Normality and Duality on Topological Groups

Víctor Pérez Valdés
Graduate School of Mathematical Sciences, The University of Tokyo

Abstract

In this text, we present a result on normality using techniques of duality theory of topological groups. A group topology ν on a topological group (G, τ) is said to be a compatible topology with the duality $(G, G^\wedge)$, if the dual groups of G endowed with τ and ν coincide, i.e. $(G, \nu)^\wedge = (G, \tau)^\wedge$.

The result we show states that for the group $G := \mathbb{Z}^{\mathbb{R}}$, there do not exist normal locally quasi-convex topologies compatible with the duality $(G, G^\wedge)$.

1 Introduction

Topological groups constitute a subclass of topological spaces which have an inner algebraic group structure that makes the multiplication and the inversion continuous functions.
Many infinite groups that we find in several areas of mathematics are topological groups. For instance, $\mathbb{R}^n$, $\mathbb{C}^n$ and their subgroups, the circle $\mathbb{S}^1$ of complex numbers with modulus one, or the group $GL(n, \mathbb{R})$ of $n \times n$ invertible matrices.
The algebraic group structure provides the space with topological properties stronger than those due plainly to the topological structure.
For instance, every topological group is completely regular; in other words, the separation axioms T_0 and *Tychonoff* are equivalent within the class of topological groups.
Another important result following this line is the Birkhoff-Kakutani Theorem, which states that every T_0 first-countable topological group is metrizable.
This fact does not hold for general topological spaces. In fact, the Sorgenfrey line $\mathbb{R}_S$ is an example of a normal, first-countable and non-metrizable space.

In the following paper, we study properties on topological groups focusing on duality theory. We do a quick review of basic results, as the Pontryagin Duality Theorem, which states that every abelian locally compact Hausdorff topological group G is topologically isomorphic to its bidual $G^{\wedge\wedge}$ via the canonical map

$$\begin{aligned} \alpha : G &\longrightarrow G^{\wedge\wedge} \\ g &\longmapsto \alpha(g) : G^\wedge \longrightarrow \mathbb{T} \\ &\qquad\qquad \gamma \longmapsto \alpha(g)(\gamma) := \gamma(g). \end{aligned}$$

Next, we study topologies that are compatible with the duality $(G, G^\wedge)$ of a given group (G, τ). Those are group topologies with the property that generate the same dual as τ. In particular, we analyze the family of locally quasi-convex compatible topologies.

The concept of a locally quasi-convex abelian group was defined by Vilenkin [15] and studied by Banaszczyk [2] and allows us to generalize strong theorems of Functional Analysis to the broader class of abelian topological groups. Nonetheless, the reader may take note of that this process is not straightforward. In particular, in [5] it is described an historical route for the notion of locally quasi-convex groups and an aproximation of the Mackey-Arens Theorem in the wider class of topological abelian groups is proved.

Finally, we prove that for the group $G := \mathbb{Z}^{\mathbb{R}}$ there **do not exist** normal locally quasi-convex topologies that are compatible with the duality $(G, G^\wedge)$.
This result have been published in a joint article with professor E. Martín-Peinador in February 2018 [11].

2 Duality Theory

Here we do a quick review of duality theory on topological groups. From now on, we suppose that every topological group is Hausdorff and abelian.

Given a topological group G, we can consider the space of continuous characters, i.e. the space of continuous homomorphisms $\gamma : G \to \mathbb{T}$, where $\mathbb{T}$ denotes the one dimensional torus of unitary complex numbers with the product and topology inherited from $\mathbb{C}$. We will denote this space by $G^\wedge := \mathrm{CHom}(G, \mathbb{T})$.
Is straightforward to verify that $G^\wedge$ has a group structure if we consider the pointwise product of characters, i.e. the product defined by $(\gamma_1 * \gamma_2)(x) = \gamma_1(x)\gamma_2(x), \ (\gamma_1, \gamma_2 \in G^\wedge, \ x \in G)$. With this operation, $G^\wedge$ is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$.

Definition 2.1. *The group $G^\wedge$ is called* ***the dual group of*** *G.*

As an example, let us show that $\mathbb{Z}^\wedge \simeq \mathbb{T}$.

Example 2.2. **($\mathbb{Z}^\wedge \simeq \mathbb{T}$)**
First, since $\mathbb{Z}$ has the discrete topology, we observe that $\mathbb{Z}^\wedge = Hom(\mathbb{Z}, \mathbb{T})$. Since every homomorphism $\gamma \in \mathbb{Z}^\wedge$ is completely determinated by the image of 1, for every $\xi \in \mathbb{T}$ we can define $\gamma_\xi : \mathbb{Z} \to \mathbb{T}$ such that $\gamma_\xi(1) = \xi$. Now let

$$\begin{aligned} \psi : \mathbb{T} &\longrightarrow \mathbb{Z}^\wedge \\ \xi &\longmapsto \gamma_\xi \end{aligned}$$

It is straightforward that ψ is a group isomorphism, proving that $\mathbb{Z}^\wedge \simeq \mathbb{T}$.

