非局所項をもつある半線形楕円型固有値問題について
(Eigenvalue problem of semilinear elliptic equation with non-local term)

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Abstract

In this paper we consider the Gel'fand problem with non-local term $\Delta v+\lambda e^{v}/\int_{\Omega}e^{v}dx=0$ on n-dimensional bounded domain Ω with Dirichlet boundary condition. If it is star-shaped, then we have an upper bound of λ for the existence of the solution. We also have infinitely many bendings in λ of the connected component of the solution set in $\lambda-v$ if Ω is a ball and $3\leq n\leq 9$.

1 Introduction

We consider the following Gel'fand problem with non-local term:

$$\begin{cases}-\Delta v=\lambda\frac{e^{v}}{\int_{\Omega}e^{v}dx} & \text{in } \Omega\\ v=0 & \text{on } \partial\Omega,\end{cases} \tag{1}$$

where λ is a positive constant and Ω is a bounded domain in $\mathbf{R}^{n}$ with smooth boundary $\partial\Omega$. We define the solution set $\mathcal{C}$ and the section of $\mathcal{C}$ cut by $\lambda>0$ by

$$\mathcal{C}=\{(\lambda,v)\mid v=v(x) \text{ is a classical solution to (1) for } \lambda>0\}.$$

and

$$\mathcal{C}^{\lambda}=\left\{v\in C^{2}(\Omega)\cap C(\overline{\Omega})\mid v=v(x) \text{ solves (1)}\right\},$$

respectively. The first theorem is concerned with the star-shaped domain, so that $x \cdot \nu > 0$ holds for each $x \in \partial\Omega$. The second one is concerned with the unit ball.

Theorem 1 *If Ω is star-shaped with respect to the origin, then there is an upper bound of λ for the existence of the solution to (1). Thus we have $\overline{\lambda} \in (0, +\infty)$ such that $\mathcal{C}^\lambda \neq \emptyset$ and $\mathcal{C}^\lambda = \emptyset$ for $0 < \lambda < \overline{\lambda}$ and $\lambda > \overline{\lambda}$, respectively. Moreover $\mathcal{C}_0$ is unbounded in $\lambda - v$ plane, and $\sharp\mathcal{C}^\lambda = 1$ for $0 < \lambda \ll 1$, where $\mathcal{C}_0$ stands for the connected component of $\mathcal{C}$ satisfying $(0,0) \in \overline{\mathcal{C}_0}$.*

Theorem 2 *If Ω is the unit ball $B = \{x \in \mathbf{R}^n \mid |x| < 1\}$, then $\mathcal{C}$ is a one-dimensional open manifold parametrized as*

$$\mathcal{C} = \{(\lambda(s), v(\cdot, s)) \mid 0 < s < +\infty\}$$

with the endpoints $(0,0)$ and the weak solution $(2\omega_n, 2\log\frac{1}{|x|})$, so that

$$\lim_{s\downarrow 0} (\lambda(s), v(\cdot, s)) = (0,0)$$

and

$$\lim_{s\uparrow+\infty} (\lambda(s), v(\cdot, s)) = \left(2\omega_n, 2\log\frac{1}{|x|}\right)$$

in $\mathbf{R} \times C(\overline{B})$ and $\mathbf{R} \times W^{2,p}(B)$ for $p \in [1, n/2)$, respectively, where ω_n denotes the $(n-1)$ dimensional volume of the unit ball in $\mathbf{R}^n$. If $3 \leq n \leq 9$, then $\mathcal{C}$ bends infinitely many times in λ. Thus there is a sequence $\{s_k\}$ labeled by $k = 1, 2, \cdots$ with $0 < s_1 < s_2 < \cdots < s_k < \cdots$ such that $s \in [s_{2k-1}, s_{2k}] \mapsto \lambda(s)$ and $s \in [s_{2k}, s_{2k+1}] \mapsto \lambda(s)$ decreasing and increasing, respectively. Furthermore, it holds that

$$\begin{aligned} \lambda(s_2) < \lambda(s_4) < \cdots < \lambda(s_{2k}) < \lambda(s_{2k+2}) < \cdots < 2\omega_n \\ < \cdots < \lambda(s_{2k+1}) < \lambda(s_{2k-1}) < \cdots < \lambda(s_3) < \lambda(s_1) \end{aligned}$$

and there are infinitely many solutions to (1) for $\lambda = 2\omega_n$ in particular. If $n \geq 10$, on the other hand, then no bending occurs to $\mathcal{C}$ and hence $s \in [0, \infty) \mapsto \lambda(s)$ is increasing and each $\lambda \in (0, 2\omega_n)$ takes a unique solution to (1).

