非局所項をもつある半線形楕円型固有値問題について

(Eigenvalue problem of semilinear elliptic equation with non-local term)

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Abstract

In this paper we consider the Gel'fand problem with non-local term $\Delta v + \lambda e^v/\int_{\Omega} e^v dx = 0$ on $n$-dimensional bounded domain $\Omega$ with Dirichlet boundary condition. If it is star-shaped, then we have an upper bound of $\lambda$ for the existence of the solution. We also have infinitely many bendings in $\lambda$ of the connected component of the solution set in $\lambda - v$ if $\Omega$ is a ball and $3 \leq n \leq 9$.

1 Introduction

We consider the following Gel'fand problem with non-local term:

$$\begin{cases}
-\Delta v = \lambda \frac{e^v}{\int_{\Omega} e^v dx} & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1)

where $\lambda$ is a positive constant and $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. We define the solution set $\mathcal{C}$ and the section of $\mathcal{C}$ cut by $\lambda > 0$ by

$$\mathcal{C} = \{ (\lambda, v) \mid v = v(x) \text{ is a classical solution to (1) for } \lambda > 0 \}.$$

and

$$\mathcal{C}^\lambda = \{ v \in C^2(\Omega) \cap C(\overline{\Omega}) \mid v = v(x) \text{ solves (1)} \},$$
respectively. The first theorem is concerned with the star-shaped domain, so that \( x \cdot \nu > 0 \) holds for each \( x \in \partial \Omega \). The second one is concerned with the unit ball.

**Theorem 1** If \( \Omega \) is star-shaped with respect to the origin, then there is an upper bound of \( \lambda \) for the existence of the solution to (1). Thus we have \( \lambda \in (0, +\infty) \) such that \( C^\lambda \neq \emptyset \) and \( C^\lambda = \emptyset \) for \( 0 < \lambda < \lambda \) and \( \lambda > \lambda \), respectively. Moreover \( C_0 \) is unbounded in \( \lambda - \nu \) plane, and \( \|C^\lambda \| = 1 \) for \( 0 < \lambda \ll 1 \), where \( C_0 \) stands for the connected component of \( C \) satisfying \( (0,0) \in C_0 \).

**Theorem 2** If \( \Omega \) is the unit ball \( B = \{ x \in \mathbb{R}^n \mid |x| < 1 \} \), then \( C \) is a one-dimensional open manifold parametrized as

\[
C = \{(\lambda(s), v(\cdot, s)) \mid 0 < s < +\infty\}
\]

with the endpoints \( (0,0) \) and the weak solution \( (2\omega_n, 2 \log \frac{1}{|x|}) \), so that

\[
\lim_{s \downarrow 0} (\lambda(s), v(\cdot, s)) = (0,0)
\]

and

\[
\lim_{s \uparrow +\infty} (\lambda(s), v(\cdot, s)) = \left( 2\omega_n, 2 \log \frac{1}{|x|} \right)
\]

in \( \mathbb{R} \times C(B) \) and \( \mathbb{R} \times W^{2,p}(B) \) for \( p \in [1, n/2) \), respectively, where \( \omega_n \) denotes the \( (n-1) \) dimensional volume of the unit ball in \( \mathbb{R}^n \). If \( 3 \leq n \leq 9 \), then \( C \) bends infinitely many times in \( \lambda \). Thus there is a sequence \( \{s_k\} \) labeled by \( k \) with \( 0 < s_1 < s_2 < \cdots < s_k < \cdots \) such that \( s \in [s_{2k-1}, s_{2k}] \mapsto \lambda(s) \) and \( s \in [s_{2k}, s_{2k+1}] \mapsto \lambda(s) \) decreasing and increasing, respectively. Furthermore, it holds that

\[
\lambda(s_2) < \lambda(s_4) < \cdots < \lambda(s_{2k}) < \lambda(s_{2k+2}) < \cdots < 2\omega_n
\]

\[
\cdots < \lambda(s_{2k+1}) < \lambda(s_{2k+1}) < \cdots < \lambda(s_3) < \lambda(s_1)
\]

and there are infinitely many solutions to (1) for \( \lambda = 2\omega_n \) in particular. If \( n \geq 10 \), on the other hand, then no bending occurs to \( C \) and hence \( s \in [0, +\infty) \mapsto \lambda(s) \) is increasing and each \( \lambda \in (0, 2\omega_n) \) takes a unique solution to (1).

