$$

Let X_L be a vector field on TQ and if X_L satisfies the condition

$$\Omega_L(u)(X_L(u), w) = \mathbf{d}E_L(u) \cdot w \tag{3}$$

for all $u \in TQ$ and $w \in T_uTQ$, then X_L is said to be a *Lagrangian vector field* or a *Lagrangian system* for L (see also Abraham and Marsden [1978]).

From the condition (3), we get an ***intrinsic Euler-Lagrange equations*** for the Lagrangian system:

$$\mathbf{i}_{X_L} \Omega_L = \mathbf{d}E_L. \tag{4}$$

Since we assume that the Lagrangian L is hyperregular, the Lagrangian vector field X_L on TQ is uniquely determined by

$$X_L = \left(\Omega_L^\flat\right)^{-1} \mathbf{d}E_L. \tag{5}$$

Energy conservation. Let $u(t)$, $t \in [t_1, t_2]$ be the integral curve of the Lagrangian vector field X_L on TQ. Then, the energy E_L is conserved such that

$$\begin{aligned}
\frac{d}{dt} E_L(u(t)) &= \mathbf{d}E_L(u(t)) \cdot X_L(u(t)) \\
&= \Omega_L(u(t)) \left(X_L(u(t)), X_L(u(t))\right) \\
&= 0,
\end{aligned}$$

where the skew-symmetric property of Ω_L is employed.

Local expressions of Euler-Lagrange equations on TQ. In finite dimensions, using local coordinates (q^i, v^i) for $u = (q, v) \in TQ$, the local expression of the energy E_L may be given by

$$E_L(q^i, v^i) = \frac{\partial L}{\partial v^i} v^i - L(q^i, v^i)$$

and the differential of E_L may be given by

$$\mathbf{d}E_L = \frac{\partial E_L}{\partial q^i} \, dq^i + \frac{\partial E_L}{\partial v^i} \, dv^i,$$

where

$$\frac{\partial E_L}{\partial q^i} = \frac{\partial^2 L}{\partial q^i \partial v^j} \, v^j - \frac{\partial L}{\partial q^i}, \qquad \frac{\partial E_L}{\partial v^i} = \frac{\partial^2 L}{\partial v^i \partial v^j} v^j.$$

The Lagrangian vector field X_L may be locally denoted by

$$X_L = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{v}^i \frac{\partial}{\partial v^i}$$

and hence

$$\begin{aligned} &\mathbf{i}_{X_L} \Omega_L \\ &= \left[\left(\frac{\partial^2 L}{\partial v^j \partial q^i} - \frac{\partial^2 L}{\partial v^i \partial q^j} \right) \dot{q}^j - \frac{\partial^2 L}{\partial v^i \partial v^j} \dot{v}^j \right] dq^i + \left(\frac{\partial^2 L}{\partial v^i \partial v^j} \dot{q}^j \right) dv^i. \end{aligned}$$

Then, it follows from equation (4) that one has

$$\begin{cases} \left(\dfrac{\partial^2 L}{\partial v^j \partial q^i} - \dfrac{\partial^2 L}{\partial v^i \partial q^j} \right) \dot{q}^j - \dfrac{\partial^2 L}{\partial v^i \partial v^j} \dot{v}^j = \dfrac{\partial^2 L}{\partial q^i \partial v^j} v^j - \dfrac{\partial L}{\partial q^i}, \\ \dfrac{\partial^2 L}{\partial v^i \partial v^j} \dot{q}^j = \dfrac{\partial^2 L}{\partial v^i \partial v^j} v^j. \end{cases}$$

