

# Some remarks on the relative polar variety and the Brasselet number \*

Hellen Santana

## Introduction

Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be an analytic function defined in a neighborhood of the origin and  $\Sigma f$  the critical locus of  $f$ . The Milnor fiber  $F_{f,0}$  is given by  $f^{-1}(\delta) \cap B_\epsilon$ , where  $\delta$  is a regular value of  $f$ ,  $0 < |\delta| \ll \epsilon \ll 1$ . In [15], Milnor proved that, if  $f$  has an isolated singularity,  $F_{f,0}$  has the homotopy type of a wedge of  $\mu(f)$  spheres of dimension  $n - 1$ , where  $\mu(f)$  is the Milnor number of  $f$ . Also,  $\mu(f)$  is the number of Morse points in a Morsefication of  $f$  in a neighborhood of the origin.

In [6], Hamm generalized Milnor's results for complete intersections with isolated singularity  $F = (f_1, \dots, f_k) : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0)$ ,  $1 < k < n$ , proving that the Milnor fiber  $F^{-1}(\delta) \cap B_\epsilon$ ,  $0 < |\delta| \ll \epsilon \ll 1$ , has the homotopy type of a wedge of  $\mu(F)$  spheres of dimension  $n - k$ . In this context, Lê [7] and Greuel [5] proved that  $\mu(F) + \mu(F') = \dim_{\mathbb{C}} \left( \frac{\mathcal{O}_{\mathbb{C}^n, 0}}{I} \right)$ , where  $F' : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{k-1}, 0)$  is the map with components  $f_1, \dots, f_{k-1}$  and  $I$  is the ideal generated by  $f_1, \dots, f_{k-1}$  and the  $(k \times k)$ -minors  $\frac{\partial(f_1, \dots, f_k)}{\partial(x_{i_1}, \dots, x_{i_k})}$ . Notice that the number  $\dim_{\mathbb{C}} \left( \frac{\mathcal{O}_{\mathbb{C}^n, 0}}{I} \right)$  is the number of critical points of a Morsefication of  $f_k$  appearing on the Milnor fibre of  $F'$ .

If  $f$  is defined over a complex analytic space  $X$  and  $f$  has an isolated singularity at the origin, a generalization for the Milnor number is the Euler obstruction of the function  $f$ , introduced in [2], by Brasselet, Massey, Parameswaran and Seade. In [17], Seade, Tibăr and Verjovsky proved that, up to sign, this number is the number of Morse critical points of a stratified Morsefication of  $f$  appearing in the regular part of  $X$ .

In a more general context, if  $f$  is defined over a complex analytic space  $X$  equipped with a good stratification  $\mathcal{V}$  of  $X$  relative to  $f$  (see Definition 2.1) and the function  $f$  does not have isolated singularity at the origin, a way to describe the generalized Milnor fiber  $X \cap f^{-1}(\delta) \cap B_\epsilon$  is to use the Brasselet number of  $f$  at the origin,  $B_{f,X}(0)$ , introduced by Dutertre and Grulha, in [3], and that generalizes the Milnor number to this more general setting and the local Euler obstruction, introduced by MacPherson, in [10], in his proof for the Deligne-Grothendieck conjecture. In [3], the authors presented several formulas to compute Brasselet numbers counting number of stratified Morse critical points. For example, they present a Lê-Greuel type formula for the Brasselet number: if  $g : X \rightarrow \mathbb{C}$  is prepolular with respect to  $\mathcal{V}$  at the origin (see Definition 2.4) and  $0 < |\delta| \ll \epsilon \ll 1$ , then

---

\*Research partially supported by FAPESP - Brazil, Grants 2015/25191-9 and 2017/18543-1.

*Key-words:* Brasselet number, relative polar variety, local Euler obstruction, stratified Morse critical points

$$B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{d-1}n_q,$$

where  $n_q$  is the number of Morse critical points of a partial Morsefication of  $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$  appearing in the regular part of  $X$ , and  $X^g = X \cap \{g = 0\}$ .

