CUTTING AND PASTING OF MORSE FUNCTIONS

DOMINIK J. WRAZIDLO

Institute of Mathematics for Industry, Kyushu University

Abstract. Cobordism groups of various types of Morse functions have been studied separately by several authors including Ikegami, Kalmàr, Saeki, Yamamoto, and the author. In this article, we propose a conceptually new approach for studying cobordism groups of several types of Morse functions within a single unifying framework. Our method is crucially based on certain cutting and pasting relations for manifolds that have been used before to define SKK-groups of manifolds. We provide an explicit isomorphism between the cobordism group of Morse functions and SKK-groups. Moreover, we sketch an application of our framework to cobordism theory for Morse functions with boundary, and raise some problems for future study concerning Morse functions with index constraints and circle-valued Morse functions.

1. Introduction

The purpose of this paper is to discuss a structural connection between cobordism groups of Morse functions on the one hand, and SKK-groups of manifolds on the other hand. Conceptually, we combine Morse theory with the cutting and pasting relations for manifolds that appear in the definition of SKK-groups. We expect that our approach allows to study cobordism groups of various types of Morse functions within one unifying framework. Details and further applications will be worked out in [27].

In general, cobordism groups of differentiable maps with prescribed types of singularities can almost always be studied by means of stable homotopy theory. The topic originates from René Thom's study of smooth oriented cobordism groups of manifolds [23]. Considering manifolds and cobordisms to be embedded into Euclidean space, Thom was able to study cobordism groups of manifolds by means of the Pontrjagin-Thom construction. In [15], Rimányi-Szűcs used a sort of Pontrjagin-Thom construction to derive fundamental results for the cobordism theory of differentiable maps with certain prescribed types of singularities. Their results have been further extended by several authors including Ando [1], Kalmàr [9], Sadykov [17], and Szűcs [22].

2010 *Mathematics Subject Classification.* 57R45, 57R90, 57R60, 57R65, 58K15, 57R56.

Key words and phrases. Morse function, cobordism of smooth maps, SKK-group.

This is a survey article based on the author's conference talk at the RIMS Symposium "Research on topology and differential geometry using singularity theory of differentiable maps" (Dec. 4 to Dec. 7, 2018).

Given a real-valued function $h\colon U \to \mathbb{R}$ defined on an n-manifold, a critical point x of h is called non-degenerate if there is a chart of U centered at x in which h takes the form

$$(x_1, \ldots, x_n) \mapsto h(x) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2. \tag{1.1}$$

In this paper, by a Morse function on a closed smooth manifold we mean a smooth real-valued function which has only non-degenerate singularities. A precise definition of the notion of cobordism of Morse functions involves the notion of fold maps, and will be given in Definition 2.1. For studying cobordism groups of Morse functions it seems most convenient to use more geometric-topological methods like Stein factorization and Levine's cusp elimination technique. In the following, let us discuss existing results concerning the study of cobordism groups of various types of Morse functions.

(a) Saeki [18] has studied the cobordism group of so-called special generic functions, namely Morse functions with only minima and maxima as their critical points. In dimension 6 and higher, these groups turn out to be isomorphic to the groups of h-cobordism classes of oriented homotopy spheres (see [12]). In [24], the author has imposed more general index constraints that allow Morse functions to have critical points of certain indefinite indices apart from minima and maxima, and the author has studied cobordism groups of such "constrained" Morse functions. As an interesting consequence, it follows that exotic Kervaire spheres are distinguished from other exotic spheres as elements of these groups in infinitely many dimensions (see also [25]).

(b) In 2004, Ikegami [3] determined the complete structure of cobordism groups $\mathcal{M}_n$ ($\mathcal{N}_n$) of Morse functions on (un-)oriented n-manifolds for any $n \geq 1$. This generalized previous results of Ikegami-Saeki [4] for Morse functions on oriented surfaces, and of Kalmàr [8] for Morse functions on unoriented surfaces. We point out that in the oriented version of Ikegami's structure theorem (see Theorem 2.3), the Kervaire semi-characteristic [14] appears in dimensions of the form $n = 4k + 1$.

(c) In [5], Ikegami-Saeki extended the work of Ikegami [3] to cover the case of circle-valued Morse functions. As an application to target oriented topological types of generic map germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^2, 0)$, $m \geq 2$, they identify an invariant given by the sum of signs associated to the cusps of a suitable stable perturbation.

(d) Later, Saeki-Yamamoto [19, 20, 21] introduced several notions of cobordism for Morse functions on manifolds with boundary (see Section 5.2), and computed the so-called admissible cobordism group of Morse functions on surfaces by means of the cohomology of the universal complex of singular fibers, and a combinatorial argument using labeled Reeb graphs. Recently, Yamamoto [28] has used similar techniques to compute the fold cobordism group of Morse functions on surfaces. In Section 5, we will indicate how the perspective taken in this paper leads to the computation of higher-dimensional admissible cobordism groups of Morse functions on manifolds with boundary (see [26]). We expect our method to be applicable as well to the computation of higher-dimensional fold cobordism groups (see Theorem 5.7).

The purpose of this paper is to discuss a new structural connection between cobordism groups of Morse functions on the one hand, and so-called SKK-groups of manifolds on the other hand. For simplicity, we focus on the case that all manifolds are oriented, while a version of our results for unoriented manifolds could be derived in a similar way.

Historically, the concept of SKK-groups was motivated by the observation of Jänich [6, 7] that the index of elliptic operators behaves invariant under a natural cutting and pasting operation on manifolds. This operation cuts a closed n-manifold along a submanifold Σ of codimension 1 with trivial normal bundle, and pastes back together the two resulting copies of Σ in the boundary by means of some gluing automorphism $\Sigma \to \Sigma$. The resulting abstract notion of SK-invariants (from German "Schneiden und Kleben" = "cutting and pasting") was studied systematically in [11] by viewing SK-invariants as homomorphisms on a universal SK-group SK_n with values in some abelian group. As a generalization, the notion of SKK-invariants (from German "SK-Kontrollierbar" = "SK-controllable") and the corresponding universal SKK-group SKK_n incorporate a correction term that may depend on the gluing automorphism $L \to L$. In dimensions of the form $n = 4k + 1$, the Kervaire semi-characteristic [14] turns out to be an SKK-invariant, and appears in fact in the structure theorem for the SKK-group (see Theorem 3.4). We observe that Ikegami's structure theorem for the cobordism group of Morse functions on oriented manifolds (see Theorem 2.3) involves the Kervaire semi-characteristic as well, which suggests a structural connection between the groups $\mathcal{M}_n$ and SKK_n. In [27], we construct an isomorphism that clarifies the precise relation. Let us present this isomorphism in the following.

