

# Weighted Bergman inner products on subspaces of bounded symmetric domains

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## Abstract

Let  $H_r(\mathbb{F})$  ( $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ ) be the space of Hermitian matrices, and we consider the bounded symmetric domain  $D \subset H_r(\mathbb{F})^{\mathbb{C}}$ . In this article we present a result on the computation of the weighted Bergman inner product on  $D$  of a polynomial on the subspace  $H_{r'}(\mathbb{F}) \oplus H_{r''}(\mathbb{F})$  and an exponential function on  $H_r(\mathbb{F})$ . Also, as an application, we present a result on explicit construction of intertwining operators from representations of  $Sp(r, \mathbb{R})$  to those of the subgroup  $U(r', r'')$ .

## 1 Introduction

Throughout the paper let  $\mathbb{F} := \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , and write

$$\begin{aligned} H_r &= H_r(\mathbb{F}) := \{X \in M_r(\mathbb{F}) \mid X = X^*\}, \\ H_r^+ &= H_r^+(\mathbb{F}) := \{X \in H_r(\mathbb{F}) \mid \text{positive definite}\}, \\ H_r^{\mathbb{C}} &= H_r^{\mathbb{C}}(\mathbb{F}) := H_r(\mathbb{F}) \otimes_{\mathbb{R}} \mathbb{C}, \quad M_{p,q}^{\mathbb{C}} = M_{p,q}^{\mathbb{C}}(\mathbb{F}) := M_{p,q}(\mathbb{F}) \otimes_{\mathbb{R}} \mathbb{C}. \end{aligned}$$

For  $X \in H_r^{\mathbb{C}}(\mathbb{F})$ , let  $\overline{X}$  be the  $\mathbb{C}$ -conjugate of  $X$  with respect to the real form  $H_r(\mathbb{F})$ , and for  $X \in M_{p,q}(\mathbb{F})$ , let  $X^*$  be the  $\mathbb{F}$ -conjugate transpose of  $X$ . We extend  $X \mapsto X^*$   $\mathbb{C}$ -linearly to the map  $M_{p,q}^{\mathbb{C}}(\mathbb{F}) \rightarrow M_{q,p}^{\mathbb{C}}(\mathbb{F})$ . Also we write  $d = \dim_{\mathbb{R}} \mathbb{F}$  so that  $d \in \{1, 2, 4\}$ , and write  $n = \dim_{\mathbb{R}} H_r(\mathbb{F}) = r + \frac{d}{2}r(r-1)$ .

Next let  $D \subset H_r^{\mathbb{C}}$  be the *bounded symmetric domain* given by

$$D := \{X \in H_r^{\mathbb{C}} \mid I - X\overline{X} \text{ is positive definite}\}.$$

Then for  $f, g \in \mathcal{O}(D)$ , the *weighted Bergman inner product* is given by

$$\langle f, g \rangle_{\lambda} := C_{\lambda} \int_D f(X) \overline{g(X)} \det(I - X\overline{X})^{\lambda - \frac{2n}{r}} dX,$$

where we set

$$C_{\lambda} := \left( \int_D \det(I - X\overline{X})^{\lambda - \frac{2n}{r}} dX \right)^{-1} = \frac{1}{\pi^n} \prod_{j=1}^r \frac{\Gamma\left(\lambda - \frac{d}{2}(j-1)\right)}{\Gamma\left(\lambda - \frac{n}{r} - \frac{d}{2}(j-1)\right)}.$$

Here we omit the definition of  $\det(I - X\overline{X})$  for  $\mathbb{F} = \mathbb{H}$  case. Then this converges for all polynomials if  $\lambda > \frac{2n}{r} - 1$ . Let  $\mathcal{H}_{\lambda}(D) \subset \mathcal{O}(D)$  be the corresponding Hilbert space. Then

the universal covering group  $\tilde{G}$  of  $G = \begin{cases} Sp(r, \mathbb{R}) & (\mathbb{F} = \mathbb{R}) \\ U(r, r) & (\mathbb{F} = \mathbb{C}) \\ SO^*(4r) & (\mathbb{F} = \mathbb{H}) \end{cases}$  acts unitarily on  $\mathcal{H}_\lambda(D)$ . For example, when  $\mathbb{F} = \mathbb{C}$ ,  $G = U(r, r)$ , the action is given by, for  $\lambda_1 + \lambda_2 = \lambda$ ,

$$\begin{aligned} \tau_{\lambda_1, \lambda_2} \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \right) f(X) \\ := \det(A^* + XB^*)^{-\lambda_1} \det(CX + D)^{-\lambda_2} f((AX + B)(CX + D)^{-1}). \end{aligned}$$

We note that  $\det(A^* + XB^*)^{-\lambda_1} \det(CX + D)^{-\lambda_2}$  is not well-defined on  $G \times D$  if  $\lambda_1, \lambda_2 \notin \mathbb{Z}$ , but is well-defined on the universal covering space  $\tilde{G} \times D$ .

Next we consider the decomposition of  $\mathcal{P}(H_r^{\mathbb{C}}) := \{\text{holomorphic polynomials on } H_r^{\mathbb{C}}\}$ . First we recall the determinant  $\det(X)$  on  $H_r^{\mathbb{C}}(\mathbb{F})$ . This is a polynomial of degree  $r$  satisfying

$$\det(gXg^*) = \det(gg^*) \det(X) \quad (g \in GL(r, \mathbb{F})^{\mathbb{C}}, X \in H_r^{\mathbb{C}}(\mathbb{F})).$$

When  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ , this is the usual determinant, and even for  $\mathbb{F} = \mathbb{H}$  case such polynomial also exists. Then for  $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$  with  $m_1 \geq \dots \geq m_r \geq 0$ , we define  $\Delta_{\mathbf{m}}(X) \in \mathcal{P}(H_r^{\mathbb{C}})$  by

$$\Delta_{\mathbf{m}}(X) := \det(X)^{m_r} \prod_{k=1}^{r-1} \det((x_{ij})_{1 \leq i, j \leq k})^{m_k - m_{k+1}},$$

and let

$$\mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}}) := \text{span} \left\{ \Delta_{\mathbf{m}}(gXg^*) \mid g \in GL(r, \mathbb{F})^{\mathbb{C}} \right\} \subset \mathcal{P}(H_r^{\mathbb{C}}).$$

Then we have the following.

**Theorem 1.1** (Hua-Kostant-Schmid, [3, Theorem XI.2.4]).

$$\mathcal{P}(H_r^{\mathbb{C}}) = \bigoplus_{m_1 \geq \dots \geq m_r \geq 0} \mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}}).$$

We define another inner product (the *Fischer inner product*) on  $\mathcal{P}(H_r^{\mathbb{C}})$  by

$$\langle f, g \rangle_F := \overline{g} \left( \frac{\partial}{\partial X} \right) f(X) \Big|_{X=0} = \frac{1}{\pi^n} \int_{H_r^{\mathbb{C}}} f(X) \overline{g(X)} e^{-\text{tr}(X\overline{X})} dX$$

(see [3, Section XI.1]), and for  $\lambda \in \mathbb{C}$ ,  $\mathbf{m} = (m_1, \dots, m_r) \in (\mathbb{Z}_{\geq 0})^r$ , we write

$$(\lambda)_{\mathbf{m}} = \prod_{j=1}^r \left( \lambda - \frac{d}{2}(j-1) \right)_{m_j}, \quad (\lambda)_m = \lambda(\lambda+1) \cdots (\lambda+m-1).$$

Then we have the following.

**Theorem 1.2** (Faraut-Korányi (1990), [3, Corollary XIII.2.3]). *For  $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$ ,  $g = \sum_{\mathbf{m}} g_{\mathbf{m}} \in \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}}) = \mathcal{P}(H_r^{\mathbb{C}})$  we have*

$$\langle f, g \rangle_\lambda = \sum_{m_1 \geq \dots \geq m_r \geq 0} \frac{1}{(\lambda)_{\mathbf{m}}} \langle f_{\mathbf{m}}, g_{\mathbf{m}} \rangle_F.$$

Next we recall the reproducing kernel of  $\langle \cdot, \cdot \rangle_F$ . It is given by  $e^{\text{tr}(X\bar{A})}$ , that is,

$$\langle f(X), e^{\text{tr}(X\bar{A})} \rangle_F = f(A) \quad (f \in \mathcal{P}(H_r^{\mathbb{C}}))$$

holds. Therefore, for  $f, g \in \mathcal{P}(H_r^{\mathbb{C}})$  we have

$$\langle f, g \rangle_\lambda = \left\langle f(X), \left\langle g(A), e^{\text{tr}(A\bar{X})} \right\rangle_{F,A} \right\rangle_{\lambda, X} = \left\langle \left\langle f(X), e^{\text{tr}(X\bar{A})} \right\rangle_{\lambda, X}, g(A) \right\rangle_{F,A}.$$

That is,  $f \mapsto \langle f, e^{\text{tr}(\bar{A})} \rangle_\lambda$  does not lose any information. Moreover, by the result of Faraud-Korányi, we get the following.