Other examples of dual groups are the following.

Example 2.3. $\mathbb{Z} \simeq \mathbb{T}^\wedge$ *via the isomorphism* $m \mapsto \gamma_m(x) = x^m$

Example 2.4. $\mathbb{R} \simeq \mathbb{R}^\wedge$ *via the isomorphism* $x \mapsto \gamma_x(\xi) = e^{2\pi i x \xi}$. *In general it can be proved that* $(\mathbb{R}^n)^\wedge \simeq \mathbb{R}^n$, $\forall n \in \mathbb{N}$.

For more examples, see [8, 23.27].

We defined the dual group in an algebraic way, but we can actually equip it with a topology compatible with the group operation; in other words, we can provide a topological group structure for $G^\wedge$. This topology is called the compact-open topology and can be defined in the following way:

Given a topological group G, for every $S \subset G$ and every $M \subset \mathbb{T}$ let

$$U(S, M) := \{\gamma \in G^\wedge : \gamma(S) \subset M\}.$$

Definition 2.5. *The **compact-open topology** (also called the topology of the uniform convergence in compact sets), in $G^\wedge$ is the topology generated by*

$$\Sigma = \{U(K, V) : K \subset G \textit{ compact}, V \subset \mathbb{T} \textit{ open}\}.$$

This topology is usually denoted by τ_{co}.

We have the following:

Theorem 2.6. *[12, Proposition 29] Let G be a topological group. Then, its dual group equipped with the compact-open topology $(G^\wedge, \tau_{co})$ is a topological group.*

Since it is possible to define a topology for the dual group that makes it a topological group, we can think about if the group isomorphisms of the examples 2.2, 2.3 and 2.4 are also homeomorphisms, considering the compact-open topology in the dual groups.
This can actually be proved; i.e., we have that the following groups are isomorphic as topological groups: $\mathbb{T} \cong \mathbb{Z}^\wedge$, $\mathbb{Z} \cong \mathbb{T}^\wedge$, $\mathbb{R}^n \cong (\mathbb{R}^n)^\wedge$, $\forall n \in \mathbb{N}$.

In a natural way, we can define the bidual of a topological group.

Definition 2.7. *Given a topological group G, the **bidual of** G is defined as the dual of $(G^\wedge, \tau_{co})$ and it is denoted by $G^{\wedge\wedge}$. In other words, $G^{\wedge\wedge} = (G^\wedge, \tau_{co})^\wedge$.*

Going back to the previous isomorphisms, we can interpret them in terms of the bidual groups. Concretely, we have that $\mathbb{T}^{\wedge\wedge} \cong \mathbb{T}, \mathbb{Z}^{\wedge\wedge} \cong \mathbb{Z}, (\mathbb{R}^n)^{\wedge\wedge} \cong \mathbb{R}^n$, $\forall n \in \mathbb{N}$. That is, these groups are "reflexive". In fact, these are not the only examples of reflexive groups. The Pontryagin-van Kampen Duality Theorem, which is one of the most useful tools in duality theory, states that every locally compact abelian group is reflexive. Concretely:

Theorem 2.8. ***(Pontryagin-van Kampen Duality Theorem)*** *Let G be an abelian topological group. Then, if G is locally compact, it is isomorphic to its bidual under the canonical map*

$$\begin{aligned} \alpha : G &\longrightarrow G^{\wedge\wedge} \\ g &\longmapsto \alpha(g) : G^\wedge \longrightarrow \mathbb{T} \\ &\qquad\qquad \gamma \longmapsto \alpha(g)(\gamma) := \gamma(g) \end{aligned}$$

Initially, this theorem was proved by L. Pontryagin in 1934 in the cases when G is second countable, compact or discrete. Later, E. van Kampen (1935) and A. Weil (1940) proved the general case independently. A proof can be seen in [12] chapter 6.

3 Topologies compatible with the duality $(G, G^\wedge)$

In this section we introduce the Bohr topology τ^+ of a topological group (G, τ). It is the coarsest group topology on G which gives arise to the same dual group as (G, τ).
Next, we will focus on a family of topologies that plays an important role in duality theory, the family of the locally quasi-convex topologies that are compatible with the duality $(G, G^\wedge)$.