For a Morse function $f\colon M \to \mathbb{R}$ on an oriented closed n-manifold M, we denote by $\nu_i(f)$ the number of critical points of f of index i. Moreover, we set $\mu_i(f) = \nu_{n-i}(f) - \nu_i(f)$, and define the integer

$$\Sigma(M, f) = \begin{cases} \nu_0(f) + \cdots + \nu_k(f), & n = 2k+1, \\ \mu_{k-1}(f) + \frac{\sigma(M) - \chi(M)}{2}, & n = 2k. \end{cases} \tag{1.2}$$

Note that when $n \not\equiv 0 \bmod 4$, the integer $\Sigma(M, f)$ depends only on the numbers $\nu_i(f)$ because $\chi(M) = \sum_{i=0}^{n} (-1)^i \nu_i(f)$ and $\sigma(M) = 0$. However, when $n \equiv 0 \bmod 4$, then there is an additional dependence in (1.2) on the signature $\sigma(M)$, which is required to make $\Sigma(M, f)$ an integer.

Theorem 1.1 (W. [27], 2018). *There is an isomorphism of groups*

$$\mathcal{M}_n \xrightarrow{\cong} SKK_n \oplus \mathbb{Z}^{\lfloor (n-1)/2 \rfloor},$$
$$[f\colon M \to \mathbb{R}] \mapsto ([M] + \Sigma(M, f) \cdot [S^n], \mu_0(f), \ldots, \mu_{\lfloor (n-1)/2 \rfloor - 1}(f)).$$

Our Theorem 1.1 provides a specific isomorphism that is not directly obtained by combining Ikegami's structure theorem for $\mathcal{M}_n$ (see Theorem 2.3) with the structure theorem for SKK_n (see Theorem 3.4). In fact, note that $\sigma(M)$, the signature of M, does not appear in those structure theorems.

Our proof of Theorem 1.1, which will be outlined in Section 4, combines many parts of the proofs of the original structure theorems, but there will be several new aspects. For instance, the signature of manifolds will appear as a cobordism invariant in the construction of the homomorphism $\mathcal{M}_n \to SKK_n$. Moreover, the proof of injectivity of this homomorphism will be crucially based on our method of cutting and pasting of Morse functions (see Theorem 4.4). Our result turns out to be useful in that it can serve as a model for studying many different variants of cobordism groups of Morse functions, as described in Section 5.

Furthermore, we point out that our approach has the potential to pave the way to directions for future study as follows. A natural generalization of the topic is to raise the dimension of the target space of smooth maps. Thus, one future goal will consist in studying cobordism theory for fold maps into higher dimensional target spaces by means of our approach. For instance, recent work of Kalmàr [10] clarifies the structure of cobordism groups of fold maps into the plane. In view of our framework proposed in Section 5, it then seems natural to search for a structure that substitutes the concept of SKK-groups in the case of higher target dimensions. Such higher analogs of SKK-groups might then be related to certain extended topological quantum field theories (TQFTs) in a similar way as SKK-invariants are related to so-called invertible TQFTs by the short exact sequence derived in [16].

The paper is organized as follows. In Section 2 we discuss Ikegami's structure result for oriented cobordism groups of Morse functions. In Section 3 we review the definition of SKK-groups based on the concept of cutting and pasting of manifolds, and discuss the structure result of SKK-groups in Theorem 3.4. The proof of our Theorem 1.1 is outlined in Section 4. Finally, in Section 5, we propose in an informal way a unifying framework for studying cobordism groups of various types of Morse functions, and illustrate our ideas by means of our results concerning Morse functions on compact manifolds with boundary.

All manifolds and maps between manifolds considered in this note will be differentiable of class C^∞. For a closed oriented n-manifold M^n, the manifold with opposite orientation will be denoted by $-M^n$.

Acknowledgements. The author is grateful to Professor Osamu Saeki for invaluable comments and discussions.

During the creation of this work, the author was a JSPS International Research Fellow (Postdoctoral Fellowships for Research in Japan (Standard)).

2. Cobordism groups of Morse functions

We start by introducing the fundamental notion of a cobordism of Morse functions.

Definition 2.1 (cobordism of Morse functions)**.** Two Morse functions $f_0\colon M_0 \to \mathbb{R}$ and $f_1\colon M_1 \to \mathbb{R}$ on oriented closed n-manifolds M_0 and M_1 are *cobordant* (see Figure 1) if

- there exists a cobordism W^{n+1} from M_0 to M_1, i.e., W is an oriented compact manifold with boundary $M_0 \sqcup -M_1$, and

- there exists a map $F\colon W \to [0,1] \times \mathbb{R}$ which has only *fold points* as its singular points, where by a fold point we mean a critical point $x \in W$ of F for which F takes in suitable charts centered at x and $F(x)$, respectively, the form

$$(x_0, x_1, \ldots, x_n) \mapsto (x_0, -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2), \tag{2.1}$$

  and there exist collar neighborhoods $[0,\varepsilon) \times M_0 \subset W$ of $M_0 \subset W$ and $(1-\varepsilon, 1] \times M_1 \subset W$ of $M_1 \subset W$ such that $F|_{[0,\varepsilon)\times M_0} = \mathrm{id}_{[0,\varepsilon)} \times f_0$ and $F|_{(1-\varepsilon,1]\times M_1} = \mathrm{id}_{(1-\varepsilon,1]} \times f_1$.

Remark 2.2. It is a basic fact from Morse theory that for every non-degenerate critical point x of some real-valued function $h\colon U \to \mathbb{R}$, the integer $i \in \{0, \ldots, n\}$ that appears in the standard quadratic form $-x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$ in (1.1) is independent of the choice of the coordinate chart, and is called the *(Morse) index* of h at x. Similarly, for every fold point x of a map $F\colon W \to \mathbb{R}^2$, the integer $\max\{i, n-1-i\} \in \{\lceil n/2 \rceil \ldots, n\}$ that derives from the standard quadratic form $-x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$ in (2.1) is independent of the choice of the coordinate charts, and is called the *(absolute) index* of F at x. Compared to the index of a non-degenerate critical point, the indeterminacy between the indices i and $n-1-i$ of a fold point comes from the fact that in order to produce the normal form we are allowed to choose coordinate charts both in the domain centered at the fold point x, and in the codomain centered at the image point $F(x)$.

A map $F\colon W \to [0,1] \times \mathbb{R}$ with only fold singularities as in Definition 2.1 is called a *fold map*. If F behaves near the boundary of W as required by Definition 2.1, then it is well-known that the singular set $S(F) \subset W$ of the fold map F is a 1-dimensional submanifold which is closed as a subset, and F restricts to an immersion $S(F) \to \mathbb{R}^2$ (see Figure 1). The absolute index of fold points (see Remark 2.2) is constant along fold lines, i.e., components of $S(F)$.