**Corollary 1.3.** For  $f = \sum_{\mathbf{m}} f_{\mathbf{m}} \in \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}}) = \mathcal{P}(H_r^{\mathbb{C}})$  we have

$$\langle f, e^{\text{tr}(\bar{A})} \rangle_\lambda = \sum_{m_1 \geq \dots \geq m_r \geq 0} \frac{1}{(\lambda)_{\mathbf{m}}} f_{\mathbf{m}}(A).$$

Now let  $r = r' + r''$ , and define  $\text{Proj}_1 : H_r^{\mathbb{C}} \rightarrow H_{r'}^{\mathbb{C}}, \text{Proj}_2 : H_r^{\mathbb{C}} \rightarrow H_{r''}^{\mathbb{C}}$  by

$$\text{Proj}_1 \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} := x, \quad \text{Proj}_2 \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} := z.$$

Then again by the Hua-Kostant-Schmid theorem we have

$$\mathcal{P}(H_{r'}^{\mathbb{C}}) = \bigoplus_{k_1 \geq \dots \geq k_{r'} \geq 0} \mathcal{P}_{\mathbf{k}}(H_{r'}^{\mathbb{C}}), \quad \mathcal{P}(H_{r''}^{\mathbb{C}}) = \bigoplus_{l_1 \geq \dots \geq l_{r''} \geq 0} \mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}}).$$

The aim of this paper is to compute

$$\langle f_1(\text{Proj}_1(\cdot)) f_2(\text{Proj}_2(\cdot)), e^{\text{tr}(\bar{A})} \rangle_\lambda \quad (f_1 \in \mathcal{P}_{\mathbf{k}}(H_{r'}^{\mathbb{C}}), f_2 \in \mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})).$$

This is computable if  $\mathbf{k} = (k, \dots, k)$  so that  $\mathcal{P}_{(\mathbf{k}, \dots, \mathbf{k})}(H_{r'}^{\mathbb{C}}) = \mathbb{C} \det(x)^k$ . In the rest of this paper we assume this.

## 2 Main result and applications

We fix some notations. For  $X, A \in H_r^{\mathbb{C}}$ , we write  $X = \begin{pmatrix} x & y \\ y^* & z \end{pmatrix}, A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$  with  $x, a \in H_{r'}^{\mathbb{C}}, y, b \in M_{r', r''}^{\mathbb{C}}, z, c \in H_{r''}^{\mathbb{C}}$ . Next, for  $k \in \mathbb{Z}_{\geq 0}$  we write  $\underline{k}_r := \underbrace{(k, \dots, k)}_r$ . Also, we

identify  $\mathbf{l} = (l_1, \dots, l_{r''}) \in \mathbb{Z}^{r''}$  and  $(l_1, \dots, l_{r''}, 0, \dots, 0) \in \mathbb{Z}^r$ , so that

$$(\lambda)_{\underline{k}_r + \mathbf{l}} = (\lambda)_{(k+l_1, \dots, k+l_{r''}, k, \dots, k)} = \prod_{j=1}^{r''} \left( \lambda - \frac{d}{2}(j-1) \right)_{k+l_j} \prod_{j=r''+1}^r \left( \lambda - \frac{d}{2}(j-1) \right)_k$$

holds. Next, for  $\mathbf{l} \in \mathbb{Z}^{r''}$  with  $l_1 \geq \dots \geq l_{r''} \geq 0$  and for  $f(z) \in \mathcal{P}_{\mathbf{l}}(H_{r''}^{\mathbb{C}})$ , we define  $\tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c) \in \mathcal{P}_{\mathbf{m}}(H_{r'}^{\mathbb{C}}) \otimes \mathcal{P}_{\mathbf{n}}(H_{r''}^{\mathbb{C}})$  by

$$f(z + c) = \sum_{m_1 \geq \dots \geq m_{r'} \geq 0} \sum_{n_1 \geq \dots \geq n_{r''} \geq 0} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c).$$

This is a finite sum since  $\tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c) = 0$  if  $|\mathbf{m}| + |\mathbf{n}| \neq |\mathbf{l}|$ . Then the main result is as follows.

**Theorem 2.1.** Let  $\operatorname{Re} \lambda > \frac{2n}{r} - 1$ ,  $k \in \mathbb{Z}_{\geq 0}$ ,  $\mathbf{l} \in \mathbb{Z}^{r''}$ ,  $l_1 \geq \dots \geq l_{r''} \geq 0$ . Then for  $f \in \mathcal{P}_1(H_{r''}^{\mathbb{C}})$ , we have

$$\begin{aligned} & \left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\bar{A})} \right\rangle_{\lambda, X} \\ &= \frac{\det(a)^k}{(\lambda)_{\underline{k}_r+1}} \det(c - b^* a^{-1} b)^{-\lambda + \frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \det(c - b^* a^{-1} b)^{\lambda + k - \frac{n}{r}} f(c) \end{aligned} \quad (\text{A})$$

$$= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+1}} \det(a)^k \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{\left(\lambda - \frac{d}{2}r'\right)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^* a^{-1} b - c, c) \quad (\text{B})$$

$$= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+1}} \det(a)^k \sum_{\mathbf{n}} \frac{\left(\lambda + k - \frac{d}{2}r'\right)_{\mathbf{n}}}{\left(\lambda - \frac{d}{2}r'\right)_{\mathbf{n}}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^* a^{-1} b, c - b^* a^{-1} b) \quad (\text{C})$$

$$= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+1} \left(\lambda - \frac{d}{2}r'\right)_1} \det(a)^k \sum_{\mathbf{m}, \mathbf{n}} (-k)_{\mathbf{m}} \left(\lambda + k - \frac{d}{2}r'\right)_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^* a^{-1} b, c). \quad (\text{D})$$

The proof is given in Sections 4 and 5. Especially, by (D) we have

$$\begin{aligned} & \left\| \det(x)^k f(z) \right\|_{\lambda, X}^2 = \left\langle \det(x)^k f(z), \det(x)^k f(z) \right\rangle_{\lambda, X} \\ &= \left\langle \left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\bar{A})} \right\rangle_{\lambda, X}, \det(a)^k f(c) \right\rangle_{F, A} \\ &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+1} \left(\lambda - \frac{d}{2}r'\right)_1} \sum_{\mathbf{m}, \mathbf{n}} (-k)_{\mathbf{m}} \left(\lambda + k - \frac{d}{2}r'\right)_{\mathbf{n}} \left\langle \det(a)^k \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^* a^{-1} b, c), \det(a)^k f(c) \right\rangle_{F, A} \\ &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+1} \left(\lambda - \frac{d}{2}r'\right)_1} (-k)_{\underline{0}_{r''}} \left(\lambda + k - \frac{d}{2}r'\right)_1 \left\langle \det(a)^k f(c), \det(a)^k f(c) \right\rangle_{F, A} \\ &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}+1}}{(\lambda)_{\underline{k}_r+1} \left(\lambda - \frac{d}{2}r'\right)_1} \left\| \det(a)^k f(c) \right\|_{F, A}^2. \end{aligned}$$

**Corollary 2.2.** Let  $\operatorname{Re} \lambda > \frac{2n}{r} - 1$ ,  $k \in \mathbb{Z}_{\geq 0}$ ,  $\mathbf{l} \in \mathbb{Z}^{r''}$ ,  $l_1 \geq \dots \geq l_{r''} \geq 0$ . Then for  $f \in \mathcal{P}_1(H_{r''}^{\mathbb{C}})$ , we have

$$\left\| \det(x)^k f(z) \right\|_{\lambda, X}^2 = \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}+1}}{(\lambda)_{\underline{k}_r+1} \left(\lambda - \frac{d}{2}r'\right)_1} \left\| \det(a)^k f(c) \right\|_{F, A}^2.$$

Now we assume  $\mathbf{l} = (l, \dots, l)$ ,  $f(z) = \det(z)^l$ . We define  $\Psi^{\mathbf{m}}(z) \in \mathcal{P}_{\mathbf{m}}(H_{r''}^{\mathbb{C}})$  by

$$\sum_{m_1 \geq \dots \geq m_{r''} \geq 0} \Psi^{\mathbf{m}}(z) = e^{\operatorname{tr}(z)}$$

When  $r'' = 1$  we have  $\Psi^m(z) = \frac{1}{m!} z^m$ . Also, for  $\mathbf{m} = (m_1, \dots, m_{r''})$ , we write  $\mathbf{m}^{\vee} :=$