Definition 3.1. *Given a topological group (G, τ), a group topology ν on G is said to be a **compatible topology with the duality** $(G, G^\wedge)$ if the dual groups endowed by τ and ν coincide, in other words, if the equality $(G, \tau)^\wedge = (G, \nu)^\wedge$ is satisfied.*

Definition 3.2. *Given a topological group (G, τ), the **Bohr topology of** G is defined as the weak topology associated to the family $(G, \tau) = \mathrm{CHom}(G, \mathbb{T})$. In other words, the topology generated by the set*

$$\Sigma := \{\gamma^{-1}(U) : U \subset \mathbb{T} \text{ open}, \gamma \in G^\wedge\}.$$

This topology is usually denoted by τ^+ (clearly, it depends on the initial topology τ).

Let's see an example.

Example 3.3. Consider the group of integer numbers with the discrete topology $(\mathbb{Z}, \delta)$. Then, the Bohr topology δ^+ of $\mathbb{Z}$, is the topology where the subbasis of neighbourhoods of 0 is given by

$$\Sigma_0 = \{W_\alpha : \alpha \in \mathbb{T}\}, \qquad W_\alpha := \{m \in \mathbb{Z} : \alpha^m \in \mathbb{T}_+\}, \quad \text{where } \mathbb{T}_+ := \{x \in \mathbb{T} : \mathrm{Re}(x) \geq 0\}.$$

Some examples of open sets in $(\mathbb{Z}, \delta^+)$ are:
$W_1 = \{m \in \mathbb{Z} : 1^m \in \mathbb{T}_+\} = \mathbb{Z}$.
$W_{-1} = \{m \in \mathbb{Z} : (-1)^m \in \mathbb{T}_+\} = 2\mathbb{Z}$.
$W_\omega = \{m \in \mathbb{Z} : \omega^m \in \mathbb{T}_+\} = 3\mathbb{Z}, \quad \omega := e^{\frac{2}{3}\pi i}$.

Now we present an important result that states that the Bohr topology of a topological group is a group topology if the group is MAP.

Definition 3.4. *A topological group G is said to be a **MAP group** if its dual separates points of G. In other words, if for every $x, y \in G, x \neq y$ $\exists \gamma \in G^\wedge$ s.t. $\gamma(x) \neq \gamma(y)$.*

Proposition 3.5. *Let (G, τ) be a MAP topological group. Then, the Bohr topology of G is a group topology for G. In other words, (G, τ^+) is a topological group.*

Note 3.6. *Using the Peter-Weyl Theorem, it can be proved that an abelian locally compact group is a MAP group [12, Theorem 21]. Thus, from the previous proposition the Bohr topology τ of an abelian locally compact topological group (G, τ) is a group topology for G.*

Now, we recall another important property of the Bohr topology. Namely, the Bohr topology is a group topology compatible with the duality $(G, G^\wedge)$.

Proposition 3.7. *[5, Theorem 3.7] Let (G, τ) be a topological group and τ^+ its Bohr topology. Then, $(G, \tau)^\wedge = (G, \tau^+)^\wedge$.*

Remark 3.8. *Given a topological group G, from the previous proposition we have that the Bohr topology of G belongs to the family of the group topologies of G compatible with the duality $(G, G^\wedge)$. Moreover, we have that it is the coarsest topology of this family. This is straightforward from the definition, but it is an important property. For this reason, we consolidate it as a corollary.*

Corollary 3.9. *The Bohr topology τ^+ of a topological group (G, τ) is the coarsest topology of the family of the group topologies of G compatible with the duality $(G, G^\wedge)$.*

In the second section, we described what we mean by duality in the class of abelian topological groups. This topic is well known for the class of topological vector spaces. Starting with Banach, along the second half of the last century there was an outburst of important results in this field. In particular, there is one remarkable subfamily in the class of topological vector spaces; the subclass of the locally convex vector spaces, (we recall that every Banach space is a locally convex topological vector space. In general, every Fréchet space is a locally convex topological vector space).

The definition by Vilenkin [15] of *locally quasi-convex sets* and *locally quasi-convex groups* and the work of Banaszczyk in his monograph [2] were the main tool to generalize many of those strong Theorems of Functional Analysis to the broader class of abelian topological groups. The reader may be warned that the passage of results of topological vector spaces to abelian topological groups is not an elementary process. Clearly, convexity makes no sense in the realm of groups.
In [5] it is described an historical route for the notion of locally quasi-convex groups. In particular, an aproximation of the Mackey-Arens Theorem in the wider class of topological abelian groups is proved.

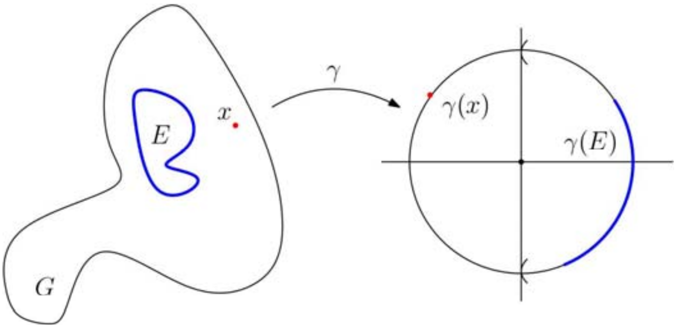
We recall the statement of the Mackey-Arens Theorem.