Since the Lagrangian L is hyperregular, i.e., $\det \left[\frac{\partial^2 L}{\partial v^i \partial v^j} \right] \neq 0$, it immediately follows that one gets the local expressions of the Euler-Lagrange equations on TQ as

$$\begin{cases} \dfrac{\partial^2 L}{\partial v^i \partial v^j} \dfrac{dv^j}{dt} = -\dfrac{\partial^2 L}{\partial v^i \partial q^j} \dot{q}^j + \dfrac{\partial L}{\partial q^i}, \\ \dfrac{dq^i}{dt} = v^i, \end{cases} \tag{6}$$

which are lifted from the Euler-Lagrange equations on Q in (2). Then, the Lagrangian vector field X_L is uniquely determined by the Lagrangian L as

$$\begin{cases} \dfrac{dv^i}{dt} = \left(\dfrac{\partial^2 L}{\partial v^i \partial v^j} \right)^{-1} \left(-\dfrac{\partial^2 L}{\partial v^j \partial q^k} \dot{q}^k + \dfrac{\partial L}{\partial q^j} \right), \\ \dfrac{dq^i}{dt} = v^i. \end{cases} \tag{7}$$

Second-order vector field. Recall $\tau_Q : TQ \to Q;\ (q, v) \mapsto q$ is the tangent bundle projection and let $\tau_{TQ} : TTQ \to TQ;\ (q, v, \dot{q}, \dot{v}) \mapsto (q, v)$ be also the canonical projection. Let us define a submanifold of TTQ by

$$T^{(2)}Q = \{ w \in TTQ \mid T\tau_Q(w) = \tau_{TQ}(w) \},$$

where $T\tau_Q : TTQ \to TQ;\ (q, v, \dot{q}, \dot{v}) \mapsto (q, \dot{q})$ is the tangent map of τ_Q. Hence, it follows that, for an element

$$w = (q, v, \dot{q}, \dot{v}) \in TTQ,$$

if it is an element of the submanifold $T^{(2)}Q \subset TTQ$, then it satisfies the *second-order condition*

$$v = \dot{q}.$$

So, we call $T^{(2)}Q$ a second-order submanifold of TTQ.

If a vector field X on TQ satisfies $T\tau_Q \circ X = \mathrm{id}$, then X is called the *second-order vector field*, which is defined as $X : TQ \to T^{(2)}Q$; $(q, \dot{q}) \mapsto (q, \dot{q}, \ddot{q})$. Hence, it is obvious that the Lagrangian vector field X_L in (7) is second-order because

$$\ddot{q}^i = \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)^{-1} \left(-\frac{\partial^2 L}{\partial \dot{q}^j \partial q^k}\, \dot{q}^k + \frac{\partial L}{\partial q^j}\right).$$

Let $(q(t), \dot{q}(t))$ be an integral curve of X_L and $q(t) = \tau_Q \circ (q(t), \dot{q}(t))$ is a *base integral curve* of $(q(t), \dot{q}(t))$. The integral curve of X_L can be uniquely determined by the base integral curve $q(t)$ with a given initial condition in TQ.

2 Dirac structures on tangent bundles

In this section, we shall introduce an induced Dirac structure on the tangent bundle from a given distribution on Q and Lagrangian L. In particular, we shall assume that L is hyperregular. In this case, we can develop the Lagrange-Dirac dynamical system on TQ.

Dirac structures. Recall the definition of a Dirac structure, see Courant and Weinstein [1988].

Let V be a vector space, let $\langle \cdot\,, \cdot \rangle$ be the natural paring between V and its dual space V^*, and consider the symmetric paring on $V \oplus V^*$ defined by

$$\langle\!\langle (v, \alpha), (\bar{v}, \bar{\alpha}) \rangle\!\rangle = \langle \alpha, \bar{v} \rangle + \langle \bar{\alpha}, v \rangle,$$

for $(v, \alpha), (\bar{v}, \bar{\alpha}) \in V \oplus V^*$. A *linear Dirac structure* on V is a subspace $D \subset V \oplus V^*$ such that $D = D^{\perp}$, where the subspace $D^{\perp}$ is orthogonal to D relative to the pairing $\langle\!\langle\, , \rangle\!\rangle$.