$(m_{r''}, \dots, m_1)$ . Then by [3, Proposition XII.1.3] we have

$$\begin{aligned} f(z+c) &= \det(z+c)^l = \det(c)^l \det\left(I + \sqrt{c}^{-1}z\sqrt{c}^{-1}\right)^l \\ &= \det(c)^l \sum_{m_1 \geq \dots \geq m_{r''} \geq 0} (-1)^{|\mathbf{m}|} (-l)_{\mathbf{m}} \Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}z\sqrt{c}^{-1}\right) \\ &\in \bigoplus_{m_1 \geq \dots \geq m_{r''} \geq 0} \mathcal{P}_{\mathbf{m}}(H_{r''}^{\mathbb{C}})_z \otimes \mathcal{P}_{\underline{L}_{r''} - \mathbf{m}^\vee}(H_{r''}^{\mathbb{C}})_w, \end{aligned}$$

and therefore  $\tilde{f}_{\mathbf{m}, \underline{L}_{r''} - \mathbf{m}^\vee}(z, c) = (-1)^{|\mathbf{m}|} (-l)_{\mathbf{m}} \det(c)^l \Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}z\sqrt{c}^{-1}\right)$  holds. When  $r'' = 1$  this coincides with the usual binomial expansion

$$(z+c)^l = c^l \sum_{m=0}^l \frac{(-1)^m (-l)_m}{m!} \left(\frac{z}{c}\right)^m = \sum_{m=0}^l \frac{(-1)^m (-l)_m}{m!} z^m c^{l-m}.$$

Therefore by (D) we have

$$\begin{aligned} &\left\langle \det(x)^k \det(z)^l, e^{\text{tr}(X\bar{A})} \right\rangle_{\lambda, X} \\ &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r+1} \left(\lambda - \frac{d}{2}r'\right)_1} \det(a)^k \sum_{\mathbf{m}, \mathbf{n}} (-k)_{\mathbf{m}} \left(\lambda + k - \frac{d}{2}r'\right)_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^*a^{-1}b, c) \\ &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r + \underline{L}_{r''}} \left(\lambda - \frac{d}{2}r'\right)_{\underline{L}_{r''}}} \det(a)^k \sum_{\mathbf{m}} (-k)_{\mathbf{m}} \left(\lambda + k - \frac{d}{2}r'\right)_{\underline{L}_{r''} - \mathbf{m}^\vee} \\ &\quad \times (-1)^{|\mathbf{m}|} (-l)_{\mathbf{m}} \det(c)^l \Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right) \\ &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}_{r''}} \left(\lambda + k - \frac{d}{2}r'\right)_{\underline{L}_{r''}}}{(\lambda)_{\underline{k}_r + \underline{L}_{r''}} \left(\lambda - \frac{d}{2}r'\right)_{\underline{L}_{r''}}} \det(a)^k \det(c)^l \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}} (-l)_{\mathbf{m}}}{\left(-\lambda - k - l + \frac{d}{2}r' + \frac{n''}{r''}\right)_{\mathbf{m}}} \\ &\quad \times \Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right) \\ &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k} + \underline{L}_{r''}} \det(a)^k \det(c)^l}{(\lambda)_{\underline{k}_r + \underline{L}_{r''}} \left(\lambda - \frac{d}{2}r'\right)_{\underline{L}_{r''}}} \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}} (-l)_{\mathbf{m}}}{\left(-\lambda - k - l + \frac{n}{r}\right)_{\mathbf{m}}} \Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right) \\ &= \frac{\left(\lambda + l - \frac{d}{2}r'\right)_{\underline{k}_{r''}}}{(\lambda)_{\underline{k}_r + \underline{L}_{r''}}} \det(a)^k \det(c)^l \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}} (-l)_{\mathbf{m}}}{\left(-\lambda - k - l + \frac{n}{r}\right)_{\mathbf{m}}} \Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right) \\ &= \frac{\left(\lambda + l - \frac{d}{2}r'\right)_{\underline{k}_{r''}} \left(\lambda + k - \frac{d}{2}r'\right)_{\underline{L}_{r''}}}{(\lambda)_{\underline{k} + \underline{L}_{r''}}} \det(a)^k \det(c)^l \\ &\quad \times \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}} (-l)_{\mathbf{m}}}{\left(-\lambda - k - l + \frac{n}{r}\right)_{\mathbf{m}}} \Psi^{\mathbf{m}}\left(\sqrt{c}^{-1}b^*a^{-1}b\sqrt{c}^{-1}\right). \end{aligned}$$

Now we write

$$\sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}} (\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} \Psi^{\mathbf{m}}(z) =: {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z\right).$$

When  $r'' = 1$ , this is the usual hypergeometric function  $\sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m m!} z^m$ . Then the above result is summarized as follows.

**Corollary 2.3.** *Let  $\operatorname{Re} \lambda > \frac{2n}{r} - 1$ ,  $k, l \in \mathbb{Z}_{\geq 0}$ . Then we have*

$$\begin{aligned} \left\langle \det(x)^k \det(z)^l, e^{\operatorname{tr}(X\bar{A})} \right\rangle_{\lambda, X} &= \frac{\left(\lambda + l - \frac{d}{2}r'\right)_{k, r''} \left(\lambda + k - \frac{d}{2}r''\right)_{l, r'}}{(\lambda)_{k+l, r}} \\ &\times \det(a)^k \det(c)^l {}_2F_1 \left( \begin{matrix} -k, -l \\ -\lambda - k - l + \frac{n}{r} \end{matrix}; \sqrt{c}^{-1} b^* a^{-1} b \sqrt{c}^{-1} \right). \end{aligned}$$

**Remark 2.4.** *This  ${}_2F_1$  coincides with a special case of Heckman-Opdam’s hypergeometric functions of type  $BC_{r''}$  (Beerends-Opdam, 1993 [1]).*

This work is motivated by branching laws for the symmetric pairs

$$(G, G_1) := \begin{cases} (Sp(r, \mathbb{R}), U(r', r'')) & (\mathbb{F} = \mathbb{R}), \\ (U(r, r), U(r', r'') \times U(r'', r')) & (\mathbb{F} = \mathbb{C}), \\ (SO^*(4r), U(2r', 2r'')) & (\mathbb{F} = \mathbb{H}). \end{cases}$$

We look at the example  $\mathbb{F} = \mathbb{C}$ . In this case, as representations of  $U(r') \times U(r')$  and  $U(r'') \times U(r'')$  we have

$$\begin{aligned} \mathcal{P}(H_{r'}^{\mathbb{C}}) &= \bigoplus_{k_1 \geq \dots \geq k_{r'} \geq 0} \mathcal{P}_k(H_{r'}^{\mathbb{C}}) \simeq \bigoplus_{k_1 \geq \dots \geq k_{r'} \geq 0} V_{\mathbf{k}}^{(r')\vee} \boxtimes V_{\mathbf{k}}^{(r')}, \\ \mathcal{P}(H_{r''}^{\mathbb{C}}) &= \bigoplus_{l_1 \geq \dots \geq l_{r''} \geq 0} \mathcal{P}_1(H_{r''}^{\mathbb{C}}) \simeq \bigoplus_{l_1 \geq \dots \geq l_{r''} \geq 0} V_1^{(r'')\vee} \boxtimes V_1^{(r'')}. \end{aligned}$$

Here we omit the definition of notations. According to this decomposition, it is known that the restriction of the unitary representation  $\mathcal{H}_{\lambda_1+\lambda_2}(D_{r,r})$  of  $\tilde{U}(r, r)$  to the subgroup  $\tilde{U}(r', r'') \times \tilde{U}(r'', r')$  is decomposed as

$$\begin{aligned} \mathcal{H}_{\lambda_1+\lambda_2}(D_{r,r})|_{U(r', r'') \times U(r'', r')} \\ \simeq \sum_{\substack{k_1 \geq \dots \geq k_{r'} \geq 0 \\ l_1 \geq \dots \geq l_{r''} \geq 0}}^{\oplus} \mathcal{H}_{\lambda_1+\lambda_2}(D_{r', r''}, V_{\mathbf{k}}^{(r')\vee} \boxtimes V_1^{(r'')}) \hat{\boxtimes} \mathcal{H}_{\lambda_1+\lambda_2}(D_{r'', r'}, V_1^{(r'')\vee} \boxtimes V_{\mathbf{k}}^{(r')}) \end{aligned}$$

(see [6, Theorem 8.3]). Now we take following functions

$$\begin{aligned} f_1(x) &\in \left( \mathcal{P}_{\mathbf{k}}(H_{r'}^{\mathbb{C}}) \otimes \overline{V_{\mathbf{k}}^{(r')\vee} \boxtimes V_{\mathbf{k}}^{(r')}} \right)^{U(r') \times U(r')}, \\ f_2(z) &\in \left( \mathcal{P}_1(H_{r''}^{\mathbb{C}}) \otimes \overline{V_1^{(r'')\vee} \boxtimes V_1^{(r'')}} \right)^{U(r'') \times U(r'')} \end{aligned}$$

which are unique up to scalar multiple, and define a vector-valued polynomial  $F(A) \in \mathcal{P}(H_r^{\mathbb{C}}, V_{\mathbf{k}}^{(r')\vee} \boxtimes V_{\mathbf{k}}^{(r')}) \boxtimes V_1^{(r'')\vee} \boxtimes V_1^{(r'')})$  by

$$F(A) := \left\langle e^{\operatorname{tr}(XA)}, f_1(x) f_2(z) \right\rangle_{\lambda, X}.$$

Then the following holds.