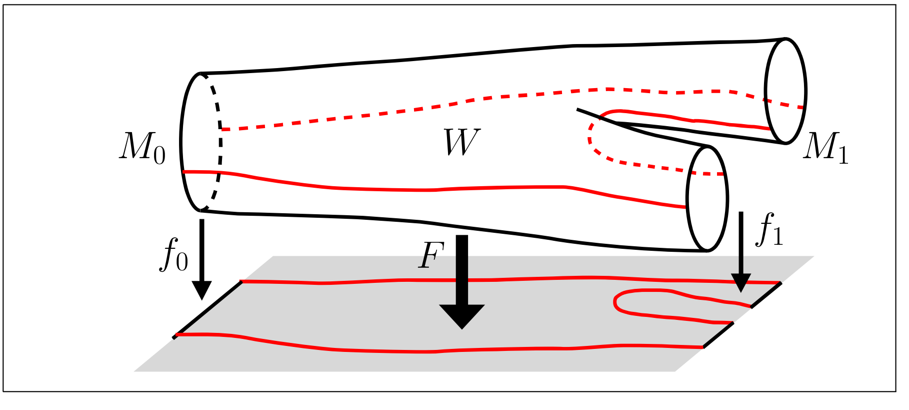

FIGURE 1. Example of a cobordism from $f_0\colon S^1 \to \mathbb{R}$ to $f_1\colon S^1 \sqcup S^1 \to \mathbb{R}$.

It can be checked that the notion of cobordism in the sense of Definition 2.1 determines an equivalence relation on the set of all Morse functions on oriented closed n-manifolds. Let $\mathcal{M}_n$ denote the set of equivalence classes $[f\colon M \to \mathbb{R}]$ represented by Morse functions on oriented closed n-manifolds. Disjoint union "$\sqcup$" induces a group law on the set $\mathcal{M}_n$ as follows. The identity element is represented by unique map $\emptyset \to \mathbb{R}$, and the inverse of a class $[f\colon M \to \mathbb{R}]$ is represented by $-f\colon -M \to \mathbb{R}$, $x \mapsto -f(x)$, where $-M^n$ denotes the manifold M equipped with the opposite orientation. We call $\mathcal{M}_n$ the *oriented cobordism group of Morse functions (on closed n-manifolds).*

Let us discuss Ikegami's structure result [3] for the cobordism group $\mathcal{M}_n$. For this purpose, recall that for a Morse function $f\colon M^n \to \mathbb{R}$ on an oriented closed n-manifold M we denote the number of critical points of f of index i by $\nu_i(f)$, and set

$$\mu_i(f) = \nu_{n-i}(f) - \nu_i(f) \ (= \nu_i(-f) - \nu_i(f)).$$

Let Ω_n^{SO} denote the smooth oriented cobordism group of dimension n. We shall need a torsion group defined by

$$J_n = \begin{cases} \mathbb{Z}/2, & n \equiv 1 \pmod 4, \\ 0, & n \not\equiv 1 \pmod 4. \end{cases}$$

Moreover, using the Kervaire semi-characteristic [14], we define $\Gamma(M,f) \in J_n$ by

$$\Gamma(M^{4k+1}, f) \equiv \sum_{i=0}^{2k} \nu_i(f) + \sum_{i=0}^{2k} \dim H_i(M^{4k+1}; \mathbb{Q}) \pmod 2.$$

Theorem 2.3 (Ikegami [3], 2004)**.** *There is an isomorphism of groups*

$$\begin{gathered} \mathcal{M}_n \xrightarrow{\cong} \Omega_n^{SO} \oplus \mathbb{Z}^{\lfloor n/2 \rfloor} \oplus J_n, \\ [f\colon M \to \mathbb{R}] \mapsto ([M], \mu_0(f), \ldots, \mu_{\lfloor n/2 \rfloor - 1}(f), \Gamma(M,f)). \end{gathered}$$

3. Cutting and pasting of manifolds; SKK-groups

The material of this section is taken from the manuscript [11], where U. Karras, M. Kreck, W.D. Neumann, and E. Ossa study the concepts of SK-groups and SKK-groups in a systematic way.

Let us introduce the fundamental notion of SKK-relation on n-manifolds.

Definition 3.1 (SKK-relation)**.** Two closed oriented n-manifolds X and Y are called *SKK-related*, and we write $X \overset{SKK}{\longrightarrow} Y$ (see Figure 2) if there exist compact oriented n-manifolds M, M', N, N' with boundaries $\partial M = \partial N$ and $\partial M' = \partial N'$, and orientation preserving diffeomorphisms $\varphi\colon \partial M \to \partial M'$ and $\psi\colon \partial N \to \partial N'$ such that

$$\begin{aligned} X &= \left(M \cup_\varphi -M'\right) \sqcup \left(N \cup_\psi -N'\right), \\ Y &= \left(M \cup_\psi -M'\right) \sqcup \left(N \cup_\varphi -N'\right). \end{aligned}$$

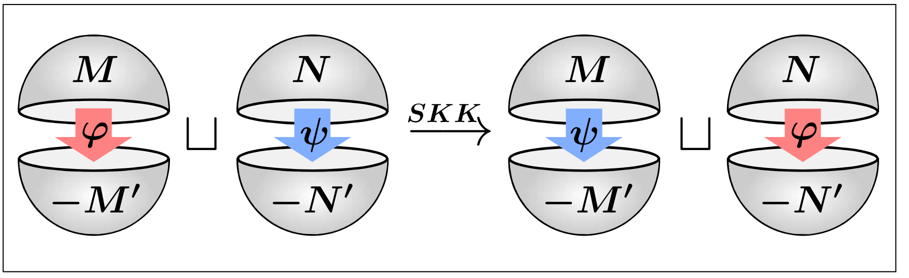

FIGURE 2. SKK-related oriented n-manifolds X and Y.

Let $\mathfrak{M}_n$ denote the set of oriented diffeomorphism classes of closed oriented n-manifolds. We regard $\mathfrak{M}_n$ as an abelian semigroup via $[M]+[N]=[M \sqcup N]$ and $0=[\emptyset]$.

While the SKK-relation on $\mathfrak{M}_n$ given by Definition 3.1 is obviously symmetric, it might not be an equivalence relation. Nevertheless, we can use the SKK-relation to define an equivalence relation $\sim_{SKK}$ via stabilization as follows. Given tow closed oriented n-manifolds M and N, we say $[M] \sim_{SKK} [N]$ if there exist closed oriented n-manifolds X and Y such that $X \overset{SKK}{\longrightarrow} Y$ and $[M]+[X]=[N]+[Y]$ in $\mathfrak{M}_n$. Then, it is straightforward to check that "$\sim_{SKK}$" is an equivalence relation on $\mathfrak{M}_n$, and the quotient $\mathfrak{M}_n/\sim_{SKK}$ inherits an abelian semigroup structure from $\mathfrak{M}_n$. We define SKK_n to be the Grothendieck group of $\mathfrak{M}_n/\sim_{SKK}$. In particular, note that an element of SKK_n is not always represented by a manifold, but can in general be written as a difference $[M]-[N]$.