They also proved results about the topology of functions with isolated singularity defined over an analytic complex Whitney stratified variety  $X$ . If  $X$  is equidimensional, let  $f, g : X \rightarrow \mathbb{C}$  be analytic functions with isolated singularity at the origin such that  $g$  is prepolar at the origin with respect to the good stratification induced by  $f$  (see (1)) and  $f$  is prepolar at the origin with respect to the good stratification induced by  $g$ , then

$$B_{f,X}(0) - B_{g,X}(0) = (-1)^{d-1}(n_q - m_q).$$

where  $X^f = X \cap \{f = 0\}$ ,  $n_q$  is the number of Morse critical points of a Morsefication of  $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$  appearing in the regular part of  $X$  and  $m_q$  is the number of Morse critical points of a Morsefication of  $f|_{X \cap g^{-1}(\delta) \cap B_\epsilon}$  appearing in the regular part of  $X$ , for  $0 < |\delta| \ll \epsilon \ll 1$ .

Computing these numbers of stratified Morse critical points is directly connected to relative polar varieties. Consider  $l$  a linear form in  $\mathbb{C}^n$ ,  $\mathcal{W}$  a Whitney stratification of an open subset  $U$  of  $X$ ,  $\{W_i \setminus \{l = 0\}, W_i \cap \{l = 0\} \setminus \{0\}, \{0\}\}$  the good stratification of  $U$  induced by  $l$  and a function-germ  $g : (U, 0) \rightarrow (\mathbb{C}, 0)$ . If  $l$  sufficiently generic, the relative polar variety (curve)  $\Gamma_{g,l}$  defined by Lê and Teissier, in [8], coincides with the relative polar curve defined by Massey, in [12], and with the relative polar varieties, defined by Massey, in [14] (see [11] and [13]). Each of these polar varieties are useful not only to compute polar multiplicities ([9]) and intersection numbers ([14]), but also to describe critical loci of pair of functions defined over  $X$  ([12], [3]), which is the approach we are interested the most.

In this work, we use relative polar varieties to compute a number of stratified Morse critical points of a specific type of Morsefication of a function-germ, aiming to obtain informations about the Brasselet number of this germ.

## 1 Local Euler obstruction and Euler obstruction of a function

We begin presenting the local Euler obstruction, a singular invariant defined by MacPherson and used as one of the main tools in his proof for the Deligne-Grothendieck conjecture about the existence and uniqueness of Chern classes for singular varieties.

Let  $(X, 0) \subset (\mathbb{C}^n, 0)$  be an equidimensional reduced complex analytic germ of dimension  $d$  in a open set  $U \subset \mathbb{C}^n$ . Consider a complex analytic Whitney stratification  $\mathcal{V} = \{V_\lambda\}$  of  $U$  adapted to  $X$  such that  $\{0\}$  is a stratum. We choose a small representative of  $(X, 0)$ , denoted by  $X$ , such that  $0$  belongs to the closure of all strata. We write  $X = \cup_{i=0}^q V_i$ , where  $V_0 = \{0\}$  and  $V_q = X_{reg}$ , where  $X_{reg}$  is the regular part of  $X$ . We suppose that  $V_0, V_1, \dots, V_{q-1}$  are connected and that the analytic sets  $\overline{V_0}, \overline{V_1}, \dots, \overline{V_q}$  are reduced. We write  $d_i = \dim(V_i)$ ,  $i \in \{1, \dots, q\}$ . Note that  $d_q = d$ .

Let  $G(d, N)$  be the Grassmannian manifold,  $x \in X_{reg}$  and consider the Gauss map  $\phi : X_{reg} \rightarrow U \times G(d, N)$  given by  $x \mapsto (x, T_x(X_{reg}))$ .

**Definition 1.1.** The closure of the image of the Gauss map  $\phi$  in  $U \times G(d, N)$ , denoted by  $\tilde{X}$ , is called **Nash modification** of  $X$ . It is a complex analytic space endowed with an analytic projection map  $\nu : \tilde{X} \rightarrow X$ .

Consider the extension of the tautological bundle  $\mathcal{T}$  over  $U \times G(d, N)$ . Since  $\tilde{X} \subset U \times G(d, N)$ , we consider  $\tilde{T}$  the restriction of  $\mathcal{T}$  to  $\tilde{X}$ , called the **Nash bundle**, and  $\pi : \tilde{T} \rightarrow \tilde{X}$  the projection of this bundle.