**Theorem 2.5** ([11, Theorem 3.10 (1)]). *The following map intertwines the  $\tilde{U}(r', r'') \times \tilde{U}(r'', r')$ -action.*

$$\mathcal{F} : \mathcal{H}_{\lambda_1+\lambda_2}(D_{r,r}) \rightarrow \mathcal{H}_{\lambda_1+\lambda_2}(D_{r',r''}, V_{\mathbf{k}}^{(r')\vee} \boxtimes V_1^{(r'')}) \hat{\boxtimes} \mathcal{H}_{\lambda_1+\lambda_2}(D_{r'',r'}, V_1^{(r'')\vee} \boxtimes V_{\mathbf{k}}^{(r')}),$$

$$\mathcal{F}\varphi(y_1, y_2) = F\left(\frac{\partial}{\partial X}\right)\varphi(X)\Big|_{x=z=0} \left(X = \begin{pmatrix} x & y_1 \\ y_2 & z \end{pmatrix}\right).$$

Especially, if  $\mathbf{k} = (k, \dots, k)$ ,  $\mathbf{l} = (l, \dots, l)$ , then up to scalar multiple we have

$$F\begin{pmatrix} a & b_1 \\ b_2 & c \end{pmatrix} = \det(a)^k \det(c)^l {}_2F_1\left(\begin{matrix} -k, -l \\ -\lambda - k - l + \frac{n}{r} \end{matrix}; \sqrt{c}^{-1}b_2a^{-1}b_1\sqrt{c}^{-1}\right),$$

and therefore the following holds.

**Corollary 2.6.** *The following map intertwines the  $\tilde{U}(r', r'') \times \tilde{U}(r'', r')$ -action.*

$$\mathcal{F} : \mathcal{H}_{\lambda_1+\lambda_2}(D_{r,r}) \rightarrow \mathcal{H}_{(\lambda_1+k)+(\lambda_2+l)}(D_{r',r''}) \hat{\boxtimes} \mathcal{H}_{(\lambda_1+l)+(\lambda_2+k)}(D_{r'',r'}),$$

$$\mathcal{F}\varphi(y_1, y_2) = F\left(\frac{\partial}{\partial X}\right)\varphi(X)\Big|_{x=z=0} \left(X = \begin{pmatrix} x & y_1 \\ y_2 & z \end{pmatrix}\right).$$

**Remark 2.7.** *There are previous works on differential symmetry breaking operators by, e.g.,*

- Cohen (1975) [2]:  $(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}), SL(2, \mathbb{R}))$ ,
- Peng-Zhang (2004) [12]:  $(G \times G, G)$  ( $G$ : Hermitian),
- Juhl (2009) [5]:  $(SO_0(1, n), SO_0(1, n - 1))$ ,
- Ibukiyama-Kuzumaki-Ochiai (2012) [4]:  $(Sp(2n, \mathbb{R}), Sp(n, \mathbb{R}) \times Sp(n, \mathbb{R}))$ ,
- Kobayashi-Ørsted-Somberg-Souček (2015) [8]:  $(SO_0(p, q), SO_0(p, q - 1))$ ,
- Kobayashi-Pevzner (2016) [9, 10]:  $(SO_0(2, n), SO_0(2, n - 1))$ ,  $(Sp(n, \mathbb{R}), Sp(n - 1, \mathbb{R}) \times Sp(1, \mathbb{R}))$ ,  $(U(n, 1) \times U(n, 1), U(n, 1))$ ,
- Kobayashi-Kubo-Pevzner (2016) [7]:  $(O(1, n), O(1, n - 1))$  (vector-valued case).

*The result in this paper deals with  $(Sp(r, \mathbb{R}), U(r', r''))$ ,  $(U(r, r), U(r', r'') \times U(r'', r'))$  and  $(SO^*(4r), U(2r', 2r''))$  cases.*

We have another motivation. Although the weighted Bergman inner product

$$\langle f, g \rangle_\lambda = C_\lambda \int_D f(X)\overline{g(X)} \det(I - X\overline{X})^{\lambda - \frac{2n}{r}} dX$$

originally converges only when  $\text{Re } \lambda > \frac{2n}{r} - 1$ , this is meromorphically continued for all  $\lambda \in \mathbb{C}$  since

$$\langle f, g \rangle_\lambda = \sum_{m_1 \geq \dots \geq m_r \geq 0} \frac{1}{\prod_{j=1}^r \left(\lambda - \frac{d}{2}(j - 1)\right)_{m_j}} \langle f_{\mathbf{m}}, g_{\mathbf{m}} \rangle_F$$

holds, and is positive definite if  $\lambda > \frac{d}{2}(r - 1)$ . Moreover,  $\mathcal{O}_\lambda(D)_{\tilde{K}} = \mathcal{P}(H_r^{\mathbb{C}})$  is reducible if and only if  $\lambda$  is a pole, and if  $\lambda \in \frac{d}{2}(j - 1) - \mathbb{Z}_{\geq 0}$  holds, then

$$M_j(\lambda) := \bigoplus_{\substack{m_1 \geq \dots \geq m_r \geq 0 \\ m_j \leq \frac{d}{2}(j-1) - \lambda}} \mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}}) \quad (j = 1, 2, \dots, r)$$

are non-trivial submodules. Also, when  $\lambda = \frac{d}{2}(j - 1)$  ( $j = 1, 2, \dots, r$ ), this pairing becomes positive definite if it is restricted on the submodule  $M_j\left(\frac{d}{2}(j - 1)\right)$ , and makes it infinitesimally unitary. For example, let  $\mathbb{F} = \mathbb{C}$  so that  $d = 2$ . Then  $\mathcal{O}_\lambda(D)_{\tilde{K}}$  is reducible if and only if  $\lambda \in \mathbb{Z}_{\leq r-1}$  holds, and has a sequence of  $(\mathbf{u}(r, r), \tilde{U}(r) \times \tilde{U}(r))$ -submodules

$$\mathcal{O}_\lambda(D)_{\tilde{K}} \supset M_r(\lambda) \supset M_{r-1}(\lambda) \supset \dots \supset M_{\max\{\lambda, 0\}+1}(\lambda) \supset \{0\}.$$

$\mathcal{O}_\lambda(D)_{\tilde{K}}$  for  $\lambda > r - 1$  and  $M_j(j - 1)$  ( $j = 1, 2, \dots, r$ ) are infinitesimally unitary.

In which submodule a polynomial is contained is determined by the order of poles of  $\langle f, g \rangle_\lambda$ . That is, for  $f \in \mathcal{P}(H_r^{\mathbb{C}})$  and for  $\mathbf{p} \in \mathbb{Z}^r$ ,  $p_1 \geq \dots \geq p_r \geq 0$ , if

$$(\lambda)_{\mathbf{p}} \langle f, e^{\text{tr}(\bar{A})} \rangle_\lambda = \prod_{j=1}^r \left( \lambda - \frac{d}{2}(j - 1) \right)_{p_j} \langle f, e^{\text{tr}(\bar{A})} \rangle_\lambda$$

is holomorphic for all  $\lambda \in \mathbb{C}$ , then for  $j = 1, \dots, r$  we can show that  $f \in M_j(\lambda)$  holds for  $\lambda \in \frac{d}{2}(j - 1) - p_j - \mathbb{Z}_{\geq 0}$ .

Now, for  $k \in \mathbb{Z}_{\geq 0}$ ,  $\mathbf{l} \in \mathbb{Z}^{r''}$ ,  $l_1 \geq \dots \geq l_{r''} \geq 0$ , we define  $p(k, \mathbf{l}) \in \mathbb{Z}^r$  by

$$p(k, \mathbf{l}) := \begin{cases} \left( k + l_1, \dots, k + l_{r''}, \overbrace{k, \dots, k}^{r'-r''}, \min\{k, l_1\}, \dots, \min\{k, l_{r''}\} \right) & (r' \geq r''), \\ \left( k + l_1, \dots, k + l_{r'}, \min\{k + l_{r'+1}, l_1\}, \dots, \min\{k + l_{r'}, l_{r''-r'}\}, \right. \\ \quad \left. \min\{k, l_{r''-r'+1}\}, \dots, \min\{k, l_{r''}\} \right) & (r' < r''). \end{cases}$$

Then the following holds.

**Corollary 2.8.** For  $f \in \mathcal{P}_1(H_{r''}^{\mathbb{C}})$ ,

$$(\lambda)_{p(k, \mathbf{l})} \langle \det(x)^k f(z), e^{\text{tr}(X\bar{A})} \rangle_{\lambda, X}$$

is entirely holomorphic with respect to  $\lambda \in \mathbb{C}$ .