Example 3.2. *It follows from the construction of SKK_n that M^n represents $0 \in SKK_n$ (that is, $[M] \sim_{SKK} [\emptyset]$) if and only if there exist X and Y such that $X \overset{SKK}{\longrightarrow} Y$ and $M \sqcup X \cong Y$. For instance, Figure 3 shows explicitly why the torus $M=T^2$ represents $0 \in SKK_2$. On the other hand, the structure result for SKK_n below (see Theorem 3.4) implies that S^2 represents a generator of $SKK_2 \cong \mathbb{Z}$.*

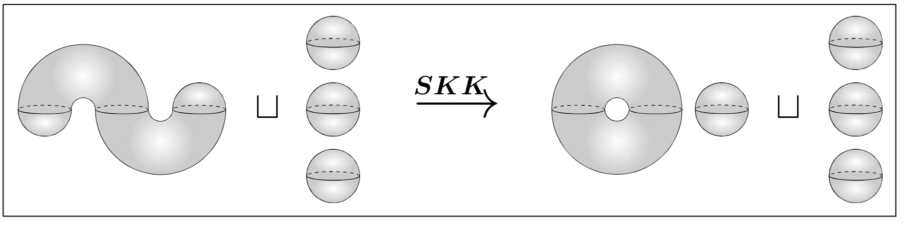

FIGURE 3. The surface $X=\bigsqcup_{k=1}^4 S^2$ is SKK-related to the surface $T^2 \sqcup X$.

Remark 3.3. Note that (in contrast to cobordism groups) the inverse of an element $[M] \in SKK_n$ is not necessarily represented by $-M$ (M with the reversed orientation). However, the structure result for SKK_n below (see Theorem 3.4) implies that $-[M]=[-M]+m \cdot [S^n]$ for some $m \in \mathbb{Z}$.

Next, let us state the structure result for SKK_n, in which the cyclic group

$$I_n = \begin{cases} \mathbb{Z}, & n \equiv 0 \pmod 2, \\ \mathbb{Z}/2, & n \equiv 1 \pmod 4, \\ 0, & n \equiv 3 \pmod 4 \end{cases} \tag{3.1}$$

appears.

Theorem 3.4 (Jänich, Karras-Kreck-Neumann-Ossa [11]). *There is a split exact sequence of abelian groups*

$$0 \longrightarrow I_n \xrightarrow{\alpha} SKK_n \xrightarrow{\beta} \Omega_n^{SO} \longrightarrow 0,$$

where α maps $1 \in \mathbb{Z}$ and $\overline{1} \in \mathbb{Z}/2$ to $[S^n]$, and $\beta([M]) = [M]$.

Moreover, a splitting of α is induced by

$$[M] \mapsto \begin{cases} \textit{Euler characteristic of } M, & n \equiv 0 \pmod 4, \\ \textit{Kervaire semi-characteristic of } M, & n \equiv 1 \pmod 4, \\ \textit{half of Euler characteristic of } M, & n \equiv 2 \pmod 4. \end{cases}$$

Remark 3.5. Note that there is an unoriented version SKK_n^O of SKK_n, and the corresponding structure result is based on the cyclic group

$$I_n^O = \begin{cases} \mathbb{Z}, & n \equiv 0 \pmod 2, \\ 0, & n \equiv 1 \pmod 2. \end{cases}$$

Namely, there is a split exact sequence of abelian groups

$$0 \longrightarrow I_n^O \xrightarrow{\alpha} SKK_n^O \xrightarrow{\beta} \Omega_n^O \longrightarrow 0,$$

and for n even, a splitting of α is induced by the Euler characteristic.

4. Sketch of proof of Theorem 1.1

Let us start with the construction of the map

$$\mathcal{M}_n \to SKK_n, \quad [f\colon M \to \mathbb{R}] \mapsto [M] + \Sigma(M, f) \cdot [S^n].$$

The following lemma shows that the above assignment is well-defined. Since the group laws on $\mathcal{M}_n$ and SKK_n are both induced by disjoint union, "$\sqcup$", it then follows automatically that our map is a group homomorphism.

Lemma 4.1. *If the Morse functions $f_0\colon M_0 \to \mathbb{R}$ and $f_1\colon M_1 \to \mathbb{R}$ are cobordant, then*

$$[M_0] + \Sigma(M_0, f_0) \cdot [S^n] = [M_1] + \Sigma(M_1, f_1) \cdot [S^n] \quad \in SKK_n.$$

Proof. Fix a cobordism $F\colon W \to [0,1] \times \mathbb{R}$ from f_0 to f_1 with the properties stated in Definition 2.1. Without loss of generality, we may assume that $\mathrm{pr}_{[0,1]} \circ F\colon W \to [0,1]$ is a Morse function with exactly one critical point, say of index i. Note that by classical Morse theory, W is the trace of the surgery on an embedding $S^{i-1} \times D^{n-i+1} \subset M_0$. Then, in order to compare the summands in SKK_n that are induced by f_0 and f_1, we show the following observations:

(1) $[M_1] = [M_0] + (-1)^i \cdot [S^n]$.

We omit the proof, which is based on the methods used in the manuscript [11], and is independent of singularity theory.

Remark 4.2. For later reference, we remark that $2 \cdot [S^n] = 0$ whenever n is odd. In fact, this follows by taking $i = n+1$, $M_0 = S^n$, and $M_1 = \emptyset$ in (1).

(2) $\Sigma(M_1, f_1) \cdot [S^n] = \Sigma(M_0, f_0) \cdot [S^n] - (-1)^i \cdot [S^n]$.

In order to show this claim, we observe first that the non-degenerate critical point of $\mathrm{pr}_{[0,1]} \circ F$ of index i can arise in the ways (a) and (b) illustrated in Figure 4.

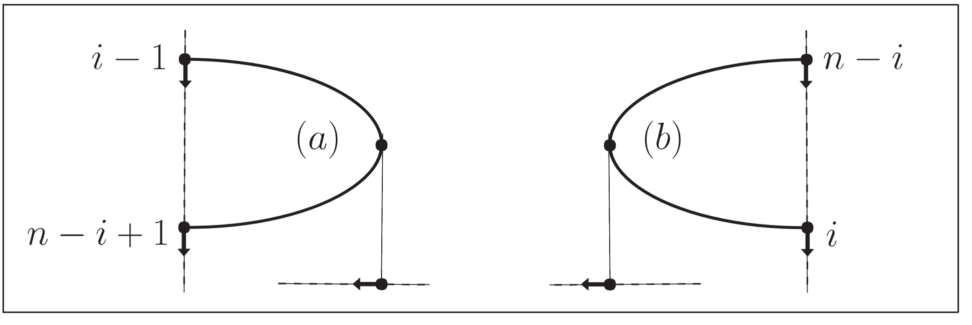

FIGURE 4. Morse critical points of $\mathrm{pr}_{[0,1]} \circ F$ of index i can arise from fold points of F that have either absolute index $\max\{i-1, n-i+1\}$ when $i > 0$ (case (a)), or absolute index $\max\{i, n\}$ (case (b)).

