数理解析研究所講究録 1372 巻 2004 年 63-69

A Representation of Solutions of Linear Difference Equations with Constant Coefficients

電気通信大学 内藤敏機 (Toshiki Naito)
The University of Electro-Communications
朝鮮大学校 申 正善 (Jong Son Shin)
Korea University

1 Introduction

Let $\mathbb{C}^d$ be a d-dimensional complex Euclidean space. In this paper we deal with the linear difference equation with constant coefficients of the form

$$x_{n+1} = Bx_n + b, \quad x_0 = w \in \mathbb{C}^d \tag{1}$$

where $b \in \mathbb{C}^d$, and B is a $d \times d$ complex matrix of the form

$$B = e^{\tau A}, \quad \tau > 0.$$

Recently, Kato, Naito and Shin [4] has given a representation of solutions of the difference equation (1) by using the matrix A. It induces naturally a representation of solutions of inhomogeneous periodic linear differential equations which shows asymptotic behaviors of solutions very clearly. Moreover, the initial values of solutions are completely classified according to the behavior of solutions.

In this paper we give two representations of solutions of the difference equation (1) ; one is given in terms of the matrix B and the other is given in terms of the matrix A. They look different from each other ; we will clear up the mechanism which is combining two representations.

2 Representations of Solutions of Difference Equations

If M is a $d \times d$ matrix with the spectrum $\sigma(M) = \{\gamma_1, \cdots, \gamma_t\}$, $\mathbb{C}^d$ is decomposed as the direct sum of generalized eigenspaces of M :

$$\mathbb{C}^d = \mathrm{M}_1 \oplus \cdots \oplus \mathrm{M}_t,$$

where $\mathrm{M}_i := \mathcal{N}((M-\gamma_i E)^{c_i})$ is the generalized eigenspace with the index c_i, and E is the unit matrix. Let $Q_i : \mathbb{C}^d \to \mathrm{M}_i$ be the projection of $\mathbb{C}^d$ onto M_i according to this decomposition. We call in short Q_i the projection onto the generalized eigenspace of γ_i.

Let $\lambda \in \sigma(A)$, $\mu \in \sigma(B)$ and $\mu = e^{\tau\lambda}$ throughout this paper. Let Q and P be the projections to the generalized eigenspaces of μ and λ, respectively. Denote by $k(\lambda)$ the index of λ and $d(\mu)$ the one of μ. Note that $\mu \neq 0$ since B is nonsingular. Put

$$A_{k,\lambda} = \frac{\tau^k}{k!}(A-\lambda E)^k, \quad B_{k,\mu} = \frac{1}{k!\mu^k}(B-\mu E)^k,$$

$$\varepsilon(z) = (e^z-1)^{-1}, \quad a(z) = (z-1)^{-1}, \quad (z \neq 1), \quad \varepsilon^{(k)}(z) = \frac{d^k}{dz^k}\varepsilon(z), \quad a^{(k)}(z) = \frac{d^k}{dz^k}a(z).$$

To give the two representations of the solution $x_n(w,b)$ of the difference equation (1), we will introduce vector's quantities that depend on w and b, while not on n.

If $\mu = e^{\tau\lambda} \neq 1$, then

$$X(w,b,A-\lambda E) = w + \sum_{k=0}^{k(\lambda)-1} \varepsilon^{(k)}(\tau\lambda)A_{k,\lambda}b,$$

$$X(w,b,B-\mu E,n) = \begin{cases} w + \displaystyle\sum_{k=0}^{n-1} \mu^k a^{(k)}(\mu)B_{k,\mu}b & (n \geq 1) \\ w & (n=0). \end{cases}$$

If $\mu = e^{\tau\lambda} = 1$, then

$$Y(w,b,A-\lambda E) = A_{1,\lambda}w + \sum_{k=0}^{k(\lambda)-1} B_k A_{k,\lambda}b,$$

$$Z(w,b,B-\mu E) = B_{1,\mu}w + b = (B-E)w + b,$$

where B_k is the Bernoulli number. We define the factorial function $(x)_k$ of the kth degree :

$$(x)_k = x(x-1)(x-2)\cdots(x-k+1).$$

Let us consider two representations of solutions of the difference equation (1). The solution $x_n(w,b)$ of the difference equation (1) is certainly given by

$$x_n(w,b) = B^n w + \sum_{k=0}^{n-1} B^k b = e^{n\tau A}w + \sum_{k=0}^{n-1} e^{k\tau A}b.$$