From this we can determine the submodules containing  $\mathcal{P}_{\underline{k}, r'}(H_{r''}^{\mathbb{C}}) \otimes \mathcal{P}_1(H_{r''}^{\mathbb{C}})$ .

### 3 Proof of the corollary on poles

In this section we prove Corollary 2.8. First we prove this for  $r' \geq r''$  case. Then by (B) of Theorem 2.1 we have

$$\begin{aligned} \langle \det(x)^k f(z), e^{\text{tr}(X\bar{A})} \rangle_{\lambda, X} &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}, r''}}{(\lambda)_{\underline{k}, r+1}} \det(a)^k \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{\left(\lambda - \frac{d}{2}r'\right)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^* a^{-1} b - c, c) \\ &= \frac{1}{(\lambda)_{\underline{k}, r+1}} \det(a)^k \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{\left(\lambda - \frac{d}{2}r'\right)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^* a^{-1} b - c, c). \end{aligned}$$

Now we recall that for  $f(z) \in \mathcal{P}_1(H_{r''}^{\mathbb{C}})$  we write

$$f(z+c) = \sum_{\mathbf{m}, \mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c) \in \bigoplus_{\mathbf{m}, \mathbf{n}} \mathcal{P}_{\mathbf{m}}(H_{r''}^{\mathbb{C}}) \otimes \mathcal{P}_{\mathbf{n}}(H_{r''}^{\mathbb{C}}).$$

Then we can show that  $\sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c) \neq 0$  implies  $m_j \leq l_j$ , and  $(-k)_{\mathbf{m}} \neq 0$  implies  $m_j \leq k$ . Now for  $\lambda \in \mathbb{C}$ ,  $\mathbf{s} \in \mathbb{C}^r$ ,  $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$  we write

$$(\lambda + \mathbf{s})_{\mathbf{m}} := \prod_{j=1}^r \left( \lambda - \frac{d}{2}(j-1) + s_j \right)_{m_j}.$$

Then we have

$$\begin{aligned} & \left\langle \det(x)^k f(z), e^{\text{tr}(X\bar{A})} \right\rangle_{\lambda, X} \\ &= \frac{1}{(\lambda)_{\underline{k}, r+1}} \det(a)^k \sum_{\mathbf{m} \leq \min\{\underline{k}, r, 1\}} \frac{(-k)_{\mathbf{m}}}{\left(\lambda - \frac{d}{2}r'\right)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^*a^{-1}b - c, c) \\ &= \frac{1}{(\lambda)_{\underline{k}, r+1} \left(\lambda - \frac{d}{2}r'\right)_{\min\{\underline{k}, r, 1\}}} \det(a)^k \\ & \quad \times \sum_{\mathbf{m} \leq \min\{\underline{k}, r, 1\}} (-k)_{\mathbf{m}} \left(\lambda - \frac{d}{2}r' + \mathbf{m}\right)_{\min\{\underline{k}, r, 1\} - \mathbf{m}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^*a^{-1}b - c, c) \\ &= \frac{\det(a)^k}{(\lambda)_{P(k, 1)}} \sum_{\mathbf{m} \leq \min\{\underline{k}, r, 1\}} (-k)_{\mathbf{m}} \left(\lambda - \frac{d}{2}r' + \mathbf{m}\right)_{\min\{\underline{k}, r, 1\} - \mathbf{m}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^*a^{-1}b - c, c), \end{aligned}$$

and therefore  $(\lambda)_{P(k, 1)} \left\langle \det(x)^k f(z), e^{\text{tr}(X\bar{A})} \right\rangle_{\lambda, X}$  is holomorphic.

Next we prove the corollary for  $r' < r''$  case. In this case, since  $\sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^*a^{-1}b - c, c)$  in Theorem 2.1 (B) are not linearly independent, we use (C), so that

$$\begin{aligned} & \left\langle \det(x)^k f(z), e^{\text{tr}(X\bar{A})} \right\rangle_{\lambda, X} \\ &= \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}, r''}}{(\lambda)_{\underline{k}, r+1}} \det(a)^k \sum_{\mathbf{n}} \frac{(\lambda + k - \frac{d}{2}r')_{\mathbf{n}}}{\left(\lambda - \frac{d}{2}r'\right)_{\mathbf{n}}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^*a^{-1}b, c - b^*a^{-1}b) \\ &= \frac{\det(a)^k}{(\lambda)_{\underline{k}, r+1}} \sum_{\mathbf{n}} \frac{\left(\lambda - \frac{d}{2}r'\right)_{\underline{k}, r'' + \mathbf{n}}}{\left(\lambda - \frac{d}{2}r'\right)_{\mathbf{n}}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^*a^{-1}b, c - b^*a^{-1}b) \\ &= \frac{\det(a)^k}{(\lambda)_{\underline{k}, r+1}} \sum_{\mathbf{n}} \left(\lambda - \frac{d}{2}r' + \mathbf{n}\right)_{\underline{k}, r''} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^*a^{-1}b, c - b^*a^{-1}b). \end{aligned}$$

Now for  $z, c \in H_{r''}^{\mathbb{C}}$ , if  $\text{rank } z \leq r'$  then we can show that  $\tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c) \neq 0$  holds only if  $m_{r'+1} = m_{r''+2} = \dots = m_r = 0$ , and moreover  $\sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c) \neq 0$  implies

$$\begin{cases} l_{r'+j} \leq n_j \leq l_j & (1 \leq j \leq r'' - r'), \\ 0 \leq n_j \leq l_j & (r'' - r' < j \leq r''). \end{cases}$$

Now we write  $\mathbf{l}' := (l_1, \dots, l_{r'})$ ,  $\mathbf{l}'' := (l_{r'+1}, \dots, l_{r''}, 0, \dots, 0)$ . Then we can show that if  $\mathbf{l}'' \leq \mathbf{n} \leq \mathbf{l}$  then

$$\left( \lambda - \frac{d}{2} r' + \mathbf{n} \right)_{\underline{k}_{r''}} = \prod_{j=1}^{r''} \left( \lambda - \frac{d}{2} (r' + j - 1) + n_j \right)_k$$

is divisible by

$$\prod_{\substack{1 \leq j \leq r'' \\ k+l_{r'+j} > l_j}} \left( \lambda - \frac{d}{2} (r' + j - 1) + l_j \right)_{k+l_{r'+j}-l_j} = \left( \lambda - \frac{d}{2} r' + \mathbf{1} \right)_{\max\{\underline{k}_{r''} + \mathbf{l}'' - \mathbf{1}, \underline{0}_{r''}\}}.$$

Therefore we have

$$\begin{aligned} & \left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\bar{A})} \right\rangle_{\lambda, X} \\ &= \frac{\det(a)^k}{(\lambda)_{\underline{k}_r + \mathbf{1}}} \sum_{\mathbf{l}'' \leq \mathbf{n} \leq \mathbf{l}} \left( \lambda - \frac{d}{2} r' + \mathbf{n} \right)_{\underline{k}_{r''}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^* a^{-1} b, c - b^* a^{-1} b) \\ &= \frac{\left( \lambda - \frac{d}{2} r' + \mathbf{1} \right)_{\max\{\underline{k}_{r''} + \mathbf{l}'' - \mathbf{1}, \underline{0}_{r''}\}}}{(\lambda)_{\underline{k}_{r'} + \mathbf{l}'}} \left( \lambda - \frac{d}{2} r' \right)_{\underline{k}_{r''} + \mathbf{l}''} \det(a)^k \\ & \quad \times \sum_{\mathbf{l}'' \leq \mathbf{n} \leq \mathbf{l}} \underbrace{\frac{\left( \lambda - \frac{d}{2} r' + \mathbf{n} \right)_{\underline{k}_{r''}}}{\left( \lambda - \frac{d}{2} r' + \mathbf{1} \right)_{\max\{\underline{k}_{r''} + \mathbf{l}'' - \mathbf{1}, \underline{0}_{r''}\}}}}_{\in \mathbb{C}[\lambda]} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(b^* a^{-1} b, c - b^* a^{-1} b) \\ & \in \frac{1}{(\lambda)_{\underline{k}_{r'} + \mathbf{l}'}} \frac{1}{\left( \lambda - \frac{d}{2} r' \right)_{\min\{\underline{k}_{r''} + \mathbf{l}'', \mathbf{1}\}}} \mathbb{C}[\lambda] \otimes \mathcal{P}(H_r^{\mathbb{C}}) = \frac{1}{(\lambda)_{p(k, \mathbf{l})}} \mathbb{C}[\lambda] \otimes \mathcal{P}(H_r^{\mathbb{C}}), \end{aligned}$$

and hence  $(\lambda)_{p(k, \mathbf{l})} \left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\bar{A})} \right\rangle_{\lambda, X}$  is holomorphic. This completes the proof of Corollary 2.8.