Then, we can check the claim by examining the definition of $\Sigma(M, f)$ (see (1.2)) in the following two cases:

- If $n = 2k+1$, then $\Sigma(M, f) = \nu_0(f) + \cdots + \nu_k(f)$. In this case, it is not hard to show by means of Figure 4 that $\Sigma(M_1, f_1) = \Sigma(M_0, f_0) \pm 1$ in either of the cases (a) and (b). Then, the claim follows in view of Remark 4.2.
- If $n = 2k$, then $\Sigma(M, f) = \mu_{k-1}(f) + \frac{\sigma(M) - \chi(M)}{2}$. We note that $\mu_{k-1}(f_0) = \mu_{k-1}(f_1)$. Moreover, we have $\sigma(M_0) = \sigma(M_1)$ because the signature is an oriented cobordism invariant. Finally, it is not hard to check in either of the cases (a) and (b) that $\chi(M_1) = \chi(M_0) + 2 \cdot (-1)^i$.

This completes the proof of our lemma. □

Surjectivity of our homomorphism $\mathcal{M}_n \to SKK_n \oplus \mathbb{Z}^{\lfloor (n-1)/2 \rfloor}$ is implied by the following lemma.

Lemma 4.3. *If* $[M] \in SKK_n$ *and* $a_0, \dots, a_{\lfloor (n-1)/2 \rfloor -1} \in \mathbb{Z}$, *then there is a Morse function* $f \colon M \to \mathbb{R}$ *such that* $\Sigma(M, f) \cdot [S^n] = 0$ *and* $\mu_j(f) = a_j$, $j = 0, \dots, \lfloor (n-1)/2 \rfloor - 1$.

Proof. The main idea is to construct f from some initially chosen Morse function $g \colon M \to \mathbb{R}$ by creating new pairs of critical points with *successive* indices. Firstly, we can adjust $\mu_0(g) = \nu_n(g) - \nu_0(g)$ to a_0 by creating new pairs of critical points of indices $0, 1$ or $n-1, n$. Next, we adjust $\mu_1(g) = \nu_{n-1}(g) - \nu_1(g)$ to a_1 by creating new pairs of critical points of indices $1, 2$ or $n-2, n-1$. (This does not affect our previous achievement that $\mu_0(g) = a_0$.) This process can be repeated until we have $\mu_j(g) = a_j$, $j = 0, \dots, \lfloor (n-1)/2 \rfloor - 1$. Finally, in the case $n = 2k+1$, we can create a critical point pair of indices $k, k+1$ to modify $\nu_k(g)$ if necessary to make $\Sigma(M, g) = \nu_0(g) + \cdots + \nu_k(g)$ even (compare Remark 4.2). In the case $n = 2k$, we can create additional critical point pairs of indices $k-1, k$ and $k, k+1$ to adjust $\mu_{k-1}(g) = -\frac{\sigma(M) - \chi(M)}{2}$ in such a way that $\Sigma(M, g) = 0$. □

The proof of injectivity of our homomorphism $\mathcal{M}_n \to SKK_n \oplus \mathbb{Z}^{\lfloor (n-1)/2 \rfloor}$ will require the following

Theorem 4.4 (cutting and pasting of Morse functions)**.** *Given closed oriented n-manifolds X and Y such that* $X \overset{SKK}{\longrightarrow} Y$ *(see Definition 3.1), there exist Morse functions* $g_X \colon X \to \mathbb{R}$ *and* $g_Y \colon Y \to \mathbb{R}$ *that are cobordant in the sense of Definition 2.1.*

Proof. By Definition 3.1, we can use the obvious terminology to write

$$X = (M_1 \cup_\varphi -M_1') \sqcup (M_2 \cup_\psi -M_2'),$$
$$Y = (M_1 \cup_\psi -M_1') \sqcup (M_2 \cup_\varphi -M_2').$$

We fix collars $\partial M_i \times [0, \varepsilon) \subset M_i$, $i = 1, 2$, and extend the projections $\partial M_i \times [0, \varepsilon) \to [0, \varepsilon)$ to Morse functions $g_i \colon M_i \to [0, \infty)$. Similarly, we fix collars $\partial M_i' \times [0, \varepsilon) \subset M_i'$, $i = 1, 2$, and extend the projections $\partial M_i' \times [0, \varepsilon) \to [0, \varepsilon)$ to Morse functions $g_i' \colon M_i' \to [0, \infty)$. As indicated in Figure 5, we are then able to construct in the sense of Definition 2.1 a nullcobordism of

$$\left(g_1' \cup_\psi -g_1\right) \sqcup \left(g_1 \cup_\varphi -g_1'\right) \sqcup \left(g_1' \cup_\partial -g_2'\right) \sqcup \left(g_2' \cup_\varphi -g_2\right) \sqcup \left(g_2 \cup_\psi -g_2'\right) \sqcup \left(g_2' \cup_\partial -g_1'\right),$$

and the claim follows. □

In order to show injectivity of our map $\mathcal{M}_n \to SKK_n \oplus \mathbb{Z}^{\lfloor (n-1)/2 \rfloor}$, let us suppose that the Morse function $f \colon M \to \mathbb{R}$ satisfies $[M] + \Sigma(M, f) \cdot [S^n] = 0$ in SKK_n, and $\mu_0(f) = \cdots = \mu_{\lfloor (n-1)/2 \rfloor - 1}(f) = 0$. In particular, there exist integers $a, b \geq 0$ and closed oriented n-manifolds X and Y such that $X \overset{SKK}{\longrightarrow} Y$ (see Definition 3.1) and

$$M \sqcup \bigsqcup_{i=1}^{a} S^n \sqcup X \cong \bigsqcup_{i=1}^{b} S^n \sqcup Y.$$

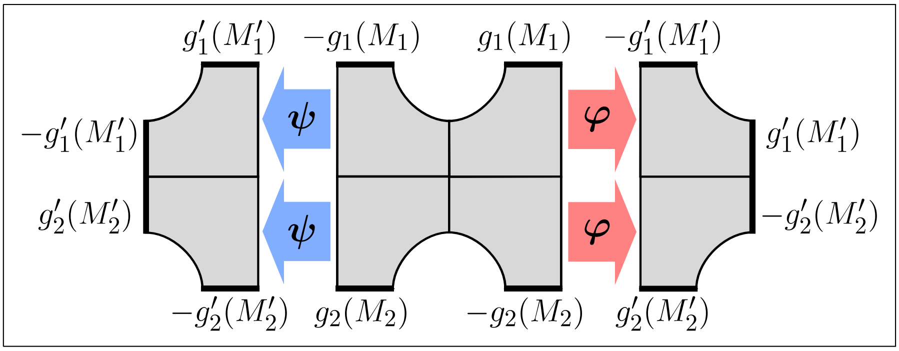

FIGURE 5. Construction of a cobordism of Morse functions via the method of cutting and pasting of Morse functions.