In this context, denoting by  $\varphi$  the natural projection of  $U \times G(d, N)$  at  $U$ , we have the following diagram:

$$\begin{array}{ccc} \tilde{T} & \longrightarrow & \mathcal{T} \\ \pi \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & U \times G(d, N) \\ \nu \downarrow & & \downarrow \varphi \\ X & \longrightarrow & U \subseteq \mathbb{C}^N \end{array}$$

Considering  $\|z\| = \sqrt{z_1 \bar{z}_1 + \dots + z_N \bar{z}_N}$ , the 1-differential form  $w = d\|z\|^2$  over  $\mathbb{C}^N$  defines a section in  $T^*\mathbb{C}^N$  and its pullback  $\varphi^*w$  is a 1-form over  $U \times G(d, N)$ . Denote by  $\tilde{w}$  the restriction of  $\varphi^*w$  over  $\tilde{X}$ , which is a section of the dual bundle  $\tilde{T}^*$ .

Choose  $\epsilon$  small enough for  $\tilde{w}$  be a non zero section over  $\nu^{-1}(z)$ ,  $0 < \|z\| \leq \epsilon$ , let  $B_\epsilon$  be the closed ball with center at the origin with radius  $\epsilon$  and denote:

1.  $Obs(\tilde{T}^*, \tilde{w}) \in \mathbb{H}^{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z})$  as the obstruction for extending  $\tilde{T}$  from  $\nu^{-1}(S_\epsilon)$  to  $\nu^{-1}(B_\epsilon)$ ;
2.  $O_{\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon)}$  as the fundamental class in  $\mathbb{H}_{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z})$ .

**Definition 1.2.** The **local Euler obstruction** of  $X$  at 0,  $Eu_X(0)$ , is given by the evaluation

$$Eu_X(0) = \langle Obs(\tilde{T}^*, \tilde{w}), O_{\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon)} \rangle.$$

In [1], Brasselet, Lê and Seade proved a formula to make the calculation of the Euler obstruction easier.

**Theorem 1.3.** (Theorem 3.1 of [1]) Let  $(X, 0)$  and  $\mathcal{V}$  be given as before, then for each generic linear form  $l$ , there exists  $\epsilon_0$  such that for any  $\epsilon$  with  $0 < \epsilon < \epsilon_0$  and  $\delta \neq 0$  sufficiently small, the Euler obstruction of  $(X, 0)$  is equal to

$$Eu_X(0) = \sum_{i=1}^q \chi(V_i \cap B_\epsilon \cap l^{-1}(\delta)) \cdot Eu_X(V_i),$$

where  $\chi$  is the Euler characteristic,  $Eu_X(V_i)$  is the Euler obstruction of  $X$  at a point of  $V_i$ ,  $i = 1, \dots, q$  and  $0 < |\delta| \ll \epsilon \ll 1$ .

Let us give the definition of another invariant introduced by Brasselet, Massey, Parameswaran and Seade in [2]. Let  $f : X \rightarrow \mathbb{C}$  be a holomorphic function with isolated singularity at the origin given by the restriction of a holomorphic function  $F : U \rightarrow \mathbb{C}$  and denote by  $\bar{\nabla}F(x)$  the conjugate of the gradient vector field of  $F$  in  $x \in U$ ,

$$\bar{\nabla}F(x) := \left( \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right).$$

Since  $f$  has an isolated singularity at the origin, for all  $x \in X \setminus \{0\}$ , the projection  $\hat{\zeta}_i(x)$  of  $\overline{\nabla}F(x)$  over  $T_x(V_i(x))$  is non-zero, where  $V_i(x)$  is a stratum containing  $x$ . Using this projection, the authors constructed, in [2], a stratified vector field over  $X$ , denoted by  $\overline{\nabla}f(x)$ . Let  $\tilde{\zeta}$  be the lifting of  $\overline{\nabla}f(x)$  as a section of the Nash bundle  $\tilde{T}$  over  $\tilde{X}$ , without singularity over  $\nu^{-1}(X \cap S_\epsilon)$ .

Let  $\mathcal{O}(\tilde{\zeta}) \in \mathbb{H}^{2n}(\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(X \cap S_\epsilon))$  be the obstruction cocycle for extending  $\tilde{\zeta}$  as a non zero section of  $\tilde{T}$  inside  $\nu^{-1}(X \cap B_\epsilon)$ .