Let P be a smooth manifold and let $TP \oplus T^*P$ denote the Pontryagin bundle over P, defined as the direct sum of the tangent and cotangent bundle of P. In this paper, we shall call a subbundle $D \subset TP \oplus T^*P$ a *Dirac structure* on P, if $D(x)$ is a linear Dirac structure on the vector space T_xP at each point $x \in P$.

In mechanics, the most important Dirac structure is an induced Dirac structure on the cotangent bundle T^*Q, which is defined by the canonical symplectic structure Ω on T^*Q and a given constraint distribution Δ_Q on Q as in Yoshimura and Marsden [2006a]. Such an induced Dirac structure plays an essential role in formulating the dynamics of nonholonomic mechanics, electric circuits, fluids as well as nonequilibrium thermodynamic systems in the context of *implicit Lagrangian systems*, which may allow the cases of degenerate Lagrangians as shown in Yoshimura and Marsden [2007, 2009]; Gay-Balmaz and Yoshimura [2015, 2018].

In this paper, we primarily focus on an induced Dirac structure on the tangent bundle TQ and its associated Lagrange-Dirac dynamical system on TQ, where the given Lagrangian is hyperregular.

Nonholonomic constraints. Now let us consider the case in which nonholonomic constraints are given. Let Δ_Q be a constraint distribution on Q given by

$$\Delta_Q(q) := \{(q,v) \in TQ \mid \langle \omega^r(q), v_q \rangle = 0,\ r = 1,...,m < n\},$$

where $\omega^r(q) = \omega^r_i(q)dq^i$ are given r-constraint one-forms on Q. If for any vector fields $X, Y \in \mathfrak{X}(Q)$ with values in Δ_Q, the condition

$$[X(q), Y(q)] \in \Delta_Q(q)$$

holds for each $q \in Q$, then the constraint distribution $\Delta_Q \subset TQ$ is integrable in the sense of Frobenius and the given distribution is holonomic since there exists a submanifold $N \subset Q$ such that, for each $q \in Q$,

$$T_qN = \Delta_Q(q).$$

In this paper, we consider the general case in which the constraint distribution Δ_Q is nonintegrable, namely, nonholonomic.

Lagrangian Dirac structure on the tangent bundle. Let L be a regular Lagrangian on TQ and $\Theta_L = (\mathbb{F}L)^*\Theta$ is the Lagrangian one-form, where Θ is the canonical symplectic structure on T^*Q as before. Recall that given a regular distribution Δ_Q on Q, the lifted distribution $\mathcal{C}$ on TQ is defined by

$$\mathcal{C} = (T\tau_Q)^{-1}(\Delta_Q),$$

which is locally given by, for each $(q,v) \in TQ$,

$$\mathcal{C}(q,v) = \left\{(q,v,\delta q,\delta v) \in T_{(q,v)}TQ \mid (q,\delta q) \in \Delta_Q(q)\right\}.$$

Then, we can define an induced Dirac structure $D_L \subset TTQ \oplus T^*TQ$ on TQ from the distribution Δ_Q and the Lagrangian L by, for each $u = (q,v) \in TQ$,

$$\begin{aligned} D_L(u) = \big\{&(w,\alpha) \in T_uTQ \times T^*_uTQ \mid w \in \mathcal{C}(u), \\ &\text{and } \langle \alpha, \delta u \rangle = \Omega_L(u)(w,\delta u), \text{ for all } \delta u \in \mathcal{C}(u)\big\}, \end{aligned} \tag{8}$$

where we recall $\Omega_L = -\mathbf{d}\Theta_L$ is the Lagrangian two-form.

Note that this Dirac structure D_L is dependent on the given Lagrangian L on TQ. In other words, the Dirac structure D_L is not naturally attributed with the tangent bundle, but is induced from a given distribution Δ_Q and a given Lagrangian L. In other words, the induced Dirac structure D_L on the tangent bundle TQ can be induced from the Dirac structure D_{T^*Q} on the cotangent bundle by the Legendre transform. In this sense, let us specifically call D_L a ***Lagrangian Dirac structure*** on TQ.