Next we study the spectral and related properties of the following linearized problem of (1):

$$\begin{cases} \Delta\phi + \lambda\frac{e^v}{\int_\Omega e^v dx}\phi - \lambda\frac{\int_\Omega e^v \phi dx}{(\int_\Omega e^v dx)^2}e^v = -\mu\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Let us denote by $i = i(\lambda, v)$ and $i_R = i_R(\lambda, v)$ the number of negative eigenvalues of (3) and that for radially symmetric eigenfunctions to (3), respectively. We call these numbers Morse index and radial Morse index at $(\lambda, v) \in \mathcal{C}$, respectively.

Theorem 3 *Under the circumstances described in the previous theorem, if* $3 \leq n \leq 9$ *then it holds that* $i = i_R = k$ *on the arc* $T_k T_{k+1}$ *of* $\mathcal{C}$ *for* $k = 0, 1, \cdots$, *where* $T_k = (\lambda(s_k), v(s_k))$ *with* $s_0 = 0$. *If* $n \geq 10$, *on the other hand, it always holds that* $i = i_R = 0$.

In §2, we treat the star-shaped domain and prove Theorem 1. We omit the proof of Theorems 2 and 3. See [8] and [9] for detail.

2 Star-shaped domain

Throughout the present section, Ω denotes the general star-shaped domain with respect to the origin in $\mathbf{R}^n$ for $n \geq 3$ provided with the smooth boundary $\partial\Omega$, and ν stands for the outer unit normal vector.

Proof of Theorem 1: It follows from McGough [7] that the star-shaped Ω takes $\tilde{\sigma} > 0$ such that the solution of

$$\begin{cases} -\Delta v = \sigma e^v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

with a constant $\sigma > 0$ is unique for $0 < \sigma < \tilde{\sigma}$. However, any solution $v = v(x)$ to (1) solves (3) with

$$\sigma = \frac{\lambda}{\int_\Omega e^v dx} \leq \frac{\lambda}{|\Omega|}$$

because of its positivity, where $|\Omega|$ denotes the volume of Ω. Therefore, the solution to (1) is unique for $0 < \lambda < \tilde{\lambda} = \tilde{\sigma}\,|\Omega|$. Hence we can prove the uniqueness result.
To have an upper bound λ we apply the Pohozaev identity [10].
Unboundedness of the component $\mathcal{C}_0$ follows from the standard degree argument similarly to [12] and [13]. □

The first eigenvalue of (2), denoted by $\mu_1(\lambda, v)$, is positive around the trivial solution $(\lambda, v) = (0, 0)$ similarly to (3). Therefore, it generates a branch in $\mathcal{C}$. This branch continues as far as $\mu_1(\lambda, v) > 0$ and because we have an upper bound for $\mathcal{C}_\lambda \neq \emptyset$ if Ω is star-shaped, only two possibilities arise then. That is, there is a one-dimensional manifold contained in $\mathcal{C}$ starting from $(\lambda, v) = (0,0)$ denoted by

$$\underline{\mathcal{C}} = \{(\lambda(s), v(\cdot, s)) \mid 0 < s < s_0\},$$

and we have either that $\lim_{s \to s_0} (\lambda(s), v(\cdot, s)) = (\lambda^*, v^*) \in \mathcal{C}$ exists in $\mathbf{R} \times C(\overline{\Omega})$ with

$$\mu_1(\lambda^*, v^*) = 0,$$

or else that $\limsup_{s \to s_0} \|v(\cdot, s)\|_\infty = +\infty$. For simplicity, we say that $\underline{\mathcal{C}}$ is closed and open in the former and the latter cases, respectively. Those notions are kept, if there is an upper bound of λ for the existence of the solution to (1), and then the alternatives between openness and closedness of $\underline{\mathcal{C}}$ given above, arise. In any case, the connected component $\mathcal{C}_0$ mentioned in Theorem 1 contains this $\underline{\mathcal{C}}$. We now describe its spectral properties.