## 4 Proof of the 1st identity of the main result

In this section we prove the 1st identity of Theorem 2.1. For the proof we use the *Laplace transform*. We recall that the Laplace transform  $\mathcal{L} : L^2(H_r^+) \rightarrow \mathcal{O}(H_r^+ + \sqrt{-1}H_r)$  and its inverse are given by

$$\begin{aligned} (\mathcal{L}f)(A) &:= \int_{H_r^+} e^{-\operatorname{tr}(AX)} f(A) dA & (X \in H_r^+ + \sqrt{-1}H_r), \\ (\mathcal{L}^{-1}g)(X) &:= \frac{1}{(2\pi\sqrt{-1})^n} \int_{Z + \sqrt{-1}H_r} e^{\operatorname{tr}(AX)} g(X) dX & (A \in H_r^+). \end{aligned}$$

Here the 2nd formula does not depend on the choice of  $Z \in H_r^+$ . Next, for  $\alpha \in \mathbb{C}$ ,  $\mathbf{s} \in \mathbb{C}^r$ , let

$$\begin{aligned} \Gamma_r(\alpha + \mathbf{s}) &:= (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^r \Gamma\left(\alpha + s_j - \frac{d}{2}(j-1)\right), \\ \Gamma_r(\alpha) &:= \Gamma_r(\alpha + \underline{0}_r), \end{aligned}$$

so that  $\frac{\Gamma_r(\alpha + \mathbf{m})}{\Gamma_r(\alpha)} = (\alpha)_{\mathbf{m}}$  holds for  $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$ . Then the following is known.

**Theorem 4.1** (Gindikin (1964), [3, Lemma XI.2.3]). *Let  $\operatorname{Re} \lambda + m_r > \frac{2n}{r} - 1$ . For  $f \in \mathcal{P}_{\mathbf{m}}(H_r^{\mathbb{C}})$ ,  $A, Z \in H_r^+$ ,  $X, W \in H_r^+ + \sqrt{-1}H_r$ , we have*

$$\int_{H_r^+} e^{-\operatorname{tr}(A(X+W))} f(A) \det(A)^{\lambda - \frac{n}{r}} dA = \Gamma_r(\lambda + \mathbf{m}) f((X+W)^{-1}) \det(X+W)^{-\lambda},$$

$$\frac{\Gamma_r(\lambda + \mathbf{m})}{(2\pi\sqrt{-1})^n} \int_{Z+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} f((X+W)^{-1}) \det(X+W)^{-\lambda} dX = e^{-\operatorname{tr}(AW)} f(A) \det(A)^{\lambda - \frac{n}{r}}.$$

In the 2nd formula we set  $W = 0$ . Then we have

$$\frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \det(A)^{-\lambda + \frac{n}{r}} \int_{Z+\sqrt{-1}H_r} f(X^{-1}) \det(X)^{-\lambda} dX = \frac{1}{(\lambda)_{\mathbf{m}}} f(A).$$

Then comparing this with the Faraut-Korányi’s result

$$\langle f, e^{\operatorname{tr}(\bar{A})} \rangle_{\lambda} = \frac{1}{(\lambda)_{\mathbf{m}}} f(A),$$

we get the following.

**Lemma 4.2.** *For  $\operatorname{Re} \lambda > \frac{2n}{r} - 1$ ,  $f \in \mathcal{P}(H_r^{\mathbb{C}})$ ,  $A \in H_r^+$ , we have*

$$\langle f, e^{\operatorname{tr}(\bar{A})} \rangle_{\lambda} = \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \det(A)^{-\lambda + \frac{n}{r}} \int_{Z+\sqrt{-1}H_r} f(X^{-1}) \det(X)^{-\lambda} dX.$$

*This does not depend on the choice of  $Z \in H_r^+$ .*

Now we start the proof of the 1st equality. For  $X = \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} \in H_r^{\mathbb{C}}(\mathbb{F})$ , we write  $\operatorname{Proj}_1 \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} = x \in H_{r'}^{\mathbb{C}}(\mathbb{F})$ ,  $\operatorname{Proj}_2 \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} = z \in H_{r''}^{\mathbb{C}}(\mathbb{F})$ . Then by the above lemma, for  $\operatorname{Re} \lambda > \frac{2n}{r} - 1$ ,  $A \in H_r^+$ , we have

$$\langle \det(x)^k f(z), e^{\operatorname{tr}(X\bar{A})} \rangle_{\lambda, X} = \langle \det(\operatorname{Proj}_1(X))^k f(\operatorname{Proj}_2(X)), e^{\operatorname{tr}(X\bar{A})} \rangle_{\lambda, X}$$

$$= \det(A)^{-\lambda + \frac{n}{r}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} \det(\operatorname{Proj}_1(X^{-1}))^k f(\operatorname{Proj}_2(X^{-1})) \det(X)^{-\lambda} dX.$$

Then since for  $X = \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} \in H_r^{\mathbb{C}}(\mathbb{F})$  we have

$$X^{-1} = \begin{pmatrix} (x - yz^{-1}y^*)^{-1} & -(x - yz^{-1}y^*)^{-1}yz^{-1} \\ -z^{-1}y^*(x - yz^{-1}y^*)^{-1} & (z - y^*x^{-1}y)^{-1} \end{pmatrix},$$

$$\det(X) = \det(z - y^*x^{-1}y) \det(x) = \det(x - yz^{-1}y^*) \det(z),$$

we have

$$\begin{aligned}
& \left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\bar{A})} \right\rangle_{\lambda, X} \\
&= \det(A)^{-\lambda + \frac{n}{r}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} \det\left((x - yz^{-1}y^*)^{-1}\right)^k \\
&\quad \times f\left((z - y^*x^{-1}y)^{-1}\right) \det(z - y^*x^{-1}y)^{-\lambda} \det(x)^{-\lambda} dX \\
&= \det(A)^{-\lambda + \frac{n}{r}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} f\left((z - y^*x^{-1}y)^{-1}\right) \\
&\quad \times \det(z - y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} \det(z)^k dX \\
&= \det(A)^{-\lambda + \frac{n}{r}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \iiint_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(ax)+2\operatorname{Re}_{\mathbb{F}}\operatorname{tr}(by^*)+\operatorname{tr}(cz)} \\
&\quad \times f\left((z - y^*x^{-1}y)^{-1}\right) \det(z - y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} \det(z)^k dx dy dz \\
&= \det(A)^{-\lambda + \frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \iiint_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(ax)+2\operatorname{Re}_{\mathbb{F}}\operatorname{tr}(by^*)+\operatorname{tr}(cz)} \\
&\quad \times f\left((z - y^*x^{-1}y)^{-1}\right) \det(z - y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} dx dy dz.
\end{aligned}$$

Since  $f \in \mathcal{P}_1(H_{r''}^{\mathbb{C}})$ , by using the inverse Laplace formula

$$\frac{\Gamma_{r''}(\mu+1)}{(2\pi\sqrt{-1})^{n''}} \int_{I+\sqrt{-1}H_{r''}} e^{\operatorname{tr}(cz)} f((z+w)^{-1}) \det(z+w)^{-\mu} dz = e^{-\operatorname{tr}(cw)} f(c) \det(c)^{\mu - \frac{n''}{r''}}$$

twice, for  $x \in I + \sqrt{-1}H_{r'}$ ,  $y \in \sqrt{-1}M_{r', r''}$ ,  $c \in H_{r''}^+$  we get

$$\begin{aligned}
& \frac{1}{(2\pi\sqrt{-1})^{n''}} \int_{I+\sqrt{-1}H_{r''}} e^{\operatorname{tr}(cz)} f\left((z - y^*x^{-1}y)^{-1}\right) \det(z - y^*x^{-1}y)^{-\lambda-k} dz \\
&= \frac{1}{\Gamma_{r''}(\lambda+k+1)} e^{\operatorname{tr}(cy^*x^{-1}y)} f(c) \det(c)^{\lambda+k - \frac{n''}{r''}} \\
&= \frac{1}{(\lambda+k)_1} \frac{1}{\Gamma_{r''}(\lambda+k)} e^{\operatorname{tr}(cy^*x^{-1}y)} \det(c)^{\lambda+k - \frac{n''}{r''}} f(c) \\
&= \frac{1}{(\lambda+k)_1} \frac{1}{(2\pi\sqrt{-1})^{n''}} \int_{I+\sqrt{-1}H_{r''}} e^{\operatorname{tr}(cz)} \det(z - y^*x^{-1}y)^{-\lambda-k} dz f(c).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
& \left\langle \det(x)^k f(z), e^{\operatorname{tr}(X\bar{A})} \right\rangle_{\lambda, X} \\
&= \det(A)^{-\lambda + \frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \iiint_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(ax)+2\operatorname{Re}_{\mathbb{F}}\operatorname{tr}(by^*)+\operatorname{tr}(cz)} \\
&\quad \times f\left((z - y^*x^{-1}y)^{-1}\right) \det(z - y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} dx dy dz \\
&= \frac{1}{(\lambda+k)_1} \det(A)^{-\lambda + \frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \iiint_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(ax)+2\operatorname{Re}_{\mathbb{F}}\operatorname{tr}(by^*)+\operatorname{tr}(cz)} \\
&\quad \times \det(z - y^*x^{-1}y)^{-\lambda-k} \det(x)^{-\lambda-k} dx dy dz f(c)
\end{aligned}$$



