On Gauss-Bonnet type theorems

Cuilian Zhang, Donghe Pei
School of Mathematics and Statistics, Northeast Normal University

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1 Introduction

The Gauss-Bonnet theorem for Riemannian manifolds is an important subject in the global differential geometry (see [1-4, 7]). The fundamental meaning of the Gauss-Bonnet theorem is that it connects geometry to topology.

Later pseudo-Riemannian manifolds have captured much attention in general relativity, becoming a very popular subject of investigation. A natural problem is that if there is a Gauss-Bonnet type theorem for pseudo-Riemannian manifolds. Fortunately, the Gauss-Bonnet theorem was successfully extended to pseudo-Riemannian manifolds (see [5]).

Problems related to non-lightlike vectors are similar to cases of Riemannian manifolds. However, it becomes very different when coming to lightlike vectors. Lightlike vectors in the normal space of spacelike submanifolds in pseudo-Riemannian manifolds are of much interest. S. Izumiya, M. C. Romero Fuster and the second author did a lot of research about this (see [10-12]). They introduced the notion of the lightcone Gauss map and define the corresponding curvature for orientable even-dimensional hypersurfaces in hyperbolic space, spacelike submanifolds of codimension two in Minkowski space and spacelike surfaces in anti de Sitter 4-space. Furthermore, they gave the following Gauss-Bonnet type formulas which relate the lightcone Gauss map to the topology of the considering manifold.

Theorem 1.1. *([11]) If M is a closed orientable even-dimensional hypersurface in hyperbolic n-space, then*

$$\int_M \widetilde{K}_h d\sigma_M = \frac{1}{2}\gamma_{n-1}\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M, $d\sigma_M$ is the volume form of M and the constant γ_{n-1} is the volume of the unit $(n-1)$-sphere S^{n-1}.

Theorem 1.2. *([12]) If M is a closed orientable, spacelike submanifold of codimension two in $\mathbb{R}^{n+1}_1$. Suppose that n is odd. Then*

$$\int_M \widetilde{K}^{\pm}_l d\sigma_M = \frac{1}{2}\gamma_{n-1}\chi(M)$$

where $\chi(M)$ *is the Euler characteristic of* M*,* $d\sigma_M$ *is the volume form of* M *and the constant* γ_{n-1} *is the volume of the unit* $(n-1)$*-sphere* S^{n-1}*.*

Theorem 1.3. *([10]) If* M *is a closed orientable spacelike surfaces in* AdS^4*, then*

$$\int_M \hat{K}_l^{\pm} d\sigma_M = 2\pi\chi(M)$$

where $d\sigma_M$ *is the are form and* $\chi(M)$ *is the Euler number of* M*.*

With the developing of singularity theory, mathematicians began to focus on Gauss-Bonnet type formulas for singular manifolds. M. Kossowski [13] gave a Boy-Gauss-Bonnet theorem for singular surfaces with limiting tangent bundle as follows:

Theorem 1.4. *([13]) Given a generic 1-resolvable surface* $j : M \to \mathbb{E}^3$ *with trivial radical line bundle,* $RAD \to D^\circ$*, then:*

$$(a)\ \int_M KdA = 2\pi[\chi(M^+) - \chi(M^-) + \#C^+ - \#C^-,]$$

$$(b)\ \int_M K|dA| = 2\pi\chi(M) + 2\int_{D^\circ} \omega^\circ.$$

K. Saji, M. Umehara and K. Yamada [15-16] introduced Gauss-Bonnet formulas for compact fronts admitting at most peaks using the singular curvature function on cuspidal edges of surfaces as follows:

Theorem 1.5. *([16]) Let* M^2 *be a compact oriented 2-manifold and let* $f : M^2 \to (N^3,\ g)$ *be a front which admits at most peak singularities, and* Σ_f *the singular set of* f*. Then*

$$\int_{M^2} KdA + 2\int_{\Sigma_f} \kappa_s ds = 2\pi\chi(M^2), \tag{1}$$

and

$$\int_{M^2} Kd\hat{A} - \Sigma_{p:peak}(\alpha_+(p) - \alpha_-(p)) = 2\pi(\chi(M_+) - \chi(M_-)) \tag{2}$$

hold, where ds *is the arclength measure on the singular set.*

Later they [17] abstracted the notion of coherent tangent bundles from the study on fronts and then gave four Gauss-Bonnet type formulas for 2-dimensional cases as follows:

Theorem 1.6. *([17]) They suppose that both of the singular sets* Σ *and* $\Sigma_\#$ *consist of* A_2*-points and* A_3*-points. Then applying the two abstract formulas in [15-16] for* ϕ *and* ψ*, respectively, we have the following four Gauss-Bonnet formulas:*

$$\int_{M^2} Kd\hat{A} = 2\pi[\chi(M^+) - \chi(M^-) + S^+ - S^-],$$

$$\int_{M^2} K dA = 2\pi\chi(M^2) - 2\int_{\Sigma} \kappa d\tau,$$

$$\int_{M^2} K_{\#} d\hat{A}_{\#} = 2\pi[\chi(M^+_{\#}) - \chi(M^-_{\#}) + S^+_{\#} - S^-_{\#}]$$

and

$$\int_{M^2} K_{\#} dA_{\#} = 2\pi\chi(M^2) - 2\int_{\Sigma_{\#}} \kappa_{\#} d\tau_{\#},$$

where κ (resp. $\kappa_{\#}$) is the singular curvature function along A_2-points in Σ (resp. $\Sigma_{\#}$), $d\tau$ (resp. $d\tau_{\#}$) is the length element on the singular curve with respect to I (resp. III), S^+ (resp. S^-) is the number of positive (resp. negative) A_3-points of ϕ, and $S^+_{\#}$ (resp. $S^-_{\#}$) is the number of positive (resp. negative) A_3-points of ψ.

In [8], two positive semi-definite metrics have been abstracted from the observation on cuspidal edges and cross caps and then the following Gauss-Bonnet type formulas were obtained. These two metrics are called the Kossowski metric and the Whitney metric respectively.

Theorem 1.7. *([8]) Let $d\sigma^2$ be a Kossowski metric on a compact orientable 2-manifold M^2 without boundary. Suppose that $d\sigma^2$ admits at most A_2 or A_3-singularities. Then its Gaussian curvature K satisfies*

$$2\pi\chi(M^2) = \int_{M^2} K dA + 2\int_{\Sigma} \kappa_s d\tau$$

and

$$\chi_\epsilon := \frac{1}{2\pi}\int_{M^2} K d\hat{A} = \chi(M_+) - \chi(M_-) + \#S_+ - \#S_-,$$

where Σ denotes the singular set of the metric $d\sigma^2$, and κ_s is the singular curvature, τ is the arclength parameter of the singualr curve, χ_ϵ is the Euler characteristic of the oriented coherent tangent vector bundle $(\epsilon, <,>, D)$ associated to $d\sigma^2$, M_+ (resp. M_-) is the subset where $d\hat{A}$ is positively (resp. negatively) proportional to dA, and $\#S_+$ (resp. $\#S_-$) is the number of positive (resp. negative) A_3-points.

Theorem 1.8. *([8]) Let M^2 be a compact oriented manifold without boundary, and $d\sigma^2$ a Whitney metric in M^2. Then its Gaussian curvature K satisfies*

$$\frac{1}{2\pi}\int_{M^2} K dA = \chi(M^2),$$

that is, there is no defect at intrinsic cross cap singularities for the Gauss-Bonnet fromula.

Furthermore, K. Saji, M. Umehara and K. Yamada studied more general index formula for a bundle homomorphism from the tangent bundle to a vector bundle of the same rank (see [18]) as follows:

Theorem 1.9. *[18] Let $\phi : TM^n \to \epsilon$ be a homomorphism between the tangent bundle TM^n and an oriented vector bundle ϵ of rank n on M^n. Suppose that ϕ admits only A_k-singular points. We denote by $U_k (k = 2, \cdots, n)$ the set of A_k-singular points. When k is odd, we can define the positivity and negetivity of A_k-point. We denote by U_k^+ (resp. U_k^-) the set of positive (resp. negative) A_k-singular points. When $n = 2m$ is an even number, the Euler characteristic χ_ϵ of the vector bundle ϵ satisfies the following formula*

$$\chi_\epsilon = \chi(M_+^n) - \chi(M_-^n) + \Sigma_{j=1}^m (\chi(U_{2j+1}^+) - \chi(U_{2j+1}^-)),$$

where $\chi(M_+^n)$ (resp. $\chi(M_-^n)$) is the Euler characteristic of the subset M_+^n (resp. M_-^n) of M^n at which the co-orientation induced by ϕ is (resp. is not) compatible with the orientation of TM^n, the number $\chi(U_{2j+1}^+)$ (resp. $\chi(U_{2j+1}^-)$) is the Euler characteristic of U_{2j+1}^+ (resp. U_{2j+1}^-).

Moreover, W. Domitrz and M. Zwierzynski [6] introduced the Gauss-Bonnet theorem for coherent tangent bundles over surfaces with boundary.

Based on the above analysis, we are more interested in the Gauss-Bonnet type formula for singular surfaces in pseudo-Riemannian manifolds whose normal space contains lightlike vectors. We want to give Gauss-Bonnet type theorems for these complicated manifolds. As far as we know, there is no such research. This is the work we are doing now. There are two obstacles in this subject. One is that the considering manifold has singularities. The other one is that the normal space of this manifold contains lightlike vectors. As we all know, there is a Legendrian duality between the hyperbolic space and the de Sitter space (see [9]). Therefore fronts in these two spaces are well defined and we have the singular curvature for fronts along cuspidal edges (see [14]).