Theorem 3.10. ***(Mackey-Arens Theorem, 1946)***
For a topological vector space E, there exists the maximum with respect to the inclusion of the family of the locally convex topologies that are compatible with the duality (E, E^*). This topology is called the Mackey topology of E. In particular, every metrizable locally convex vector space carries the Mackey topology (e.g. Banach spaces).

Thus, for a topological vector space V there are two important topologies that belong to the family of the locally convex topologies of V compatible with the duality (V, V^*). The minimum of the family, which is the weak topology $\sigma(V, V^*)$, and the maximum of the family, which exists thanks to the previous theorem.

The quasi-convexity notion is inspired by the Hahn-Banach Theorem, and is not defined on pure algebraic terms. The topology plays a crucial role. Concretely:

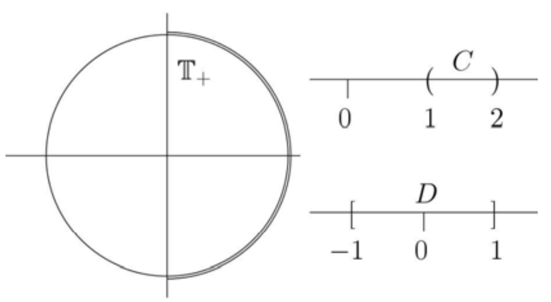
Definition 3.11. *Given a topological abelian group G, a subset $E \subset G$ is said to be* ***a quasi-convex subset of*** *G if for every $x \in G, x \notin E$ there exists $\gamma \in G^\wedge$ such that $\gamma(E) \subset \mathbb{T}_+$ and $\gamma(x) \notin \mathbb{T}_+ = \{x \in \mathbb{T} : \mathrm{Re}(x) \geq 0\}$.*



Example 3.12. *$\mathbb{T}_+$ is a quasi-convex subset of $\mathbb{T}$.*

Example 3.13. *$D := [-1, 1]$ is a quasi-convex subset of $\mathbb{R}$.*

Example 3.14. *$C := (1, 2)$ is not a quasi-convex subset of $\mathbb{R}$.*



Example 3.12 is trivial and examples 3.13 and 3.14 can be proved using the isomorphism of the example 2.4. Furthermore, it can be proved that a quasi-convex subset has to be closed and symmetric [2, Proposition 2.4].

Definition 3.15. *A group topology for a group G (or simply G itself), is said to be* ***locally quasi-convex*** *if there exists a neighbourhood basis of the neutral element formed by quasi-convex subsets.*

The next proposition shows that the locally quasi-convexity in fact extends the concept of locally convexity to the class of abelian topological groups.

Proposition 3.16. *[2, Proposition 2.4] A Hausdorff topological vector space is locally convex if and only if considered as an abelian topological group is locally quasi-convex.*

We observe that the Bohr topology is a locally quasi-convex topology. This is easy to check since the family $\{\gamma^{-1}(\mathbb{T}_+) : \gamma \in G^\wedge\}$ is a neighbourhood subbasis of e_G formed by quasi-convex subsets, and the intersection of quasi-convex subsets is again quasi-convex.

Proposition 3.17. *The Bohr topology τ^+ of a topological group (G,τ) is a locally quasi-convex topology.*

In other words, for a topological group G, if we denote by $\mathcal{LQC}(G)$ the family of locally quasi-convex group topologies that are compatible with the duality $(G, G^\wedge)$, we have, by Corollary 3.9 and Proposition 3.17, that the Bohr topology of G is the minimum of $\mathcal{LQC}(G)$. There exists the Mackey topology (the finest topology), for G? Is it unique? The answer to these questions, that corresponds to an aproximation of the Mackey-Arens Theorem to the class of abelian topological groups, was given in 1999 by M.J Chasco, E. Martín-Peinador and V. Tarieladze.

Theorem 3.18. ***(Chasco, Martín-Peinador, Tarieladze, 1999)***
Let (G,τ) be a topological group. Then, if G is locally compact, or locally quasi-convex metrizable and complete, or locally quasi-convex Baire separable, then τ is the finest topology of the family $\mathcal{LQC}(G)$. In other words, τ is the Mackey topology for (G,τ).

4 Normality through duality

In this section we present the main result; namely, for the topological group $G := \mathbb{Z}^{\mathbb{R}}$ there do not exist normal topologies in the family $\mathcal{LQC}(G)$.
To show this property we need a little bit of theory. We begin recalling the Jones' Lemma and the Hewitt-Marczewski-Pondiczery Theorem and the definition of the extent of a topological space.