Proposition 1 *If $(\lambda^*, v^*) \in \mathcal{C}$ satisfies $\mu_2(\lambda^*, v^*) > \mu_1(\lambda^*, v^*) = 0$, with $\mu_1(\lambda^*, v^*) = 0$ admiting the eigenfunction $\phi^* > 0$, then $\mathcal{C}$ is locally one-dimensional manifold parametrized as*

$$\mathcal{C}^* = \{(\lambda(s), v(s)) \mid |s| < \delta\}$$

with $(\lambda(0), v(0)) = (\lambda^*, v^*)$. Here $\mu_2(\lambda^*, v^*)$ denotes the second eigenvalue of (2) at $(\lambda, v) = (\lambda^*, v^*)$. Furthermore, $\mathcal{C}^*$ bends to the left with respect to λ at (λ^*, v^*), so that $\lambda(s) < \lambda^*$ holds for $0 < |s| < \delta$ and the mappings $s \in (-\delta, 0] \mapsto \lambda(s)$ and $s \in [0, \delta) \mapsto \lambda(s)$ are increasing and decreasing, respectively. Finally, $\mu_1(\lambda(s), v(s))$ changes sign at $s = 0$, say, $\pm\mu_1(\lambda(s), v(s)) > 0$ according as $-\delta < \pm s < 0$.

Proof: Given $(\lambda^*, v^*) \in \mathcal{C}$ with $\mu_1(\lambda^*, v^*) = 0$, let the linearized operator, the left-hand side of (2) with $(\lambda, v) = (\lambda^*, v^*)$ be A^*. Then, from the assumption we have $\mathrm{Ker}(A^*) = \langle \phi^* \rangle$ with $\phi^* = \phi^*(x) \in H_0^1(\Omega) \setminus \{0\}$ positive in Ω. Now, we take the nonlinear operator

$$\Phi(s, \sigma, w) = \Delta(v^* + s\phi^* + w) + (\lambda^* + \sigma) \frac{e^{v^* + s\phi^* + w}}{\int_\Omega e^{v^* + s\phi^* + w} dx},$$

defined for $s \in \mathbf{R}$, $\sigma \in \mathbf{R}$, and $w \in Y$, where

$$Y = \left\{ w \in C^2(\overline{\Omega}) \mid w|_{\partial\Omega} = 0, \ \int_\Omega w\phi^* dx = 0 \right\}.$$

It is obvious that $\Phi(0,0,0)=0$ and the linearized operator

$$\Phi_{\sigma,w}(0,0,0)=\begin{pmatrix} e^{v^*}/\int_\Omega e^{v^*}dx \\ -A^* \end{pmatrix} : \begin{array}{c}\mathbf{R}\\ \times \\ Y\end{array} \to C(\overline{\Omega})$$

is an isomorphism by $\phi^*>0$. Because classical solution to (1) near (λ^*,v^*) is identified with zero of Φ, the implicit function theorem then guarantees a C^2-family $\{(\lambda(s),v(s)) \mid |s|<s_0\}$ of classical solutions satisfying $(\lambda(0),v(0))=(\lambda^*,v^*)$, where $s_0>0$. It also follows from the standard perturbation theory ([4]) that the linearized operator around this $(\lambda(s),v(s))$ takes the simple eigenvalue $\mu(s)$ and the eigenfucntion $\phi(s)$ with C^2 dependence in s such that $(\mu(0),\phi(0))=(0,\phi^*)$ so that (2) is valid to

$$(\lambda,v,\mu,\phi)=(\lambda(s),v(s),\mu(s),\phi(s))$$

for $|s|<s_0$.

Differentiating with respect to s, we have from (1) that

$$\begin{cases} \Delta\dot v+\dot\lambda\frac{e^v}{\int_\Omega e^v dx}+\lambda\frac{e^v}{\int_\Omega e^v dx}\dot v-\lambda\frac{\int_\Omega e^v\dot v dx}{(\int_\Omega e^v dx)^2}e^v=0 & \text{in } \Omega \\ \dot v=0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Then, subtracting (2) from (4) with $s=0$ multiplied by $\dot v$ and ϕ^*, respectively, we get that

$$\dot\lambda(0)\frac{\int_\Omega e^{v^*}\phi^* dx}{\int_\Omega e^{v^*}dx}=0,$$

and hence $\dot\lambda(0)=0$ holds true. This implies $\dot v(0)\in \operatorname{Ker} A^*$ by (4), and we can assume that $\dot v(0)=\phi^*$ without loss of generality, because $\left(\dot\lambda(0),\dot v(0)\right)$ does not vanish from the implicit function theorem.