Using the previous diffeomorphism as well as the fact that the standard height function $S^n \to \mathbb{R}$ is nullcobordant in $\mathcal{M}_n$, we conclude from Theorem 4.4 that there exist Morse functions

$$g\colon M \sqcup \bigsqcup_{i=1}^{a} S^n \sqcup X \to \mathbb{R}, \qquad h\colon X \to \mathbb{R},$$

which are cobordant in the sense of Definition 2.1. If $F\colon W \to [0,1] \times \mathbb{R}$ denotes a cobordism from g to h, we extend W to a nullcobordism V of M by gluing it together with a cylinder $X \times [0,1]$ along $X \sqcup -X$, and with $\bigsqcup_{i=1}^{a} D^{n+1}$ along $\bigsqcup_{i=1}^{a} S^n$. There is no obstruction to extending F to a generic map $G\colon V \to [0,1] \times \mathbb{R}$. Finally, by exploiting the assumption $\mu_0(f) = \cdots = \mu_{\lfloor (n-1)/2 \rfloor - 1}(f) = 0$, we are able to eliminate all cusps of G by means of Levine's cusp elimination technique [13].

This completes our outline of the proof of Theorem 1.1.

5. ENVISIONING A UNIFYING APPROACH

In the previous sections, we focused on the cobordism relation for Morse functions on oriented closed manifolds (Definition 2.1), and studied the precise connection to cutting and pasting relations on manifolds (Definition 3.1). The purpose of the present section is to envision in an informal way a framework which provides a unified perspective on cobordism relations for various types of Morse functions from the same viewpoint. In Section 5.1, we outline our framework for studying cobordism theory of Morse functions of any given type. In conclusion, we discuss in Section 5.2 the author's recent computation of cobordism groups of Morse functions on manifolds with boundary (see [26, 27]).

5.1. **A unifying framework.** We propose the following steps for studying the structure of cobordism groups of Morse functions of a given type. Let us consider a class $\mathcal{T}$ of certain Morse functions on closed (un-)oriented manifolds for which an appropriate cobordism relation is defined (as a modification Definition 3.1). We denote the resulting

n-dimensional cobordism group of Morse functions of class $\mathcal{T}$ by $\mathcal{M}_n^{\mathcal{T}}$. For instance, $\mathcal{T}$ could be the class of special generic functions (see [18]) or, more generally, the class of k-constrained Morse functions as studied in [24]. Next, we modify the cutting and pasting relation of Definition 3.1 in order to reflect the properties of Morse functions in the class $\mathcal{T}$, but without using singularity theory. For example, when $\mathcal{T}$ is the class of k-constrained Morse functions, one might have to incorporate into the cutting and pasting relation of Definition 3.1 suitable connectedness assumptions on M, M', N, N', that depend on the parameter k. The resulting n-dimensional SKK-group corresponding to the modified gluing and pasting relations will be denoted by $SKK_n^{\mathcal{T}}$. Now, the "correct" choice of cutting and pasting relations will enable us to define a structure map

$$\mathcal{M}_n^{\mathcal{T}} \to SKK_n^{\mathcal{T}}.$$

This map turns out to be surjective (in all known cases), but we might not expect it to be injective in general. Instead, there should exist a homomorphism

$$\mathcal{M}_n^{\mathcal{T}} \to A_n^{\mathcal{T}}$$

to some abelian group $A_n^{\mathcal{T}}$ that extracts further singularity theoretic invariants from $\mathcal{T}$-cobordism classes in such a way that the pair of both homomorphisms taken together yields indeed an isomorphism

$$\mathcal{M}_n^{\mathcal{T}} \cong SKK_n^{\mathcal{T}} \oplus A_n^{\mathcal{T}}.$$

When $\mathcal{T}$ is the class of all Morse functions on oriented closed manifolds, we note that $\mathcal{M}_n^{\mathcal{T}} = \mathcal{M}_n$ and $SKK_n^{\mathcal{T}} = SKK_n$, and read off from Theorem 1.1 that $A_n^{\mathcal{T}} = \mathbb{Z}^{\lfloor (n-1)/2 \rfloor}$. In Section 5.2 below we will discuss how our framework applies to cobordism theory of Morse functions on compact manifolds with boundary (compare Problem 5.6). We leave it as open problems to study other classes of Morse functions mentioned in the introduction.

Problem 5.1. *Suppose that $\mathcal{T}$ is the class of all special generic functions on oriented closed manifolds. We suggest to modify the cutting and pasting relation of Definition 3.1 by requiring that each of the manifolds M, M', N, N' is diffeomorphic to D^n. Does the associated SKK-group $SKK_n^{\mathcal{T}}$ admit a homomorphism $\mathcal{M}_n^{\mathcal{T}} \to SKK_n^{\mathcal{T}}$? If so, it seems plausible that this is an isomorphism, and therefore, that $A_n^{\mathcal{T}} = 0$. Then, the main result of* [18] *would imply that $SKK_n^{\mathcal{T}} \cong \Sigma_n$ for $n \geq 6$, where Σ_n denotes the group of oriented homotopy n-spheres up to h-cobordism.*

Problem 5.2. *Suppose that $\mathcal{T}$ is the class of all circle-valued Morse functions as considered in* [5]. *Find a modification of the cutting and pasting relation of Definition 3.1 such that the associated SKK-group $SKK_n^{\mathcal{T}}$ admits a homomorphism $\mathcal{M}_n^{\mathcal{T}} \to SKK_n^{\mathcal{T}}$, and study its properties.*

Problem 5.3. *Study the analog of problem Problem 5.2 for the class $\mathcal{T}$ of k-constrained Morse functions (see* [24]*).*

Remark 5.4 (orientations). Suppose that our class $\mathcal{T}$ of Morse functions does not interact with orientations of the underlying manifolds. Then, we can switch between the oriented and the unoriented versions of cobordism groups of Morse functions of class $\mathcal{T}$ by adapting the cutting and pasting relations appropriately. While the oriented and the unoriented versions of the resulting SKK-groups can have different structures, we point out that the task of computing them is independent of singularity theory. On the other hand, the singularity theoretic invariants encoded in the map $\mathcal{M}_n^{\mathcal{T}} \to A_n^{\mathcal{T}}$ will not be affected.

5.2. **Morse functions on manifolds with boundary.** As an application, let us explain how the framework of Section 5.1 applies to cusp and fold cobordism relations of Morse functions on compact manifolds possibly with boundary. In the following, we will focus on the case that the underlying manifolds are oriented. The case of unoriented manifolds is then covered by Remark 5.4.