Theorem 4.1. ***(Jones' Lemma, 1937)***
Let X be a topological space and $S, D \subset X$ such that S is discrete and closed and D is dense in X. Then, if S and D satisfy $|S| \geq 2^{|D|}$, X is non normal.

Theorem 4.2. ***(Hewitt-Marczewski-Pondiczery Theorem, 1944-1947)***
Let $(X_j, \tau_j)_{j\in J}$ be a family of separable topological spaces such that $|J| \leq \mathfrak{c} := 2^{\aleph_0}$. Then, the product $\prod_{j\in J} X_j$ is also separable.

Definition 4.3. *Given a topological space X, the **extent of** X is defined as*

$$e(X) := \sup\{|Y| : Y \subset X, Y \text{ closed and discrete}\}.$$

The extent of the group $\mathbb{Z}^{\mathbb{R}}$

Our next aim is to show that the group $\mathbb{Z}^{\mathbb{R}}$ has a closed discrete subset of cardinality $\mathfrak{c}$. This proves in particular that $\mathbb{Z}^{\mathbb{R}}$ is non normal due to the Jones' Lemma and the Hewitt-Marczewski-Pondiczery Theorem (since $\mathbb{Z}$ is separable).
A. H. Stone already proved in 1948 [14, Theorem 3], that $\mathbb{Z}^{\mathbb{R}}$ is a non-normal group, but here we will show $e(\mathbb{Z}^{\mathbb{R}}) = \mathfrak{c}$; which is slightly a stronger property. This was pointed out by Engelking, and since is essential for our further considerations, we give below its complete proof, following the hints of [7, Exercise 3.1.H]. We need to introduce some notation which we do next.
Let $I := [0, 1]$. For every $t \in I$, define f_t by:

$$\begin{aligned} f_t : I &\longrightarrow \mathbb{N} \subset \mathbb{Z} \\ s &\longmapsto f_t(s) := \begin{cases} 0, & \text{if } t = s. \\ \lfloor \frac{1}{|s-t|} \rfloor, & \text{if } t \neq s. \end{cases} \end{aligned}$$

Where $\lfloor x \rfloor$ stands for the integer part of $x \in \mathbb{R}$. In other words, $m := \lfloor \frac{1}{|s-t|} \rfloor$ is the unique natural number which satisfies

$$\frac{1}{m+1} < |s-t| \leq \frac{1}{m}.$$

Now, define ω by:

$$\begin{aligned} \omega : I &\longrightarrow \mathbb{Z}^I \\ t &\longmapsto \omega(t) := f_t \end{aligned}$$

In the sequel we will use repeatedly the following elementary property:

Fact 4.4. *For every $r \in I$, f_r has only* **one** *coordinate equal to* 0*, precisely $f_r(r)$. The other coordinates are positive.*

Proposition 4.5. *The set $\omega(I)$ is a closed discrete subspace of $\mathbb{Z}^I$ such that $|\omega(I)| = \mathfrak{c}$.*

Proof. First we prove that ω is 1-1, and therefore $|\omega(I)| = \mathfrak{c}$. Indeed, if $r, s \in I$ with $r \neq s$, then $f_r(r) \neq f_s(r)$. Thus $\omega(r) \neq \omega(s)$.
Let us show that $\omega(I)$ is discrete in the product $\mathbb{Z}^I$. Denote by π_r the r-projection from $\mathbb{Z}^I$ to $\mathbb{Z}$ for every $r \in I$.
For $f_t \in \omega(I)$, consider $\pi_t^{-1}(\{0\})$. It is a neighbourhood of f_t and by Fact 4.4, $\pi_t^{-1}(\{0\}) \cap \omega(I) = \{f_t\}$.
In order to prove that $\omega(I)$ is closed in $\mathbb{Z}^I$, fix a point $y := (y_t)_{t \in I} \in \mathbb{Z}^I \setminus \omega(I)$ and let us find V, a neighbourhood of y such that $V \cap \omega(I) = \emptyset$. We distinguish two possibilities for y:

(a) There is at least one element $s \in I$ such that $y_s = 0$.
(b) $y_t \neq 0$ for all $t \in I$.

For the case (a), $\pi_s^{-1}(\{0\})$ is a neighbourhood of y and contains exactly one element of $\omega(I)$, namely f_s. Since $y \neq \omega(s)$, at least for one coordinate $r \in I \setminus \{s\}$, $y_r \neq f_s(r) = \lfloor \frac{1}{|r-s|} \rfloor$.

Clearly $V := \pi_s^{-1}(\{0\}) \cap \pi_r^{-1}(\{y_r\})$ is a neighbourhood of y such that $V \cap \omega(I) = \emptyset$.