We will obtain Theorems 1, 2 below by deforming this formula in the different manner as follows. Since

$$QB^k = (\mu E + B - \mu E)^k Q = \sum_{j=0}^{k} \binom{k}{j} \mu^{k-j}(B-\mu E)^j Q,$$

in the first manner we rearrange the sum

$$\sum_{k=0}^{n-1} B^k Qb = \sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{k}{j}\mu^{k-j}(B-\mu E)^j Qb.$$

by collecting terms containing the same power of μ ; after long calculation, we get Theorem 1. The key tool in the arrangement is the following technical relation : if $\mu \neq 1$,

$$\sum_{i=k}^{n-1}(i)_k \mu^{i-k} = \frac{d^k}{d\mu^k}\sum_{i=k}^{n-1}\mu^i = \frac{d^k}{d\mu^k}\sum_{i=0}^{n-1}\mu^i = \frac{d^k}{d\mu^k}\frac{\mu^n-1}{\mu-1} = \frac{d^k}{d\mu^k}(\mu^n-1)a(\mu).$$

This is the main reason why $X(w,b,B-\mu E,n)$ contains the term $a^{(k)}(\mu)$.
Since $B = e^{\tau A}$, we have

$$Pe^{k\tau A} = e^{k\tau(\lambda E + A - \lambda E)}P = e^{k\tau\lambda}\sum_{j=0}^{\infty}k^j A_{j,\lambda}P = e^{k\tau\lambda}\sum_{j=0}^{k(\lambda)-1}k^j A_{j,\lambda}P.$$

In the second manner we rearrange the sum

$$\sum_{k=0}^{n-1} e^{k\tau A}P = \sum_{k=0}^{n-1}\sum_{j=0}^{k(\lambda)-1} k^j e^{k\tau\lambda}A_{j,\lambda}P = \sum_{j=0}^{k(\lambda)-1}\sum_{k=0}^{n-1} k^j e^{k\tau\lambda}A_{j,\lambda}P.$$

by collecting terms containing the same power of n ; after long calculation, we get Theorem 2. The key tool in the arrangement is the following technical relation. If $e^z \neq 1$,

$$\begin{aligned}\sum_{k=0}^{n-1} k^j e^{kz} &= \sum_{k=0}^{n-1}\frac{d^j}{dz^j}e^{kz} = \frac{d^j}{dz^j}\sum_{k=0}^{n-1}e^{kz} = \frac{d^j}{dz^j}\frac{e^{nz}-1}{e^z-1} \\ &= \frac{d^j}{dz^j}(e^{nz}-1)\varepsilon(z) = \sum_{i=0}^{j}\binom{j}{i}n^i e^{nz}\varepsilon^{(j-i)}(z) - \varepsilon^{(j)}(z).\end{aligned}$$

This is the main reason why $X(w,b,A-\lambda E)$ contains the term $\varepsilon^{(k)}(\tau\lambda)$. If $e^z = 1$,

$$\sum_{j=0}^{n-1} k^j = \sum_{i=1}^{j+1}\binom{j+1}{i}\frac{B_{j+1-i}}{j+1}n^i.$$

This is the main reason why $Y(w,b,A-\lambda E)$ contains the Bernoulli number.

Theorem 1 *Let $x_n(w,b)$ be the solution of the difference equation* (1). *Then $Qx_n(w,b)$ is expressed as follows :*

1) *If $\mu \neq 1$, then*

$$\begin{aligned} & Qx_n(w,b) \\ = & \mu^n \sum_{k=0}^{d(\mu)-1} (n)_k B_{k,\mu} X(Qw, Qb, B-\mu E, d(\mu)) - X(Qw, Qb, B-\mu E, d(\mu)) + Qw \\ = & B^n X(Qw, Qb, B-\mu E, d(\mu)) - X(Qw, Qb, B-\mu E, d(\mu)) + Qw. \end{aligned}$$

2) *If $\mu = 1$, then*

$$Qx_n(w,b) = \sum_{k=0}^{d(\mu)-1} \frac{1}{k+1}(n)_k B_{k,\mu} Z(Qw, Qb, B-\mu E) + Qw.$$

Theorem 2 ([4]) *Let $x_n(w,b)$ be the solution of the difference equation* (1). *Then $Px_n(w,b)$ is expressed as follows :*