Lagrange-Dirac systems on the tangent bundle. Associated to the Lagrangian L on TQ, we can define the energy E_L on TQ by, for $u \in Q$,

$$E_L(u) := \langle \mathbb{F}L(u), u \rangle - L(u).$$

Given E_L and D_L, a curve $u(t) \in TQ$ is a solution of the ***Lagrange-Dirac system*** if it satisfies the condition

$$\big(\dot{u}(t), \mathbf{d}E_L(u(t))\big) \in D_L(u(t)). \tag{9}$$

It follows from the condition (9) that we get the ***intrinsic Lagrange-d'Alembert equations on*** TQ**:**

$$\begin{cases} \mathbf{i}_{\dot{u}_0(t)}\Omega_L(u_0(t)) - \mathbf{d}E_L(u_0(t)) \in \mathcal{C}(u_0)^{\circ}, \\ \dot{u}_0(t) \in \mathcal{C}(u_0). \end{cases}$$

By computations using local coordinates (q^i, v^i) for $u \in TQ$, we can obtain the *local expression of the Lagrange-d'Alembert equations on* TQ:

$$\begin{cases} \dfrac{\partial^2 L}{\partial v^i \partial v^j}\dfrac{dv^j}{dt} + \dfrac{\partial^2 L}{\partial v^i \partial q^j}\dot{q}^j - \dfrac{\partial L}{\partial q^i} = \lambda_r \omega_i^r, \\ \dfrac{dq^i}{dt} = v^i, \\ \omega_i^r(q)\dot{q}^i = 0. \end{cases} \tag{10}$$

Energy balance equation. Along a solution curve $u(t) = (q(t), v(t)) \in TQ$ of the Lagrange-Dirac system on TQ in (9), the energy balance equation holds as:

$$\frac{d}{dt}E_L(q(t), v(t)) = 0.$$

3 Examples

Lagrangian systems with nonholonomic constraints. We illustrate our theory with an example of the mechanical system in which the Lagrangian L on TQ has the form of kinetic minus potential energy as

$$L(q, v) = T(q, v) - U(q),$$

where $U(q)$ denotes the potential energy defined on an n-dimensional configuration manifold Q and $T(q, v) = \frac{1}{2}M_{ij}(q)v^i v^j$ the kinetic energy with an $n \times n$ mass matrix M_{ij} whose elements are dependent on $q \in Q$. Further we assume that $\det M_{ij}(q) \neq 0$ at each q. From the given Lagrangian $L(q, v)$, we have the energy as

$$E_L(q, v) = \frac{1}{2}M_{ij}(q)v^i v^j + U(q).$$

Suppose that we have nonholonomic constraints $\Delta_Q \subset TQ$ given by

$$\Delta_Q(q) := \{(q, v) \in TQ \mid \langle \omega^r(q), v_q \rangle = 0,\ r = 1, ..., m < n\},$$

where $\omega^r(q) = \omega^r_i(q)dq^i$ are given r-constraint one-forms on Q. Then, we can define the induced Lagrangian Dirac structure D_L on TQ as in (8).

By direct computations, it follows from the condition for a curve $u(t) = (q^i(t), v^i(t))$ in TQ

$$\left(\dot{u}(t), \mathbf{d}E_L(u(t))\right) \in D_L(u(t))$$

that we can obtain the Euler-Lagrange equations on TQ by using Lagrange multipliers λ_r as

$$\begin{cases} M_{ij}\dfrac{dv^j}{dt} + \dfrac{\partial M_{ij}}{\partial q^k} v^j \dot{q}^k - \dfrac{1}{2}\dfrac{\partial M_{jk}}{\partial q^i} v^j v^k = -\dfrac{\partial U}{\partial q^i} + \lambda_r \omega^r_i(q), \\ \dfrac{dq^i}{dt} = v^i, \\ \omega^r_i(q)\dot{q} = 0. \end{cases}$$