Next we study the spectral and related properties of the following linearized problem of (1):

\[
\begin{align*}
\Delta \phi + \lambda \int_\Omega e^v dx \phi - \lambda \frac{\int_\Omega e^\phi dx e^v}{(\int_\Omega e^v dx)^2} e^v &= -\mu \phi & \text{in } \Omega \\
\phi &= 0 & \text{on } \partial \Omega.
\end{align*}
\]
Let us denote by \( i = i(\lambda, v) \) and \( i_R = i_R(\lambda, v) \) the number of negative eigenvalues of (3) and that for radially symmetric eigenfunctions to (3), respectively. We call these numbers Morse index and radial Morse index at \((\lambda, v) \in C\), respectively.

**Theorem 3** Under the circumstances described in the previous theorem, if \( 3 \leq n \leq 9 \) then it holds that \( i = i_R = k \) on the arc \( T_k T_{k+1} \) of \( C \) for \( k = 0, 1, \ldots \), where \( T_k = (\lambda(s_k), v(s_k)) \) with \( s_0 = 0 \). If \( n \geq 10 \), on the other hand, it always holds that \( i = i_R = 0 \).

In §2, we treat the star-shaped domain and prove Theorem 1. We omit the proof of Theorems 2 and 3. See [8] and [9] for detail.

# 2 Star-shaped domain

Throughout the present section, \( \Omega \) denotes the general star-shaped domain with respect to the origin in \( \mathbb{R}^n \) for \( n \geq 3 \) provided with the smooth boundary \( \partial \Omega \), and \( \nu \) stands for the outer unit normal vector.

**Proof of Theorem 1:** It follows from McGough [7] that the star-shaped \( \Omega \) takes \( \bar{\sigma} > 0 \) such that the solution of

\[
\begin{aligned}
-\Delta v &= \sigma e^v \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]  

(3)

with a constant \( \sigma > 0 \) is unique for \( 0 < \sigma < \bar{\sigma} \). However, any solution \( v = v(x) \) to (1) solves (3) with

\[ \sigma = \frac{\lambda}{\int_\Omega e^v dx} \leq \frac{\lambda}{|\Omega|} \]

because of its positivity, where \(|\Omega| \) denotes the volume of \( \Omega \). Therefore, the solution to (1) is unique for \( 0 < \lambda < \bar{\lambda} = \bar{\sigma}|\Omega| \). Hence we can prove the uniqueness result.

To have an upper bound \( \lambda \) we apply the Pohozaev identity [10].

Unboundedness of the component \( C_0 \) follows from the standard degree argument similarly to [12] and [13].

The first eigenvalue of (2), denoted by \( \mu_1(\lambda, v) \), is positive around the trivial solution \((\lambda, v) = (0, 0)\) similarly to (3). Therefore, it generates a branch in \( C \). This branch continues as far as \( \mu_1(\lambda, v) > 0 \) and because we
have an upper bound for $C_{\lambda} \neq \emptyset$ if $\Omega$ is star-shaped, only two possibilities arise then. That is, there is a one-dimensional manifold contained in $C$ starting from $(\lambda, v) = (0, 0)$ denoted by

$$C = \{(\lambda(s), v(\cdot, s)) \mid 0 < s < s_0\},$$

and we have either that $\lim_{s \to s_0}(\lambda(s), v(\cdot, s)) = (\lambda^*, v^*) \in C$ exists in $\mathbb{R} \times C(\Omega)$ with

$$\mu_1(\lambda^*, v^*) = 0,$$

or else that $\limsup_{s \to s_0} \|v(\cdot, s)\|_{\infty} = +\infty$. For simplicity, we say that $C$ is closed and open in the former and the latter cases, respectively. Those notions are kept, if there is an upper bound of $A$ for the existence of the solution to (1), and then the alternatives between openness and closedness of $C$ given above, arise. In any case, the connected component $C_0$ mentioned in Theorem 1 contains this $C$. We now describe its spectral properties.