Assume now (b). For every $t \in I$, let $V_t := \pi_t^{-1}(\{y_t\})$. If $y_t < 0$ for some t, then V_t is a neighbourhood of y totally contained in $\mathbb{Z}^I \setminus \omega(I)$. So, we restrict to the case when $y_t > 0$ for all $t \in I$. Consider the set

$$\omega(I) \cap V_t = \{f_r \in \omega(I) : f_r(t) = y_t\} = \{f_r : r \in [t - \frac{1}{y_t}, t - \frac{1}{y_t+1}) \cup (t + \frac{1}{y_t+1}, t + \frac{1}{y_t}]\}$$

Write J_t as the closure, with respect to the euclidean topology induced on I, of the intervals appearing in $\omega(I) \cap V_t$, that is

$$J_t := \Big[t - \frac{1}{y_t}, t - \frac{1}{y_t+1}\Big] \cup \Big[t + \frac{1}{y_t+1}, t + \frac{1}{y_t}\Big].$$

Since $(J_t)_{t \in I}$ is a collection of closed sets with empty intersection (observe that $t \notin J_t$ for all $t \in I$), contained in a closed bounded interval, say $J \subset \mathbb{R}$; by compactness there must exist $t_1, \ldots, t_k \in I$ such that $J_{t_1} \cap \cdots \cap J_{t_k} = \emptyset$.
Let $V := V_{t_1} \cap \cdots \cap V_{t_k}$; we have that V is a neighbourhood of y in $\mathbb{Z}^I$ such that $V \cap \omega(I) = \emptyset$. Indeed, if $f_r \in V \cap w(I)$, then $r \in J_{t_1} \cap \cdots \cap J_{t_k}$, which is the empty set.
Thus, the neighbourhood V obtained in both cases (a) and (b) permits to claim that $\omega(I)$ is closed. □

As a consequence we have the following:

Corollary 4.6. *The group $\mathbb{Z}^{\mathbb{R}}$ has extent $\mathfrak{c}$.*

Proof. From Proposition 4.5 we have that $e(\mathbb{Z}^{\mathbb{R}}) \geq \mathfrak{c}$. However, since $\mathbb{Z}^{\mathbb{R}}$ has a basis of cardinality $\mathfrak{c}$, its extent cannot be greater than $\mathfrak{c}$. Thus, $e(\mathbb{Z}^{\mathbb{R}}) = \mathfrak{c}$. □

The non normality of the group $\mathbb{Z}_b^{\mathbb{R}}$

Using the previous proposition we can show that the group $\mathbb{Z}_b^{\mathbb{R}}$ has also extent $\mathfrak{c}$. Here $\mathbb{Z}_b^{\mathbb{R}}$ denotes the product space of $\mathbb{R}$-copies of the group of integer numbers endowed with the Bohr topology, i.e., $\mathbb{Z}_b := (\mathbb{Z}, \delta^+)$.
In the same article of Stone that we mentioned before [14], it is proved that if a product of nonempty T_1 spaces is normal, then all but at most a countable number of the factor spaces must be countably compact.
The topological group $\mathbb{Z}_b$ is not a countably compact space. Otherwise, since it is countable, it should be compact. But by the Baire category Theorem there do not exist countable Hausdorff compact groups.
Thus, the above mentioned result yields that $\mathbb{Z}_b^{\mathbb{R}}$ is a non-normal space.
However, we want to have the stronger result that $\mathbb{Z}_b^{\mathbb{R}}$ has extent $\mathfrak{c}$. For this purpose, we recall a fundamental lemma of non countably compact spaces.

Lemma 4.7. *Let X be a T_1 topological space. Then, if X is non countably compact, then it has a countably infinite subset $H \subset X$ that is discrete and closed.*

Proposition 4.8. *The topological group $\mathbb{Z}_b^{\mathbb{R}}$ has extent $\mathfrak{c}$.*

Proof. Since the factor spaces $\mathbb{Z}_b$ are not countably compact, by the previous lemma they must contain an infinite closed discrete subset H, which is obviously homeomorphic to $\mathbb{Z}$. Now $H^{\mathbb{R}} \subset \mathbb{Z}^{\mathbb{R}}$ is closed in $\mathbb{Z}_b^{\mathbb{R}}$. Arguing as in Proposition 4.5, $H^{\mathbb{R}}$ contains a closed discrete subset, say L, with cardinality $\mathfrak{c}$. In particular, L is also closed and discrete in the bigger space $\mathbb{Z}_b^{\mathbb{R}}$.
On the other hand $\mathbb{Z}_b^{\mathbb{R}}$ cannot contain a closed discrete subset of cardinality greater than $\mathfrak{c}$, for in that case it should also be closed and discrete in $\mathbb{Z}^{\mathbb{R}}$, which contradicts Corollary 4.6. □