Planar linkage mechanisms. We show an example of holonomic mechanical systems, i.e., a planar linkage mechanism as in Fig.1. The planar linkage is consisted of three rigid links with four ideal pin joints, where we assume that there is no friction at the joints and also that there exists the gravitational acceleration g along the negative direction of y-axis. The configuration manifold

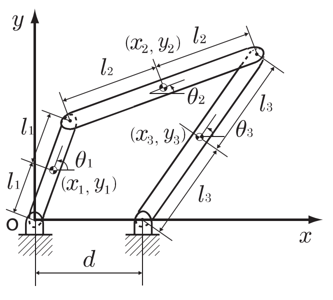

Figure 1: Rigid planar linkage mechanism

of the planar linkage mechanism is denoted by $Q = \mathbb{R}^2 \times S^1 \times \mathbb{R}^2 \times S^1 \times \mathbb{R}^2 \times S^1$, whose local coordinates are given by $q = (x_1, y_1, \theta_1, x_2, y_2, \theta_2, x_3, y_3, \theta_3)$ for each $q \in Q$. The linkage mechanism has a set of holonomic constraints given by

$$\phi(q) = \begin{bmatrix} -x_1 + l_1 \cos\theta_1 \\ -y_1 + l_1 \sin\theta_1 \\ -x_2 + 2l_1 \cos\theta_1 + l_2 \cos\theta_2 \\ -y_2 + 2l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ -x_3 + l_3 \cos\theta_3 + d \\ -y_3 + l_3 \sin\theta_3 \\ 2l_1 \cos\theta_1 + 2l_2 \cos\theta_2 - 2l_3 \cos\theta_3 - d \\ 2l_1 \sin\theta_1 + 2l_2 \sin\theta_2 - 2l_3 \sin\theta_3 \end{bmatrix} = 0.$$

In the above, $\phi = (\phi^1, ..., \phi^m)^T$, where each $\phi(q)^r$ is a function on Q. Define the Lagrangian on TQ by using $v = (v_{x_1}, v_{y_1}, v_{\theta_1}, v_{x_2}, v_{y_2}, v_{\theta_2}, v_{x_3}, v_{y_3}, v_{\theta_3}) \in T_qQ$ as

$$L(q, v) = \sum_{i=1}^{3} \frac{1}{2} m_i ({v_{x_i}}^2 + {v_{y_i}}^2) + \sum_{i=1}^{3} \frac{1}{2} I_i {v_{\theta_i}}^2 - U(q),$$

where m_i and I_i denote the mass and the moment of inertia of the i-the link respectively and where $U(q) = \sum_{i=1}^{3} m_i g y_i$.

Then, the dynamics of the planar linkage mechanism may be represented by the Lagrangian systems with holonomic constraints on TQ as

$$\begin{cases} M_{ij}\, \dot{v}^j = -\dfrac{\partial U}{\partial q^i} + \dfrac{\partial \phi^r}{\partial q^i} \lambda_r, \\ \dot{q}^i = v^i, \\ \phi^r(q) = 0, \end{cases}$$

where the mass matrix is $M = \mathrm{diag}(m_1, m_1, m_1 {l_1}^2/3, m_2, m_2, m_2 {l_2}^2/3, m_3, m_3, m_3 {l_3}^3/3)$.

4 Conclusions

In this paper, first we have reviewed conventional Hamilton's principle in mechanics to formulate the Euler-Lagrange equations. Then, we have seen the geometry of the tangent bundle TQ, in which the Lagrangian forms are induced on TQ from the canonical forms on the cotangent bundle via the Legendre transform. In particular, for the case in which a given Lagrangian is regular, we can develop the Lagrangian system on TQ. Second, we have introduced an induced Dirac structure D_L on TQ, which is called Lagrangian Dirac structure, from a given distribution Δ_Q and a regular Lagrangian L on TQ. We have shown that the solution curve of the Lagrange-Dirac system satisfies the generalized Lagrange-d'Alembert equations. We have illustrated our theory with examples of a nonholonomic mechanical system and a rigid planar linkage mechanism with holonomic constraints.

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