1) *If $e^{\tau\lambda} \neq 1$, then*

$$\begin{aligned} & Px_n(w,b) \\ = & e^{n\tau\lambda} \sum_{j=0}^{k(\lambda)-1} n^j A_{j,\lambda} X(Pw, Pb, A-\lambda E) - X(Pw, Pb, A-\lambda E) + Pw \\ = & e^{n\tau A} X(Pw, Pb, A-\lambda E) - X(Pw, Pb, A-\lambda E) + Pw. \end{aligned}$$

2) *If $e^{\tau\lambda} = 1$, then*

$$Px_n(w,b) = \sum_{j=0}^{k(\lambda)-1} \frac{1}{j+1} n^{j+1} A_{j,\lambda} Y(Pw, Pb, A-\lambda E) + Pw.$$

3 A Transform of theorems

The formula in Theorems 1, 2 represent the same term, but look very differently. We fall into temptation to translate each other. Since it takes long pages to carry out this process, we only show technical lemmas employed in the proofs.

The Stirling number of the second kind is given by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{\alpha} \prod_{i=0}^{n} \frac{n!}{\alpha_i!(i!)^{\alpha_i}},$$

where the sum is taken over the all sequences $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers α_i such that $\sum_{i=1}^n \alpha_i = k$ and $\sum_{i=1}^n i\alpha_i = n$. The Bernoulli polynomial $B_k(x)$ is defined by the Maclaurin series such that

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k,$$

and Bernourlli number is defined by $B_k := B_k(0)$. For properties of the Stirling number and the Bernoulli number, refer to [6], [7], [8].

Lemma 3.1

$$\sum_{k=0}^{j} \frac{1}{k+1}(n)_{k+1} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} = \sum_{k=0}^{n-1} k^j = \frac{B_{j+1}(n) - B_{j+1}}{j+1}, \quad (j \geq 1).$$

Using the formula of Fáa di Bruno [2],[5] for the nth derivative of a composition of two functions, we have the following result.

Lemma 3.2 *If* $\mu = e^{\tau\lambda} \neq 1$,

$$\varepsilon^{(n)}(\tau\lambda) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{(-1)^k k! \mu^k}{(\mu - 1)^{k+1}} = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \mu^k a^{(k)}(\mu), \quad n \geq 0.$$

Lemma 3.3

$$\left(\sum_{i=1}^{k(\lambda)-1} A_{i,\lambda} \right)^k = k! \sum_{i=k}^{k(\lambda)-1} \left\{ \begin{matrix} i \\ k \end{matrix} \right\} A_{i,\lambda} \ (k \leq k(\lambda) - 1),$$

$$\left(\sum_{i=1}^{k(\lambda)-1} A_{i,\lambda} \right)^k = 0 \ (k > k(\lambda) - 1).$$

Lemma 3.4

$$B_{k,\mu} P = \left(\sum_{j=k}^{k(\lambda)-1} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} A_{j,\lambda} \right) P.$$

In particular, if $k \geq k(\lambda)$, *then* $(B - \mu E)^k P = 0$.

Lemma 3.5

$$\sum_{k=0}^{k(\lambda)-1} (n)_k B_{k,\mu} P = \sum_{k=0}^{k(\lambda)-1} n^k A_{k,\lambda} P, \quad n \geq 0.$$

Lemma 3.6 *The following relations hold true.*

1) *If* $\mu = e^{\tau\lambda} \neq 1$, *then*

$$\sum_{k=0}^{k(\lambda)-1} \mu^k a^{(k)}(\mu) B_{k,\mu} P = \sum_{k=0}^{k(\lambda)-1} \varepsilon^{(k)}(\tau\lambda) A_{k,\lambda} P.$$

2) *If* $\mu = e^{\tau\lambda} = 1$, *then*

$$\sum_{k=0}^{k(\lambda)-1} \frac{(-1)^k}{k+1}(B-E)^k P = \sum_{k=0}^{k(\lambda)-1} B_k A_{k,\lambda} P,$$

$$\sum_{k=0}^{k(\lambda)-1} \frac{(-1)^k}{k+1}(B-E)^{k+1} P = A_{1,\lambda} P.$$

In view of the relation

$$X(Pw, Pb, B-\mu E, k(\lambda)) = X(Pw, Pb, A-\lambda E),$$

we have the following result.