**Definition 1.4.** The **local Euler obstruction of the function**  $f$ ,  $Eu_{f,X}(0)$  is the evaluation of  $\mathcal{O}(\tilde{\zeta})$  on the fundamental class  $[\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(X \cap S_\epsilon)]$ .

The next theorem compares the Euler obstruction of a space  $X$  with the Euler obstruction of function defined over  $X$ .

**Theorem 1.5.** (Theorem 3.1 of [2]) Let  $(X, 0)$  and  $\mathcal{V}$  be given as before and let  $f : (X, 0) \rightarrow (\mathbb{C}, 0)$  be a function with an isolated singularity at 0. For  $0 < |\delta| \ll \epsilon \ll 1$ , we have

$$Eu_{f,X}(0) = Eu_X(0) - \sum_{i=1}^q \chi(V_i \cap B_\epsilon \cap f^{-1}(\delta)) \cdot Eu_X(V_i).$$

In [17], Seade, Tibăr and Verjovsky proved that the Euler obstruction of a function  $f$  is also related to the number of Morse critical points of a Morsefication of  $f$ . Before we state their result, let us see the definition of a general point.

**Definition 1.6.** Let  $(X, 0) \subset (U, 0)$  be a germ of complex analytic space in  $\mathbb{C}^n$  equipped with a Whitney stratification and let  $f : (X, 0) \rightarrow (\mathbb{C}, 0)$  be an analytic function, given by the restriction of an analytic function  $F : (U, 0) \rightarrow (\mathbb{C}, 0)$ . Then 0 is said to be a **generic point** of  $f$  if the hyperplane  $Ker(d_0F)$  is transverse in  $\mathbb{C}^n$  to all limit of tangent spaces  $\lim_{n \rightarrow \infty} T_{x_n}(V_\alpha)$ , for all  $V_\alpha$  and sequence of points  $x_n \in V_\alpha$  converging to 0.

Now, let us see the definition of a Morsefication of a function.

**Definition 1.7.** A function  $f : (X, 0) \rightarrow (\mathbb{C}, 0)$  is said to be **Morse stratified** if for all strata  $V_\alpha$ , with  $\dim V_\alpha \geq 1$ , 0 is a generic point of the restriction  $f|_{V_\alpha}$  and for  $V_0 = \{0\}$ , 0 is a Morse point of  $f|_{V_0}$ .

A stratified Morsefication of a germ of analytic function  $f : (X, 0) \rightarrow (\mathbb{C}, 0)$  is a deformation  $\tilde{f}$  of  $f$  such that  $\tilde{f}$  is Morse stratified.

**Proposition 1.8.** (Proposition 2.3 of [17]) Let  $f : (X, 0) \rightarrow (\mathbb{C}, 0)$  be a germ of analytic function with isolated singularity at the origin. Then,

$$Eu_{f,X}(0) = (-1)^{d_{n_{reg}}},$$

where  $n_{reg}$  is the number of Morse points in  $X_{reg}$  in a stratified Morsefication of  $f$ .

In the case where we have a function with several number of isolated critical points, one can be interested in a deformation of this function which is a Morsefication around each one of these singularities. This is what Dutertre and Grulha called a partial Morsefication.

**Definition 1.9.** A **partial Morsefication** of  $g : f^{-1}(\delta) \cap X \cap B_\epsilon \rightarrow \mathbb{C}$  is a function  $\tilde{g} : f^{-1}(\delta) \cap X \cap B_\epsilon \rightarrow \mathbb{C}$  (not necessarily holomorphic) which is a local Morsefication of all isolated critical points of  $g$  in  $f^{-1}(\delta) \cap X \cap \{g \neq 0\} \cap B_\epsilon$  and which coincides with  $g$  outside a small neighborhood of these critical points.

## 2 Brasselet number

We present now the two most important tools in this work: the relative polar variety, defined by Massey in [12], and the Brasselet number, introduced by Dutertre and Grulha in [3]. We also present formulas proved by Dutertre and Grulha to compute Brasselet numbers by counting numbers of stratified Morse critical points.