Corollary 4.9. *The space $\mathbb{Z}_b^{\mathbb{R}}$ is not normal.*

Proof. Since $\mathbb{Z}_b$ is countable, it is separable, and by the Hewitt-Marczewski-Pondiczery Theorem, $\mathbb{Z}_b^{\mathbb{R}}$ is also separable. Denote by D a countable dense subspace of $\mathbb{Z}_b^{\mathbb{R}}$.
For the closed discrete subset L obtained above, we have $|L| = 2^{\aleph_0} = 2^{|D|}$. The non-normality of $\mathbb{Z}_b^{\mathbb{R}}$ follows now from Jones' Lemma. □

A direct proof of Proposition 4.8 following the argument of Proposition 4.5 should be interesting. However, the Bohr topology on $\mathbb{Z}$ is not so easy to handle, as we saw in example 3.3.

The duality $(\mathbb{Z}^{\mathbb{R}}, \mathbb{T}^{(\mathbb{R})})$

Next, we analyze the bottom and the top elements of the family $\mathcal{LQC}(G)$, where $G := \mathbb{Z}^{\mathbb{R}}$ is the product of $\mathfrak{c}$ copies of the discrete group of integer numbers.
The dual group of G is the direct sum of the dual groups of the factors, that is $G^\wedge = \mathbb{T}^{(\mathbb{R})}$. This is true for a general product of topological spaces, see for instance [10].
In the next proposition, we use a strong theorem of W.W. Comfort and L.A. Ross about precompact groups. For that reason, we remember the definition and state the result to make the lecture more comfortable to the reader.

Definition 4.10. *A topological group G is said to be* ***precompact*** *if for each open neighbourhood of the neutral element V there exists a finite subset $A \subset G$ such that $AV = G$.*

Theorem 4.11. *[6, 1.2] Let (G, τ) be a topological group. Then (G, τ) is precompact if and only if $\tau = \tau^+$. In other words, G is precompact if and only if its topology coincides with its Bohr topology.*

We can conclude the following:

Proposition 4.12. *The Bohr topology on $\mathbb{Z}^{\mathbb{R}}$ is precisely the product topology in $\mathbb{Z}_b^{\mathbb{R}}$.*

This follows as a corollary of the more general fact expressed as follows:

Proposition 4.13. *Let $\{(G_i, \tau_i), i \in J\}$ be a family of abelian topological groups, and let $G_\pi := \prod_{i \in J} G_i$ where π stands for the product topology. Then, the Bohr topology of G_π coincides with the product topology of the corresponding Bohr topologies $(\tau_i)^+$ on the factor spaces G_i.*

Proof. As said above, the dual of a product is the direct sum of the dual groups of the factor spaces, and $(G_i, \tau_i)^\wedge = (G_i, (\tau_i)^+)^\wedge =: G_i^\wedge$. Therefore:

$$(\prod_{i\in J}(G_i, \tau_i))^\wedge = (\prod_{i\in J}(G_i, (\tau_i)^+))^\wedge = \bigoplus_{i\in J} G_i^\wedge$$

On the other hand $\prod_{i\in J}(G_i, (\tau_i)^+)$ is precompact, since the product of precompact groups is precompact (see [1, Corollary 3.7.14]). Since in $\mathcal{LQC}(G_\pi)$ there is only one precompact compatible topology, we obtain that $(G, \pi^+) = \prod_{i\in J}(G_i, (\tau_i)^+)$. □

We study now the top element of the family $\mathcal{LQC}(G)$.

Proposition 4.14. *The topological group $G = \mathbb{Z}^{\mathbb{R}}$ is a separable Baire space. Therefore it carries the Mackey topology.*

Proof. It was proved by J. C. Oxtoby [13, Theorem 4], that the product of a family of Baire spaces, each of which has a countable pseudo-base (in particular a countable base), is a Baire space. Thus, G is a Baire space. On the other hand, we observe that G is a locally quasi-convex space, since $\mathbb{Z}$ is locally quasi-convex and the product of locally quasi-convex spaces is again quasi-convex.
Hence, by Theorem 3.18, G carries the Mackey topology. Thus, if we call π the product topology in $\mathbb{Z}^{\mathbb{R}}$, we have that π is precisely the Mackey topology for G. □

The previous proposition allows us to claim that the family $\mathcal{LQC}(G)$ has a top element, precisely π. On the other hand, the bottom element π^+ is, by Proposition 4.12, the product topology on $\mathbb{Z}_b^{\mathbb{R}}$. Using these facts, we can deduce the following:

Proposition 4.15. *The topologies in the family $\mathcal{LQC}(G)$ have extent $\mathfrak{c}$.*