Fix an integer $n \geq 2$. Let M^n denote a compact manifold possibly with boundary. By a Morse function on M we mean a function $f\colon M \to \mathbb{R}$ which is a submersion in a neighborhood of ∂M, and such that the critical points of both f and $f|_{\partial M}$ are all non-degenerate (see Figure 6). The concept of cobordant Morse functions (see Definition 2.1)

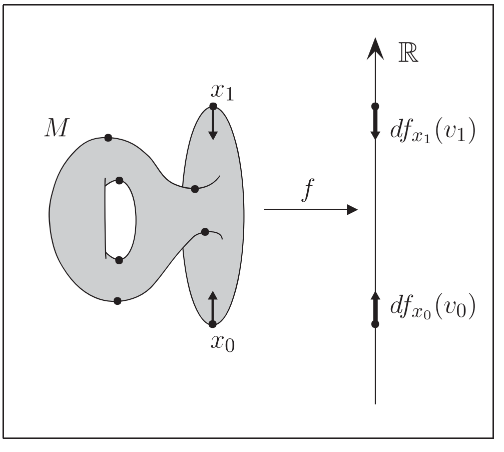

FIGURE 6. Illustration of a Morse function $f\colon M \to \mathbb{R}$ on a compact surface with boundary induced by the height function in $\mathbb{R}^3$. The critical points of $f|_{\partial M}$ are x_0 and x_1. Using the indicated inward pointing tangent vectors $v_0 \in T_{x_0}M$ and $v_1 \in T_{x_1}M$, we see that $\sigma_f(x_0) = +1$ and $\sigma_f(x_1) = -1$.

has been adapted to manifolds possibly with boundary by Saeki-Yamamoto [21] as follows.

Definition 5.5. Two Morse functions $f_0\colon M_0 \to \mathbb{R}$ and $f_1\colon M_1 \to \mathbb{R}$ on compact manifolds possibly with boundary M_0, M_1 are *cusp cobordant* (resp. *fold cobordant*) if

- there exists a cobordism (W^{n+1}, V) (with corners) from M_0 to M_1, that is, W is a compact oriented $(n+1)$-manifold with corners such that $\partial W = M_0 \cup_{\partial M_0}$

$V \cup_{-\partial M_1} -M_1$, where M_0, $-M_1$ and V are oriented codimension 0 submanifolds of ∂W such that $M_0 \cap M_1 = \emptyset$, $V \cap M_0 = \partial M_0$ and $V \cap M_1 = \partial M_1$, V^n is an oriented cobordism from ∂M_0 to ∂M_1, and W has corners precisely along ∂V,

- there exists a map $F\colon W \to [0,1] \times \mathbb{R}$ such that F and $F|_{\partial W \setminus (M_0 \sqcup M_1)}$ have only fold points and cusps (resp. only fold points) as singular points, where recall that the local normal form of a cusp is given by the map germ $(\mathbb{R}^{m+1}, 0) \to (\mathbb{R}^2, 0)$,

$$(x_0, x_1, \ldots, x_m) \mapsto (x_0, x_0x_1 + x_1^3 - x_2^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_m^2),$$

- F is a submersion in a neighborhood of $\partial W \setminus (M_0 \sqcup M_1)$, and
- there exist collars (with corners) $[0,\varepsilon) \times M_0 \subset W$ of $M_0 \subset W$ and $(1-\varepsilon, 1] \times M_1 \subset W$ of $M_1 \subset W$ such that $F|_{[0,\varepsilon)\times M_0} = \mathrm{id}_{[0,\varepsilon)} \times f_0$ and $F|_{(1-\varepsilon,1]\times M_1} = \mathrm{id}_{(1-\varepsilon,1]} \times f_1$.

It can be checked that cobordism in the sense of Definition 5.5 determines an equivalence relation on the set of all Morse functions on compact oriented n-manifolds possibly with boundary. Let $\mathcal{M}_n^{\partial,\mathrm{cusp}}$ (resp. $\mathcal{M}_n^{\partial,\mathrm{fold}}$) denote the resulting sets of equivalence classes. As usual, disjoint union "$\sqcup$" induces a group law on these sets. The following problem has been posed by Saeki-Yamamoto in [21]. While they use a definition of cobordism that is slightly different from Definition 5.5 in that they make additional C^∞ stability assumptions on the maps, the resulting cobordism groups turn out to be isomorphic.

Problem 5.6 (Saeki-Yamamoto, 2018). *Study cusp and fold cobordism groups of Morse functions $\mathcal{M}_n^{\partial,\mathrm{cusp}}$ and $\mathcal{M}_n^{\partial,\mathrm{fold}}$, as well as their unoriented versions $\overline{\mathcal{M}}_n^{\partial,\mathrm{cusp}}$ and $\overline{\mathcal{M}}_n^{\partial,\mathrm{fold}}$.*

Previously, it has been shown by Saeki-Yamamoto [21] that $\overline{\mathcal{M}}_2^{\partial,\mathrm{cusp}} \cong \mathbb{Z}/2$, and by Yamamoto [28] that $\overline{\mathcal{M}}_2^{\partial,\mathrm{fold}} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$.

Let us indicate how the framework of Section 5.1 can be applied to compute $\mathcal{M}_n^{\partial,\mathrm{fold}}$ for $n > 2$. We modify the cutting and pasting relation of Definition 3.1 by allowing the manifolds M, M', N, N' to have corners. By augmentation of the arguments in [11], we can then show that $SKK_n^{\partial,\mathrm{fold}} \cong I_n \oplus I_{n-1}$ (see (3.1)) by identifying a generator of I_n with $[S^n]$, and a generator of I_{n-1} with $[D^n]$. (Note that cobordism groups of manifolds with boundary are trivial.) The definition of $\Sigma(M, f)$ (see (1.2)) carries over to Morse functions $f\colon M \to \mathbb{R}$ on manifolds possibly with boundary. We construct the homomorphism

$$\omega_n\colon \mathcal{M}_n^{\partial,\mathrm{fold}} \to SKK_n^{\partial,\mathrm{fold}}, \quad [f\colon M \to \mathbb{R}] \mapsto [M] + \Sigma(M, f) \cdot [S^n] + \Sigma(\partial M, f|_{\partial M}) \cdot [D^n].$$

Let us introduce some more notation for Morse functions $f\colon M \to \mathbb{R}$ defined on n-manifolds possibly with boundary. Following [2], we assign to every critical point x of the Morse function $f|_{\partial M}$ a sign $\sigma_f(x) \in \{\pm 1\}$ (see Figure 6) that is uniquely determined by requiring that for an *inward* pointing tangent vector $v \in T_xM$ the tangent vector

$$\sigma_f(x) \cdot df_x(v) \in T_{f(x)}\mathbb{R} = \mathbb{R}$$

points into the *positive* direction of the real axis. In fact, this sign depends only on the germ $[f]$ of f near ∂M. Let $S_i^+(f) \subset S(f|_{\partial M})$ denote the subset of those critical points x of the Morse function $f|_{\partial M}$ of index i for which $\sigma_f(x) = +1$. We also define $\nu_i^+(f) = \# S_i^+(f)$, and $\mu_i^+(f) = \nu_i^+(-f) - \nu_i^+(-f)$.