$$\begin{aligned}
&= \frac{1}{(\lambda+k)_1} \det(A)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{I+\sqrt{-1}H_r} e^{\operatorname{tr}(AX)} \det(X)^{-\lambda-k} dX f(c) \\
&= \frac{1}{(\lambda+k)_1} \det(A)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \frac{\Gamma_r(\lambda)}{\Gamma_r(\lambda+k)} \det(A)^{\lambda+k-\frac{n}{r}} f(c) \\
&= \frac{1}{(\lambda)_{\underline{k}_r}} \frac{1}{(\lambda+k)_1} \det(c-b^*a^{-1}b)^{-\lambda+\frac{n}{r}} \det(a)^{-\lambda+\frac{n}{r}} \\
&\quad \times \det\left(\frac{\partial}{\partial c}\right)^k \det(c-b^*a^{-1}b)^{\lambda+k-\frac{n}{r}} \det(a)^{\lambda+k-\frac{n}{r}} f(c) \\
&= \frac{\det(a)^k}{(\lambda)_{\underline{k}_r+1}} \det(c-b^*a^{-1}b)^{-\lambda+\frac{n}{r}} \det\left(\frac{\partial}{\partial c}\right)^k \det(c-b^*a^{-1}b)^{\lambda+k-\frac{n}{r}} f(c).
\end{aligned}$$

This computation is originally done for  $A \in H_r^+(\mathbb{F})$ , but since the both sides are holomorphic, this equality holds for all  $A \in H_r^{\mathbb{C}}(\mathbb{F})$ . This proves the 1st equality.

## 5 Proof of the 2nd, 3rd and 4th identities of the main result

In this section we prove the 2nd, 3rd and 4th identities of Theorem 2.1. We write  $b^*a^{-1}b =: z \in H_{r''}^{\mathbb{C}}$  and  $\lambda - \frac{d}{2}r' =: \mu$ , so that  $\lambda - \frac{n}{r} = \lambda - \frac{d}{2}r' - \frac{n''}{r''} = \mu - \frac{n''}{r''}$  holds. Then our aim is to prove, for  $f \in \mathcal{P}_1(H_{r''}^{\mathbb{C}})$ ,

$$\det(c-z)^{-\mu+\frac{n''}{r''}} \det\left(\frac{\partial}{\partial c}\right)^k \det(c-z)^{\mu+k-\frac{n''}{r''}} f(c) \quad (\text{A}')$$

$$= (\mu)_{\underline{k}_r''} \sum_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{(\mu)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z-c, c) \quad (\text{B}')$$

$$= (\mu)_{\underline{k}_r''} \sum_{\mathbf{n}} \frac{(\mu+k)_{\mathbf{n}}}{(\mu)_{\mathbf{n}}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z, c-z) \quad (\text{C}')$$

$$= \frac{(\mu)_{\underline{k}_r''}}{(\mu)_1} \sum_{\mathbf{m},\mathbf{n}} (-k)_{\mathbf{m}} (\mu+k)_{\mathbf{n}} \tilde{f}_{\mathbf{m},\mathbf{n}}(z, c). \quad (\text{D}')$$

Since each side is holomorphic with respect to  $z, c$ , it is enough to prove this when  $c-z \in H_{r''}^+$ . To do this, we recall the *Riesz distribution*. For  $\operatorname{Re} \alpha > \frac{n''}{r''} - 1$ , let

$$R_{\alpha}(w) := \frac{1}{\Gamma_{r''}(\alpha)} \det(w)^{\alpha-\frac{n''}{r''}} 1_{H_{r''}^+} \in \mathcal{S}'(H_{r''}).$$

Then  $R_{\alpha}(w) \in \mathcal{S}'(H_{r''})$  is analytically continued for all  $\alpha \in \mathbb{C}$  as distributions, and for  $\alpha = -k \in -\mathbb{Z}_{\geq 0}$ ,

$$R_{-k}(w) = \det\left(\frac{\partial}{\partial w}\right)^k \delta_0$$

holds (see [3, Theorem VII.2.2]). This  $R_\alpha(w)$  is called the Riesz distribution. Especially, when  $c - z \in H_{r''}^+$ , the formula (A'),

$$\begin{aligned}
& \det(c - z)^{-\mu + \frac{n''}{r''}} \det\left(\frac{\partial}{\partial c}\right)^k \det(c - z)^{\mu + k - \frac{n''}{r''}} f(c) \\
&= \det(c - z)^{-\mu + \frac{n''}{r''}} \det\left(\frac{\partial}{\partial w}\right)^k \det(c - z + w)^{\mu + k - \frac{n''}{r''}} f(c + w) \Big|_{w=0} \\
&= \det(c - z)^{-\mu + \frac{n''}{r''}} \det\left(-\frac{\partial}{\partial w}\right)^k \det(c - z - w)^{\mu + k - \frac{n''}{r''}} f(c - w) \Big|_{w=0} \\
&= \det(c - z)^{-\mu + \frac{n''}{r''}} \int_{w \in (c - z - H_{r''}^+)} \left( \det\left(\frac{\partial}{\partial w}\right)^k \delta_0 \right) \det(c - z - w)^{\mu + k - \frac{n''}{r''}} f(c - w) dw \\
&= \det(c - z)^{-\mu + \frac{n''}{r''}} \int_{w \in (c - z - H_{r''}^+)} R_{-k}(w) \det(c - z - w)^{\mu + k - \frac{n''}{r''}} f(c - w) dw
\end{aligned}$$

is viewed as the analytic continuation of

$$\frac{\det(c - z)^{-\mu + \frac{n''}{r''}}}{\Gamma_{r''}(-k)} \int_{H_{r''}^+ \cap (c - z - H_{r''}^+)} \det(w)^{-k - \frac{n''}{r''}} \det(c - z - w)^{\mu + k - \frac{n''}{r''}} f(c - w) dw.$$

Therefore, by putting  $-k =: \alpha$ ,  $\mu + k =: \beta$  so that  $(\mu)_{\underline{k}, r''} = \frac{\Gamma_{r''}(\mu + k)}{\Gamma_{r''}(\mu)} = \frac{\Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha + \beta)}$ , it is enough to prove

$$\det(c - z)^{-\alpha - \beta + \frac{n''}{r''}} \int_{H_{r''}^+ \cap (c - z - H_{r''}^+)} \det(w)^{\alpha - \frac{n''}{r''}} \det(c - z - w)^{\beta - \frac{n''}{r''}} f(c - w) dw \quad (\text{A}'')$$

$$= \frac{\Gamma_{r''}(\alpha) \Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha + \beta)} \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{(\alpha + \beta)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z - c, c) \quad (\text{B}'')$$

$$= \frac{\Gamma_{r''}(\alpha) \Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha + \beta)} \sum_{\mathbf{n}} \frac{(\beta)_{\mathbf{n}}}{(\alpha + \beta)_{\mathbf{n}}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c - z) \quad (\text{C}'')$$

$$= \frac{\Gamma_{r''}(\alpha) \Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha + \beta) (\alpha + \beta)_1} \sum_{\mathbf{m}, \mathbf{n}} (\alpha)_{\mathbf{m}} (\beta)_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c) \quad (\text{D}'')$$

for  $\operatorname{Re} \alpha, \operatorname{Re} \beta > \frac{n''}{r''} - 1$ . We note that the formula (A'') is written in a more symmetric form

$$\det(c - z)^{-\alpha - \beta + \frac{n''}{r''}} \int_{(z + H_{r''}^+) \cap (c - H_{r''}^+)} \det(w - z)^{\alpha - \frac{n''}{r''}} \det(c - w)^{\beta - \frac{n''}{r''}} f(c + z - w) dw,$$

but we do not use this expression for the proof.

To prove these equalities, for  $\mathbf{l}, \mathbf{m}, \mathbf{n} \in \mathbb{Z}^{r''}$  with  $l_1 \geq \dots \geq l_{r''} \geq 0$ ,  $m_1 \geq \dots \geq m_{r''} \geq 0$ ,  $n_1 \geq \dots \geq n_{r''} \geq 0$ , we define vector spaces  $\mathcal{P}_{\mathbf{m}, \mathbf{n}}^-(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})$ ,  $\mathcal{P}_{\mathbf{m}, \mathbf{n}}^1(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}}) \subset \mathcal{P}(H_{r''}^{\mathbb{C}}) \otimes \mathcal{P}(H_{r''}^{\mathbb{C}})$  as

$$\begin{aligned}
\mathcal{P}_{\mathbf{m}, \mathbf{n}}^-(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}}) &:= \{f(z, c) \in \mathcal{P}_{\mathbf{m}}(H_{r''}^{\mathbb{C}}) \otimes \mathcal{P}_{\mathbf{n}}(H_{r''}^{\mathbb{C}}) \mid f(z, z) = 0\}, \\
\mathcal{P}_{\mathbf{m}, \mathbf{n}}^1(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}}) &:= \{f(z, c) \in \mathcal{P}_{\mathbf{m}}(H_{r''}^{\mathbb{C}}) \otimes \mathcal{P}_{\mathbf{n}}(H_{r''}^{\mathbb{C}}) \mid f(z, z) \in \mathcal{P}_1(H_{r''}^{\mathbb{C}})\} \\
&\quad \cap \mathcal{P}_{\mathbf{m}, \mathbf{n}}^-(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})^\perp.
\end{aligned}$$

Then the following holds.