Lemma 3.7 *The following relations hold true.*
1) *If* $\mu = e^{\tau\lambda} \neq 1$, *then*

$$\begin{aligned} &\mu^n \sum_{k=0}^{k(\lambda)-1} (n)_k B_{k,\mu} X(Pw, Pb, B-\mu E, k(\lambda)) - X(Pw, Pb, B-\mu E, k(\lambda)) \\ = \;& e^{n\tau\lambda} \sum_{k=0}^{k(\lambda)-1} n^k A_{k,\lambda} X(Pw, Pb, A-\lambda E) - X(Pw, Pb, A-\lambda E). \end{aligned}$$

2) *If* $\mu = e^{\tau\lambda} = 1$, *then*

$$\begin{aligned} &\left(\sum_{k=0}^{k(\lambda)-1} \frac{1}{k+1}(n)_{k+1} B_{k,\mu}\right) Z(Pw, Pb, B-\mu E) \\ = \;& \left(\sum_{j=0}^{k(\lambda)-1} \frac{1}{j+1} n^{j+1} A_{j,\lambda}\right) Y(Pw, Pb, A-\lambda E). \end{aligned}$$

Using the spectral mapping theorem, we have $\{\lambda \in \sigma(A) \; : \; \mu = e^{\tau\lambda}\} = \{\lambda_1, \lambda_2, \cdots, \lambda_q\}$, and

$$Q = P_1 + P_2 + \cdots + P_q,$$

where P_i is the projection onto the generalized eigenspace of λ_i. This implies that $P_i Q = P_i$. Thus it holds clearly that

$$PX(Qw, Qb, B-\mu E, d(\mu)) = X(Pw, Pb, B-\mu E, k(\lambda)).$$

The following result is the main result in this section. The proof follows from Theorem 1 and Lemma 3.7.

Theorem 3 *Let $x_n(w,b)$ be the solution of the difference equation* (1). *Then $Px_n(w,b)$ is expressed as follows :*

1) *If $\mu = e^{\tau\lambda} \neq 1$, then*

$$
\begin{aligned}
&Px_n(w,b)\\
&= \mu^n \sum_{k=0}^{k(\lambda)-1} (n)_k B_{k,\mu} X(Pw, Pb, B-\mu E, k(\lambda)) - X(Pw, Pb, B-\mu E, k(\lambda)) + Pw\\
&= e^{n\tau\lambda} \sum_{k=0}^{k(\lambda)-1} n^k A_{k,\lambda} X(Pw, Pb, A-\lambda E) - X(Pw, Pb, A-\lambda E) + Pw.
\end{aligned}
$$

2) *If $\mu = e^{\tau\lambda} = 1$, then*

$$
\begin{aligned}
Px_n(w,b) &= \left(\sum_{k=0}^{k(\lambda)-1} \frac{1}{k+1}(n)_{k+1} B_{k,\mu}\right) Z(Pw, Pb, B-\mu E) + Pw\\
&= \sum_{k=0}^{k(\lambda)-1} \frac{1}{k+1} n^{k+1} A_{k,\lambda} Y(Pw, Pb, A-\lambda E) + Pw.
\end{aligned}
$$

References

[1] R. P. Agarwal, Difference equations and inequalities, Theory, Methods, and Applications, Marcel Dekker, New York-Basel-Hong Kong, 1992

[2] C. F. Fáa di Bruno, Note sur une nouvelle formule du calcul differentiel. Quart. J. Pure Appl. Math., 1(1855), 359-360.

[3] J. Kato, T. Naito and J.S. Shin, Bounded solutions and periodic solutions to linear differential equations in Banach spaces, Proceeding in DEAA, Vietnam J. of Math., 30(2002), 561-575.

[4] J. Kato, T. Naito and J.S. Shin, A characterization of solutions in linear differential equations with periodic forcing functions, submitted.

[5] S. G. Krantz and H. R. Parks, A Primer of Real Analytic Functions, Second Edition, Birkhauser, Boston-Basel-Berlin, 2002

[6] K. S. Miller, An Introduction to the Calculus of Finite Differences and Difference Equations, Dover Publications, New York, 1966.

[7] 差分法入門, 井上正雄, 広川書店, 1968.

[8] 荒川恒男,伊吹山知義,金子昌信,「ベルヌーイ数とゼータ関数」,牧野書店, 2002.