Proof. Let ν be a compatible locally quasi-convex topology on $\mathbb{Z}^{\mathbb{R}}$. Then it must hold $\pi^+ \leq \nu \leq \pi$. The closed discrete subset of cardinality $\mathfrak{c}$ found in Proposition 4.8 for the topology π^+ is also closed and discrete for the stronger topology ν. On the other hand, a closed and discrete subset for the topology ν would also be closed and discrete in π. Taking into account Corollary 4.6, there cannot be a closed discrete subset of cardinality greater than $\mathfrak{c}$. Thus, $e(G, \nu) = \mathfrak{c}$. □

Theorem 4.16. *The family $\mathcal{LQC}(G)$ of all locally quasi-convex topologies compatible with the duality $(\mathbb{Z}^{\mathbb{R}}, \mathbb{T}^{(\mathbb{R})})$* **does not** *contain a normal topology. Consequently, it does not contain either a metrizable, or a locally compact group topology.*

Proof. Let ν be a compatible locally quasi-convex topology on $\mathbb{Z}^{\mathbb{R}}$. From $\pi^+ \leq \nu \leq \pi$, taking into account that π is separable by the Hewit-Marczewski-Pondiczery Theorem, the weaker topology ν is also separable. By Proposition 4.8 and Jones' Lemma we obtain that $(\mathbb{Z}^{\mathbb{R}}, \nu)$ is a non normal topological group.
The last statement derives from the facts that any locally compact group topology is normal, and obviously any metrizable topology is normal. □

The previous theorem is also true in a more general setting, namely, for the $\mathbb{R}$-product of a family of topological groups $(G_j, \tau_j)_{j\in J}$ that are locally compact, non countably compact and separable. Concretely, we have the following result.

Theorem 4.17. *[11, Theorem 6.9] Let $(G_j, \tau_j)_{j\in J}$ be a family of locally compact, non countably compact, separable, abelian topological groups, and set $G := \prod_{j\in J}(G_j, \tau_j)$. Then, if $|J| = \mathfrak{c}$, there are **no** normal locally quasi-convex group topologies compatible with the duality $(G, G^\wedge)$.*

We don't include the proof here, but it can be read in [11, section 6]. This theorem cannot be deduced immediately from the results we introduced above and we need strong theorems as the Glicksberg Theorem and some results about the product of Čech-complete spaces. However, the philosophy is the same as in the case $G = \mathbb{Z}^{\mathbb{R}}$.

References

[1] A. V. ARHANGEL'SKII, M.G. TKACHENKO. *Topological Groups and Related Structures*. Atlantis Press/World Scientific, Amsterdam/Paris (2008).

[2] W. BANASZCZYK. *Additive Subgroups of Topological Vector Spaces*. Lecture Notes in Math. 1466, Springer, Berlin, (1991).

[3] G. BIRKHOFF. *A note on topological groups*. Compositio Mathematica 3 (1936), 427-430.

[4] N. BOURBAKI. *Elements of Mathematics. General Topology Vol. I.* Addison-Wesley Educational Publishers Inc. (1967).

[5] M. J. CHASCO, E. MARTÍN-PEINADOR, V. TARIELADZE. *On Mackey topology for groups*. Studia Mathematica 132 (3) (1999) 257-284.

[6] W. W. COMFORT, K. A. ROSS. *Topologies induced by groups of characters*. Fund. Math. 55, 283-291 (1964).

[7] R. ENGELKING. *General Topology*. PWN, Warsaw (1977).

[8] E. HEWITT, K. ROSS. *Abstract Harmonic Analysis I, Structure of Topological Groups. Integration Theory. Group Representations*. [*Grundlehren der mathematischen Wissenschaften*]. Springer-Verlag (1963).

[9] F. B. JONES. *Concerning normal and completely normal spaces*. Bull. Amer. Math. Soc. 43 (1937), 671-677.

[10] S. KAPLAN. *Extensions of the Pontrjagin duality I: infinite products*. Duke Math. J. 15 (1948), no. 3, 649–658.

[11] E. MARTÍN-PEINADOR, V. PÉREZ VALDÉS. *A class of topological groups which do not admit normal compatible locally quasi-convex topologies*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, Vol. 112, no. 3, (2018) pp 867–876.

[12] S. A. MORRIS. *Pontryagin Duality and the structure of locally compact abelian groups*. Cambridge University Press (1977).

[13] J. C. OXTOBY. *Cartesian products of Baire spaces*. Fund. Math. 49, 157-166 (1960/1961).

[14] A. H. STONE. *Paracompact and product spaces*. Bull. Am. Soc. 54, 977-982 (1948).

[15] N. Y. VILENKIN. *The theory of characters of topological Abelian groups with a given boundedness*. Izv. Akad. Nauk SSSR Ser. Mat. 15 (1951), 439-462 (in Russian).

(Víctor Pérez Valdés)
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro, Tokyo, 153-8914
JAPAN
E-mail address: perez@ms.u-tokyo.ac.jp