**Proposition 1** If $(\lambda^*, v^*) \in C$ satisfies $\mu_2(\lambda^*, v^*) > \mu_1(\lambda^*, v^*) = 0$, with $\mu_1(\lambda^*, v^*) = 0$ admitting the eigenfunction $\phi^*>0$, then $C$ is locally one-dimensional manifold parametrized as

$$C^* = \{(\lambda(s), v(s)) \mid |s| < \delta\}$$

with $(\lambda(0), v(0)) = (\lambda^*, v^*)$. Here $\mu_2(\lambda^*, v^*)$ denotes the second eigenvalue of (2) at $(\lambda, v) = (\lambda^*, v^*)$. Furthermore, $C^*$ bends to the left with respect to $\lambda$ at $(\lambda^*, v^*)$, so that $\lambda(s) < \lambda^*$ holds for $0 < |s| < \delta$ and the mappings $s \in (-\delta, 0) \mapsto \lambda(s)$ and $s \in [0, \delta) \mapsto \lambda(s)$ are increasing and decreasing, respectively. Finally, $\mu_1(\lambda(s), v(s))$ changes sign at $s = 0$, say, $\pm \mu_1(\lambda(s), v(s)) > 0$ according as $-\delta < \pm s < 0$.

**Proof:** Given $(\lambda^*, v^*) \in C$ with $\mu_1(\lambda^*, v^*) = 0$, let the linearized operator, the left-hand side of (2) with $(\lambda, v) = (\lambda^*, v^*)$ be $A^*$. Then, from the assumption we have $\text{Ker}(A^*) = \langle \phi^* \rangle$ with $\phi^* = \phi^*(x) \in H_0^1(\Omega) \setminus \{0\}$ positive in $\Omega$. Now, we take the nonlinear operator

$$\Phi(s, \sigma, w) = \Delta(v^* + s\phi^* + w) + (\lambda^* + \sigma)\frac{e^{v^* + s\phi^* + w}}{\int_{\Omega} e^{v^* + s\phi^* + w}dx},$$

defined for $s \in \mathbb{R}$, $\sigma \in \mathbb{R}$, and $w \in Y$, where

$$Y = \left\{w \in C^2(\Omega) \mid w|_{\partial\Omega} = 0, \int_{\Omega} w\phi^* dx = 0\right\}.$$
It is obvious that $\Phi(0, 0, 0) = 0$ and the linearized operator
\[
\Phi_{\sigma, \omega}(0, 0, 0) = \begin{pmatrix} e^{v^*} / \int_{\Omega} e^{v^*} dx \\ -A^* \end{pmatrix} : \mathbb{R}^2 \rightarrow C(\Omega)
\]
is an isomorphism by $\phi^* > 0$. Because classical solution to (1) near $(\lambda^*, v^*)$ is identified with zero of $\Phi$, the implicit function theorem then guarantees a $C^2$-family $\{\lambda(s), v(s) \} : |s| < s_0$ of classical solutions satisfying $(\lambda(0), v(0)) = (\lambda^*, v^*)$, where $s_0 > 0$. It also follows from the standard perturbation theory (4) that the linearized operator around this $(\lambda(s), v(s))$ takes the simple eigenvalue $\mu(s)$ and the eigenfunction $\phi(s)$ with $C^2$ dependence in $s$ such that $(\mu(0), \phi(0)) = (0, \phi^*)$ so that (2) is valid to
\[
(\lambda, v, \mu, \phi) = (\lambda(s), v(s), \mu(s), \phi(s))
\]
for $|s| < s_0$.

Differentiating with respect to $s$, we have from (1) that
\[
\begin{cases}
\Delta \dot{v} + \dot{\lambda} \frac{e^{v}}{\int_{\Omega} e^{v} dx} + \lambda \frac{e^{v}}{\int_{\Omega} e^{v} dx} \dot{v} - \lambda \frac{e^{v} \dot{v}}{(\int_{\Omega} e^{v} dx)^2} e^{v} = 0 \text{ in } \Omega \\
\dot{v} = 0 \text{ on } \partial\Omega.
\end{cases}
\]

Then, subtracting (2) from (4) with $s = 0$ multiplied by $\dot{v}$ and $\phi^*$, respectively, we get that
\[
\dot{\lambda}(0) \frac{\int_{\Omega} e^{v^*} \phi^* dx}{\int_{\Omega} e^{v^*} dx} = 0,
\]
and hence $\dot{\lambda}(0) = 0$ holds true. This implies $\dot{v}(0) \in \text{Ker} A^*$ by (4), and we can assume that $\dot{v}(0) = \phi^*$ without loss of generality, because $(\dot{\lambda}(0), \dot{v}(0))$ does not vanish from the implicit function theorem.