Let  $X$  be a reduced complex analytic space (not necessarily equidimensional) of dimension  $d$  in an open set  $U \subseteq \mathbb{C}^n$  and let  $f : (X, 0) \rightarrow (\mathbb{C}, 0)$  be an analytic map. We write  $V(f) = f^{-1}(0)$ .

**Definition 2.1.** A **good stratification of  $X$  relative to  $f$**  is a stratification  $\mathcal{V}$  of  $X$  which is adapted to  $V(f)$  such that  $\{V_\lambda \in \mathcal{V}, V_\lambda \not\subseteq V(f)\}$  is a Whitney stratification of  $X \setminus V(f)$  and such that for any pair  $(V_\lambda, V_\gamma)$  such that  $V_\lambda \not\subseteq V(f)$  and  $V_\gamma \subseteq V(f)$ , the  $(a_f)$ -Thom condition is satisfied, that is, if  $p \in V_\gamma$  and  $p_i \in V_\lambda$  are such that  $p_i \rightarrow p$  and  $T_{p_i}V(f)|_{V_\lambda} - f|_{V_\lambda}(p_i)$  converges to some  $\mathcal{T}$ , then  $T_pV_\gamma \subseteq \mathcal{T}$ .

If  $f : X \rightarrow \mathbb{C}$  has a stratified isolated critical point and  $\mathcal{V}$  is a Whitney stratification of  $X$ , then

$$\{V_\lambda \setminus X^f, V_\lambda \cap X^f \setminus \{0\}, \{0\}, V_\lambda \in \mathcal{V}\} \quad (1)$$

is a good stratification of  $X$  relative to  $f$ , called the good stratification induced by  $f$ .

**Definition 2.2.** The **critical locus of  $f$  relative to  $\mathcal{V}$** ,  $\Sigma_{\mathcal{V}}f$ , is given by the union

$$\Sigma_{\mathcal{V}}f = \bigcup_{V_\lambda \in \mathcal{V}} \Sigma(f|_{V_\lambda}).$$

**Definition 2.3.** If  $\mathcal{V} = \{V_\lambda\}$  is a stratification of  $X$ , the **symmetric relative polar variety of  $f$  and  $g$  with respect to  $\mathcal{V}$** ,  $\tilde{\Gamma}_{f,g}(\mathcal{V})$ , is the union  $\bigcup_\lambda \tilde{\Gamma}_{f,g}(V_\lambda)$ , where  $\Gamma_{f,g}(V_\lambda)$  denotes the closure in  $X$  of the critical locus of  $(f, g)|_{V_\lambda \setminus (X^f \cup X^g)}$ ,  $X^f = X \cap \{f = 0\}$  and  $X^g = X \cap \{g = 0\}$ .

**Definition 2.4.** Let  $\mathcal{V}$  be a good stratification of  $X$  relative to a function  $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ . A function  $g : (X, 0) \rightarrow (\mathbb{C}, 0)$  is **prepolar with respect to  $\mathcal{V}$  at the origin** if the origin is a stratified isolated critical point, that is, 0 is an isolated point of  $\Sigma_{\mathcal{V}}g$ .

**Definition 2.5.** A function  $g : (X, 0) \rightarrow (\mathbb{C}, 0)$  is **tractable at the origin with respect to a good stratification  $\mathcal{V}$  of  $X$  relative to  $f : (X, 0) \rightarrow (\mathbb{C}, 0)$**  if  $\dim_0 \tilde{\Gamma}_{f,g}^1(\mathcal{V}) \leq 1$  and, for all strata  $V_\alpha \subseteq X^f$ ,  $g|_{V_\alpha}$  has no critical points in a neighbourhood of the origin except perhaps at the origin itself.

Another concept useful for this work is the notion of constructible functions. Consider a Whitney stratification  $\mathcal{W} = \{W_1, \dots, W_q\}$  of  $X$  such that each stratum  $W_i$  is connected.

**Definition 2.6.** A constructible function with respect to the stratification  $\mathcal{W}$  of  $X$  is a function  $\beta : X \rightarrow \mathbb{Z}$  which is constant on each stratum  $W_i$ , that is, there exist integers  $t_1, \dots, t_q$ , such that  $\beta = \sum_{i=1}^q t_i \cdot 1_{W_i}$ , where  $1_{W_i}$  is the characteristic function of  $W_i$ .