Differentiating (4) once more and putting $s=0$, we have

$$\begin{aligned} &\Delta\ddot v+\ddot\lambda\frac{e^v}{\int_\Omega e^v dx}-\lambda\frac{\int_\Omega e^v\phi^* dx}{\left(\int_\Omega e^v dx\right)^2}e^v+\lambda\frac{e^v\phi^{*2}}{\int_\Omega e^v dx} \\ &+\lambda\frac{e^v\ddot v}{\int_\Omega e^v dx}-\lambda\frac{\int_\Omega e^v\phi^* dx}{\left(\int_\Omega e^v dx\right)^2}e^v\phi^*=0 \qquad \text{in } \Omega \end{aligned} \tag{5}$$

with $\ddot v=0$ on $\partial\Omega$. Then, subtracting (5) from (2) multiplied by ϕ^* and $\ddot v$, respectively, we obtain that

$$\begin{aligned} &\ddot\lambda(0)\frac{\int_\Omega e^{v^*}\phi^* dx}{\int_\Omega e^{v^*}dx}= \\ &\lambda^*\left\{3\frac{\int_\Omega e^{v^*}\phi^* dx\int_\Omega e^{v^*}\phi^{*2}dx}{\left(\int_\Omega e^{v^*}dx\right)^2}-2\frac{\left(\int_\Omega e^{v^*}\phi^* dx\right)^3}{\left(\int_\Omega e^{v^*}dx\right)^3}-\frac{\int_\Omega e^{v^*}\phi^{*3}dx}{\int_\Omega e^{v^*}dx}\right\}. \end{aligned}$$

Letting $\frac{e^{v^*}dx}{\int_\Omega e^{v^*}dx} = d\mu$, we have

$$\frac{\ddot{\lambda}(0)}{\lambda^*}\int_\Omega \phi^* d\mu = 3\int_\Omega \phi^* d\mu \int_\Omega \phi^{*2} d\mu - 2\left(\int_\Omega \phi^* d\mu\right)^3 - \int_\Omega \phi^{*3} d\mu$$
$$= 3\int_\Omega \phi^* d\mu \cdot \left\{\int_\Omega \phi^{*2} d\mu - \left(\int_\Omega \phi^* d\mu\right)^2\right\} + \left(\int_\Omega \phi^* d\mu\right)^3 - \int_\Omega \phi^{*3} d\mu \le 0$$

with the equality only when ϕ^* is a constant. This is impossible, and we get that $\ddot{\lambda}(0) < 0$.

To complete the proof, we differentiate (2) and obtain

$$\Delta\dot{\phi} + \lambda\frac{e^v\phi^{*2}}{\int_\Omega e^v dx} - \lambda\frac{\int_\Omega e^v\phi^* dx}{\left(\int_\Omega e^v dx\right)^2}e^v\phi^* + \lambda\frac{e^v\dot{\phi}}{\int_\Omega e^v dx} - \lambda\frac{\int_\Omega e^v\phi^{*2}dx}{\left(\int_\Omega e^v dx\right)^2}e^v \qquad (6)$$
$$-\lambda\frac{\int_\Omega e^v\dot{\phi}dx}{\left(\int_\Omega e^v dx\right)^2}e^v + 2\lambda\frac{\left(\int_\Omega e^v\phi^* dx\right)^2}{\left(\int_\Omega e^v dx\right)^2}e^v - \lambda\frac{\int_\Omega e^v\phi^* dx}{\left(\int_\Omega e^v dx\right)^2}e^v\phi^* = -\dot{\mu}\phi^* \qquad \text{in } \Omega$$

with $\dot{\phi} = 0$ on $\partial\Omega$ by putting $s = 0$. Integrating (6) multiplied by ϕ^* we have

$$-\dot{\mu}(0)\frac{\|\phi^*\|_2^2}{\lambda^*} = \int_\Omega \phi^{*3} d\mu - 3\int_\Omega \phi^* d\mu \cdot \int_\Omega \phi^{*2} d\mu + 2\left(\int_\Omega \phi^* d\mu\right)^3,$$

similarly. The proof is complete. □

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