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 Preliminary

In this section, we will simply review some basic notions about fronts in the hyperbolic 3-space and the de Sitter 3-space.

Let $\mathbb{R}^4 = \{\mathbf{x} = (x_0,\ x_1,\ x_2,\ x_3) | x_i \in \mathbb{R},\ i = 0,\ 1,\ 2,\ 3\}$ be 4-dimensional vector space. We endow $\mathbb{R}^4$ with the pseudo scalar product as follows:

$$< \mathbf{x},\ \mathbf{y} >= -x_0 y_0 + \sum_{i=1}^{3} x_i y_i,$$

where $\mathbf{x} = (x_0,\ x_1,\ x_2,\ x_3)$ and $\mathbf{y} = (y_0,\ y_1,\ y_2,\ y_3)$. We call $(\mathbb{R}^4,\ <,>)$ the *Minkowski 4-space* denoted by $\mathbb{R}_1^4$. A vector $\mathbf{x} \in \mathbb{R}_1^4 \backslash \{\mathbf{0}\}$ is called *spacelike, timelike, or lightlike* if $< \mathbf{x},\ \mathbf{x} >$ is positive, negative or equals to zero, respectively. We call $\sqrt{| < \mathbf{x},\ \mathbf{x} > |}$ the *norm* of the vector $\mathbf{x}$ denoted by $\|\mathbf{x}\|$.

There are three model pseudo spheres in the Minkowski 4-space. They are the *de Sitter 3-space*, the *hyperbolic 3-space* and the *open lightcone* LC^* defined by

$$\begin{cases} S_1^3 = \{\mathbf{x} \in \mathbb{R}_1^4 | < \mathbf{x},\ \mathbf{x} >= 1\}, \\ H^3 = \{\mathbf{x} \in \mathbb{R}_1^4 | < \mathbf{x},\ \mathbf{x} >= -1\}, \\ LC^* = \{\mathbf{x} \in \mathbb{R}_1^4 | < \mathbf{x},\ \mathbf{x} >= 0,\ \mathbf{x} \neq \mathbf{0}\} \end{cases}$$

respectively.

For simplicity, we make the following notation:

$$Q_\epsilon = \begin{cases} H^3, & \epsilon = -, \\ S_1^3, & \epsilon = +. \end{cases}$$

In [9], S. Izumiya introduced four Legendrian dualities in $\mathbb{R}_1^4$. There are four contact manifolds as follows:

$$\begin{cases} \triangle_1 = \{(\mathbf{v},\ \mathbf{w}) \in H^3 \times S_1^3 | < \mathbf{v},\ \mathbf{w} >= 0\} \\ \triangle_2 = \{(\mathbf{v},\ \mathbf{w}) \in H^3 \times LC^* | < \mathbf{v},\ \mathbf{w} >= -1\} \\ \triangle_3 = \{(\mathbf{v},\ \mathbf{w}) \in LC^* \times S_1^3 | < \mathbf{v},\ \mathbf{w} >= 1\} \\ \triangle_4 = \{(\mathbf{v},\ \mathbf{w}) \in LC^* \times LC^* | < \mathbf{v},\ \mathbf{w} >= -2\} \end{cases}$$

The corresponding contact 1-forms are

$$\theta_i =< d\mathbf{v},\ \mathbf{w} > |\triangle_i,\ i = 1,\ 2,\ 3,\ 4$$

respectively. The corresponding Legendrian fibrations are

$$\pi_i(\mathbf{v},\ \mathbf{w}) = \mathbf{v},\ i = 1,\ 2,\ 3,\ 4$$

respectively.

Let M be an oriented 2-dimensional manifold. A map $f^\epsilon : M \to Q_\epsilon$ is called a *frontal* if there exists a map $f^{-\epsilon} : M \to Q_{-\epsilon}$ such that the map $L := (f^\epsilon, f^{-\epsilon}) : M \to \triangle_1$ is isotropic, that is

$$L^*\theta_1 = 0.$$

Moreover, a frontal $f^\epsilon : M \to Q_\epsilon$ is a *front* if the isotropic map $L = (f^\epsilon, f^{-\epsilon})$ is also an immersion. If $(f^\epsilon, f^{-\epsilon}) : M \to \triangle_1$ is isotropic, then we say that f^ϵ and $f^{-\epsilon}$ are $\triangle_1$*-dual* each other, $f^{-\epsilon}$ is a $\triangle_1$*-dual* of f^ϵ, and f^ϵ is a $\triangle_1$*-dual* of $f^{-\epsilon}$.

Now we have fronts in the hyperbolic 3-space and the de Sitter 3-space. We want to give a Gauss-Bonnet type theorem for singular surfaces in these spaces. We have obtained some related results and will continue more deeper research on this work in the future.

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School of Mathematics and Statistics, Northeast Normal University
Changchun 130024
P. R. China
E-mail address: zhangcl345@nenu.edu.cn, peidh340@nenu.edu.cn