Differentiating (4) once more and putting $s = 0$, we have
\[
\Delta \ddot{v} + \ddot{\lambda} \frac{e^{v}}{\int_{\Omega} e^{v} dx} - \lambda \frac{\int_{\Omega} e^{v} \phi^* dx}{(\int_{\Omega} e^{v} dx)^2} e^{v} + \lambda \frac{e^{v} \phi^2}{\int_{\Omega} e^{v} dx} + \lambda \frac{e^{v} \ddot{v}}{\int_{\Omega} e^{v} dx} - \lambda \frac{\int_{\Omega} e^{v} \phi^* dx}{(\int_{\Omega} e^{v} dx)^2} e^{v} \phi^* = 0 \text{ in } \Omega
\]
with $\ddot{v} = 0$ on $\partial\Omega$. Then, subtracting (5) from (2) multiplied by $\phi^*$ and $\ddot{v}$, respectively, we obtain that
\[
\ddot{\lambda}(0) \frac{\int_{\Omega} e^{v^*} \phi^* dx}{\int_{\Omega} e^{v^*} dx} = \\
\lambda^* \left\{ 3 \frac{\int_{\Omega} e^{v^*} \phi^* dx \int_{\Omega} e^{v^*} \phi^2 dx}{(\int_{\Omega} e^{v^*} dx)^2} - 2 \frac{\left( \int_{\Omega} e^{v^*} \phi^* dx \right)^3}{(\int_{\Omega} e^{v^*} dx)^3} - \frac{\int_{\Omega} e^{v} \phi^3 dx}{\int_{\Omega} e^{v} dx} \right\}.
\]
Letting $\frac{e^{v^{*}}dx}{\int_{\Omega}e^{v}dx} = d\mu$, we have

$$\frac{\lambda(0)}{\lambda^{*}} \int_{\Omega} \phi^{*} d\mu = 3 \int_{\Omega} \phi^{*} d\mu \int_{\Omega} \phi^{*2} d\mu - 2 \left( \int_{\Omega} \phi^{*} d\mu \right)^3 - \int_{\Omega} \phi^{*3} d\mu$$

$$= 3 \int_{\Omega} \phi^{*} d\mu \cdot \left\{ \int_{\Omega} \phi^{*2} d\mu - \left( \int_{\Omega} \phi^{*} d\mu \right)^2 \right\} + \left( \int_{\Omega} \phi^{*} d\mu \right)^3 - \int_{\Omega} \phi^{*3} d\mu \leq 0$$

with the equality only when $\phi^{*}$ is a constant. This is impossible, and we get that $\lambda(0) < 0$.

To complete the proof, we differentiate (2) and obtain

$$\Delta \phi + \lambda \frac{e^{v} \phi^{*2}}{\int_{\Omega} e^{v} dx} - \lambda \frac{\int_{\Omega} e^{v} \phi^{*} dx}{\left( \int_{\Omega} e^{v} dx \right)^{2}} e^{v} \phi^{*} + \frac{e^{v} \phi}{\int_{\Omega} e^{v} dx} - \lambda \frac{\int_{\Omega} e^{v} \phi^{*2} dx}{\left( \int_{\Omega} e^{v} dx \right)^{2}} e^{v} = -\dot{\mu}$$

in $\Omega$ with $\dot{\phi} = 0$ on $\partial \Omega$ by putting $s = 0$. Integrating (6) multiplied by $\phi^{*}$ we have

$$-\dot{\mu}(0) \frac{\|\phi^{*}\|^{2}}{\lambda^{*}} = \int_{\Omega} \phi^{*3} d\mu - 3 \int_{\Omega} \phi^{*} d\mu \cdot \int_{\Omega} \phi^{*2} d\mu + 2 \left( \int_{\Omega} \phi^{*} d\mu \right)^{3},$$

similarly. The proof is complete. \(\square\)

**References**


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