**Lemma 5.1.** (1) Let  $\operatorname{Re} \alpha, \operatorname{Re} \beta > \frac{n''}{r''} - 1$ . If  $f(z, c) \in \mathcal{P}_{\mathbf{m}, \mathbf{n}}^{\mathbf{1}}(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})$ , then for  $z \in H_{r''}^+$  we have

$$\begin{aligned} & \det(z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(z-w)^{\beta-\frac{n''}{r''}} f(w, z-w) dw \\ &= \frac{\Gamma_{r''}(\alpha + \mathbf{m}) \Gamma_{r''}(\beta + \mathbf{n})}{\Gamma_{r''}(\alpha + \beta + \mathbf{1})} f(z, z). \end{aligned}$$

(2) Let  $\operatorname{Re} \alpha, \operatorname{Re} \beta > \frac{n''}{r''} - 1$ . If  $f(z, c) \in \mathcal{P}_{\mathbf{m}, \mathbf{n}}^-(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})$ , then for  $z \in H_{r''}^+$  we have

$$\det(z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(z-w)^{\beta-\frac{n''}{r''}} f(w, z-w) dw = 0.$$

When  $f(z, c) \in \mathcal{P}_{\mathbf{m}, \mathbf{n}}^{\mathbf{1}}(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})$  is a highest weight vector under the diagonal action of  $GL(r'', \mathbb{F})$ , this is proved by a similar way as in [3, Theorem VII.1.7 (ii)]. Then by  $GL(r'', \mathbb{F})$ -equivariance, this holds for all  $f(z, c) \in \mathcal{P}_{\mathbf{m}, \mathbf{n}}^{\mathbf{1}}(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})$ .

By this lemma, (A'') = (B'') is proved as

$$\begin{aligned} & \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (c-z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} f(c-w) dw \\ &= \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (c-z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} \sum_{\mathbf{m}, \mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(-w, c) dw \\ &= \sum_{\mathbf{m}} \frac{\Gamma_{r''}(\alpha + \mathbf{m}) \Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha + \beta + \mathbf{m})} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(-(c-z), c) \\ &= \frac{\Gamma_{r''}(\alpha) \Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha + \beta)} \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{(\alpha + \beta)_{\mathbf{m}}} \sum_{\mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z-c, c), \end{aligned}$$

and (A'') = (C'') is proved as

$$\begin{aligned} & \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (c-z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} f(c-w) dw \\ &= \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (c-z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} f(z+c-z-w) dw \\ &= \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (c-z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} \sum_{\mathbf{m}, \mathbf{n}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c-z-w) dw \\ &= \sum_{\mathbf{n}} \frac{\Gamma_{r''}(\alpha) \Gamma_{r''}(\beta + \mathbf{n})}{\Gamma_{r''}(\alpha + \beta + \mathbf{n})} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c-z) \\ &= \frac{\Gamma_{r''}(\alpha) \Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha + \beta)} \sum_{\mathbf{n}} \frac{(\beta)_{\mathbf{n}}}{(\alpha + \beta)_{\mathbf{n}}} \sum_{\mathbf{m}} \tilde{f}_{\mathbf{m}, \mathbf{n}}(z, c-z). \end{aligned}$$

On the other hand, the proof of (A'') = (D'') is more complicated. First, by the  $GL(r'', \mathbb{F})$ -equivariance, it suffices to prove for one non-zero  $f(z) \in \mathcal{P}_1(H_{r''}^{\mathbb{C}})$ . Here we set  $f(z) = \Psi^{\mathbf{1}}(z)$ ,

$$\Psi^{\mathbf{1}}(z) \in \mathcal{P}_1(H_{r''}^{\mathbb{C}}), \quad \sum_{l_1 \geq \dots \geq l_{r''} \geq 0} \Psi^{\mathbf{1}}(z) = e^{\operatorname{tr}(z)}.$$

Then we sum up the formula (A'') with respect to  $\mathbf{l}$ . Then we get

$$\begin{aligned}
& \sum_{\mathbf{l}} \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (c-z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} \Psi^{\mathbf{l}}(c-w) dw \\
&= \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (c-z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(c-w)} dw \\
&= \det(c-z)^{-\frac{n''}{r''}} \int_{H_{r''}^+ \cap (c-z-H_{r''}^+)} \det(\sqrt{c-z}^{-1} w \sqrt{c-z}^{-1})^{\alpha-\frac{n''}{r''}} \\
&\quad \times \det(I - \sqrt{c-z}^{-1} w \sqrt{c-z}^{-1})^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(c-w)} dw \\
&= \int_{H_{r''}^+ \cap (I-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(c)-\mathrm{tr}(\sqrt{c-z} w \sqrt{c-z})} dw \\
&= \int_{H_{r''}^+ \cap (I-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(c)-\mathrm{tr}(w(c-z))} dw \\
&= \int_{H_{r''}^+ \cap (I-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(wz)+\mathrm{tr}((I-w)c)} dw.
\end{aligned}$$

Now we expand  $e^{\mathrm{tr}(xy)+\mathrm{tr}(zw)}$  as

$$\begin{aligned}
e^{\mathrm{tr}(xy)+\mathrm{tr}(zw)} &= \sum_{\mathbf{m}, \mathbf{n}} \left( \sum_{\mathbf{l}} K_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(x, z; y, w) + K_{\mathbf{m}, \mathbf{n}}^{-\mathbf{l}}(x, z; y, w) \right) \\
&\in \bigoplus_{\mathbf{m}, \mathbf{n}} \left( \bigoplus_{\mathbf{l}} \left( \mathcal{P}_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})_{x, z} \otimes \mathcal{P}_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})_{y, w} \right) \right. \\
&\quad \left. \oplus \left( \mathcal{P}_{\mathbf{m}, \mathbf{n}}^{-\mathbf{l}}(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})_{x, z} \otimes \mathcal{P}_{\mathbf{m}, \mathbf{n}}^{-\mathbf{l}}(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})_{y, w} \right) \right).
\end{aligned}$$

Then  $\tilde{\Psi}_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(z, c) = K_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(z, c; I, I) = K_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(I, I; z, c)$  holds. Therefore, by Lemma 5.1 we get

$$\begin{aligned}
& \sum_{\mathbf{l}} \det(c-z)^{-\alpha-\beta+\frac{n''}{r''}} \int_{H_{r''}^+ \cap (c-z-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(c-z-w)^{\beta-\frac{n''}{r''}} \Psi^{\mathbf{l}}(c-w) dw \\
&= \int_{H_{r''}^+ \cap (I-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} e^{\mathrm{tr}(wz)+\mathrm{tr}((I-w)c)} dw \\
&= \int_{H_{r''}^+ \cap (I-H_{r''}^+)} \det(w)^{\alpha-\frac{n''}{r''}} \det(I-w)^{\beta-\frac{n''}{r''}} \\
&\quad \times \sum_{\mathbf{m}, \mathbf{n}} \left( \sum_{\mathbf{l}} K_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(w, I-w; z, c) + K_{\mathbf{m}, \mathbf{n}}^{-\mathbf{l}}(w, I-w; z, c) \right) dw \\
&= \sum_{\mathbf{l}, \mathbf{m}, \mathbf{n}} \frac{\Gamma_{r''}(\alpha + \mathbf{m}) \Gamma_{r''}(\beta + \mathbf{n})}{\Gamma_{r''}(\alpha + \beta + \mathbf{l})} K_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(I, I; z, c) \\
&= \sum_{\mathbf{l}} \frac{\Gamma_{r''}(\alpha) \Gamma_{r''}(\beta)}{\Gamma_{r''}(\alpha + \beta)(\alpha + \beta)_{\mathbf{l}}} \sum_{\mathbf{m}, \mathbf{n}} (\alpha)_{\mathbf{m}} (\beta)_{\mathbf{n}} \tilde{\Psi}_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(z, c).
\end{aligned}$$

Then projecting both sides to  $\bigoplus_{\mathbf{m}, \mathbf{n}} \mathcal{P}_{\mathbf{m}, \mathbf{n}}^{\mathbf{l}}(H_{r''}^{\mathbb{C}} \oplus H_{r''}^{\mathbb{C}})$ , we get (A'') = (D'').

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