Theorem 5.7 (W. [27], 2018)**.** *For $n > 2$, there is a group isomorphism*

$$\mathcal{M}_n^{\partial,\mathrm{fold}} \xrightarrow{\cong} SKK_n^{\partial,\mathrm{fold}} \oplus \mathcal{A}_n^{\partial,\mathrm{fold}} = SKK_n^{\partial,\mathrm{fold}} \oplus \mathbb{Z}^{\lfloor (n-1)/2 \rfloor} \oplus \mathbb{Z}^{\lfloor (n-2)/2 \rfloor} \oplus \mathbb{Z}^{\lceil n/2 \rceil},$$
$$[f\colon M \to \mathbb{R}] \mapsto (\omega_n[f], \boldsymbol{\mu}_{\lfloor (\boldsymbol{n}-\boldsymbol{1})/\boldsymbol{2} \rfloor}(f), \boldsymbol{\mu}_{\lfloor (\boldsymbol{n}-\boldsymbol{2})/\boldsymbol{2} \rfloor}(f|\partial M), \boldsymbol{\mu}^+_{\lceil \boldsymbol{n}/\boldsymbol{2} \rceil}(f)),$$

where we make use of the vector notation $\boldsymbol{\mu}_{\boldsymbol{N}}^{(+)}(g) = (\mu_0^{(+)}(g), \ldots, \mu_{N-1}^{(+)}(g))$.

Problem 5.8. *Prove the analog of Theorem 5.7 for $n = 2$. Then, proceed as suggested in Remark 5.4 to reproduce the isomorphism $\overline{\mathcal{M}}_2^{\partial,\mathrm{fold}} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ due to Yamamoto* [28]*. In particular, give an explicit formula for the invariant $\overline{\mathcal{M}}_2^{\partial,\mathrm{fold}} \to \mathbb{Z}/2$.*

After discovering the singularity theoretic invariants $\mathcal{M}_n^{\partial,\mathrm{fold}} \to \mathcal{A}_n^{\partial,\mathrm{fold}}$ of Theorem 5.7, the author was able to answer Problem 5.6 for $\mathcal{M}_n^{\partial,\mathrm{cusp}}$ (as well as for $\overline{\mathcal{M}}_n^{\partial,\mathrm{cusp}}$ in view of Remark 5.4) as follows. For Morse functions $f\colon M \to \mathbb{R}$ defined on compact n-manifolds possibly with boundary we define in analogy with Euler characteristic formulas the integer

$$\chi^+(f) = \sum_{i=0}^{n-1} (-1)^i \cdot \nu_i^+(f).$$

Theorem 5.9 (W. [26], 2018)**.** *Assigning to Morse functions $f\colon M^n \to \mathbb{R}$ on compact oriented n-manifolds possibly with boundary the integers $\chi(M) - \chi^+(f)$, we obtain*

$$\mathcal{M}_n^{\partial,\mathrm{cusp}} \xrightarrow{\cong} \begin{cases} \mathbb{Z}/2, & n \text{ even}, \\ \mathbb{Z}, & n \text{ odd}. \end{cases}$$

References

1. Y. Ando, *Cobordisms of maps with singularities of a given class*, Alg. Geom. Topol. **8** (2008), 1989–2029.
2. C. Curley, *Non-singular extensions of Morse functions*, Topology **16** (1977), 89–97.
3. K. Ikegami, *Cobordism group of Morse functions on manifolds*, Hiroshima Math. J. **34** (2004), 211–230.
4. K. Ikegami, O. Saeki, *Cobordism group of Morse functions on surfaces*, J. Math. Soc. Japan **55** (2003), 1081–1094.
5. K. Ikegami, O. Saeki, *Cobordism of Morse maps and its application to map germs*, Math. Proc. Camb. Phil. Soc. **147** (2009), 235–254.
6. K. Jänich, *Charakterisierung der Signatur von Mannigfaltigkeiten durch eine Additivitätseigenschaft*, Invent. Math. **6** (1968), 35–40.
7. K. Jänich, *On invariants with the Novikov additive property*, Math. Ann. **184** (1969), 65–77.

8. B. Kalmár, *Cobordism group of Morse functions on unoriented surfaces*, Kyushu J. Math. **59** (2005), 351–363.
9. B. Kalmár, *Pontryagin-Thom-Szűcs type construction for non-positive codimensional singular maps with prescribed singular fibers*, The second Japanese-Australian Workshop on Real and Complex Singularities, RIMS Kôkyûroku **1610** (2008), 66–79.
10. B. Kalmár, *Fold cobordisms and a Poincaré-Hopf type theorem for the signature*, preprint (2018), **arXiv:1809.03703**.
11. U. Karras, M. Kreck, W.D. Neumann, E. Ossa, *Cutting and Pasting of Manifolds; SK-groups*. Publish or Perish, Inc., Boston, Mass., 1973. Mathematics Lecture Series, No. 1.
12. M. Kervaire, J.W. Milnor, *Homotopy Spheres: I*, *The Annals of Mathematics*, Second Series, **77** (1963), no. 3, 504–537.
13. H.I. Levine, *Elimination of cusps*, Topology **3**, Suppl. 2 (1965), 263–296.
14. G. Lusztig, J. Milnor, F.P. Peterson, *Semi-characteristics and cobordism*, Topology **8** (1969), 357–359.
15. R. Rimányi, A. Szűcs, *Pontrjagin-Thom-type construction for maps with singularities*, Topology **37** (1998), 1177–1191.
16. C. Rovi, M. Schoenbauer, *Relating cut and paste invariants and TQFTs*, preprint (2018), **arXiv:1803.02939**.
17. R. Sadykov, *Bordism groups of solutions to differential relations*, Alg. Geom. Topol. **9** (2009), 2311–2349.
18. O. Saeki, *Cobordism groups of special generic functions and groups of homotopy spheres*, Japan. J. Math. (N. S.) **28** (2002), 287–297.
19. O. Saeki, T. Yamamoto, *Co-orientable singular fibers of stable maps of 3-manifolds with boundary into surfaces*, RIMS Kôkyûroku, Research Institute for Mathematical Sciences, Kyoto University, **1948** (2015), 137–152.
20. O. Saeki, T. Yamamoto, *Singular fibers of stable maps of* 3*-manifolds with boundary into surfaces and their applications*, Algebr. Geom. Topol. **16** (2016), 1379–1402.
21. O. Saeki, T. Yamamoto, *Cobordism group of Morse functions on surfaces with boundary*, in: XIII International Workshop, Real and Complex Singularities, Universidade de São Paulo; Contemporary Mathematics **675** (2016), 279–297.
22. A. Szűcs, *Cobordism of singular maps*, Geom. Topol. **12** (2008), 2379–2452.
23. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954),17–86.
24. D.J. Wrazidlo, *Bordism of constrained Morse functions*, preprint (2018), **arXiv:1803.11177**.
25. D.J. Wrazidlo, *Detecting exotic spheres via fold maps*, RIMS Kôkyûroku, Research Institute for Mathematical Sciences, Kyoto University, **2085** (2018), 81–96. `http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/2085-09.pdf`
26. D.J. Wrazidlo, *Cusp cobordism group of Morse functions on compact manifolds with boundary*, preprint (2019).
27. D.J. Wrazidlo, *Relating SKK-relations to Morse theory*, in preparation.
28. T. Yamamoto, *Fold cobordism groups of Morse functions on surfaces with boundary*, preprint.

Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan
E-mail address: `d-wrazidlo@imi.kyushu-u.ac.jp`