**Definition 2.7.** The Euler characteristic  $\chi(X, \beta)$  of a constructible function  $\beta : X \rightarrow \mathbb{Z}$  with respect to the stratification  $\mathcal{W}$  of  $X$ , given by  $\beta = \sum_{i=1}^q t_i \cdot 1_{W_i}$ , is defined by  $\chi(X, \beta) = \sum_{i=1}^q t_i \cdot \chi(W_i)$ .

Before we state Dutertre and Grulha results, we need to introduce some definitions about normal Morse data. We cite as main references [4] and [18]. The first concept we present is the complex link, an object analogous to the Milnor fibre, important in the study of complex stratified Morse theory.

Let  $V$  be a stratum of the stratification  $\mathcal{V}$  of  $X$  and let  $x$  be a point of  $V$ . Let  $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be an analytic complex function-germ such that the differential form  $Dg(x)$  is not a degenerate covector of  $\mathcal{V}$  at  $x$ . Let  $N$  be a normal slice to  $V$  at  $x$ , that is,  $N$  is a closed complex submanifold of  $\mathbb{C}^n$  which is transversal to  $V$  at  $x$  and  $N \cap V = \{x\}$ .

**Definition 2.8.** Let  $B_\epsilon$  be the closed ball of radius  $\epsilon$  centered at  $x$ . The **complex link**  $l_V$  of  $V$  is defined by  $l_V = X \cap N \cap B_\epsilon \cap \{g = \delta\}$ , where  $0 < |\delta| \ll \epsilon \ll 1$ .

The **normal Morse datum**  $NMD(V)$  of  $V$  is the pair of spaces

$$NMD(V) = (X \cap N \cap B_\epsilon, X \cap N \cap B_\epsilon \cap \{g = \delta\}).$$

In Part II, section 2.3 of [4], the authors explained why this two notions are independent of all choices made.

**Definition 2.9.** Let  $\beta : X \rightarrow \mathbb{Z}$  be a constructible function with respect to the stratification  $\mathcal{V}$ . Its normal Morse index  $\eta(V, \beta)$  along  $V$  is defined by

$$\eta(V, \beta) = \chi(NMD(V), \beta) = \chi(X \cap N \cap B_\epsilon, \beta) - \chi(l_V, \beta).$$

In the case where the constructible function is the local Euler obstruction, the following identities are valid ([18], page 34):

$$\eta(V', Eu_{\overline{V}}) = 1, \text{ if } V' = V \text{ and } \eta(V', Eu_{\overline{V}}) = 0, \text{ if } V' \neq V.$$

We present now the definition of the Brasselet number and the main theorems of [3], used as inspiration for this work.

Let  $f : (X, 0) \rightarrow (\mathbb{C}, 0)$  be a complex analytic function germ and let  $\mathcal{V}$  be a good stratification of  $X$  relative to  $f$ . We denote by  $V_1, \dots, V_q$  the strata of  $\mathcal{V}$  that are not contained in  $\{f = 0\}$  and we assume that  $V_1, \dots, V_{q-1}$  are connected and that  $V_q = X_{reg} \setminus \{f = 0\}$ . Note that  $V_q$  could be not connected.

**Definition 2.10.** Suppose that  $X$  is equidimensional. Let  $\mathcal{V}$  be a good stratification of  $X$  relative to  $f$ . The **Brasselet number** of  $f$  at the origin,  $B_{f,X}(0)$ , is defined by

$$B_{f,X}(0) = \sum_{i=1}^q \chi(V_i \cap f^{-1}(\delta) \cap B_\epsilon) Eu_X(V_i),$$

where  $0 < |\delta| \ll \epsilon \ll 1$ .

**Remark:** If  $V_q^i$  is a connected component of  $V_q$ ,  $Eu_X(V_q^i) = 1$ .

Notice that if  $f$  has a stratified isolated singularity at the origin, then  $B_{f,X}(0) = Eu_X(0) - Eu_{f,X}(0)$  (see Theorem 1.5).

In [3], Dutertre and Grulha proved interesting formulas describing the topological relation between the Brasselet number and a number of certain critical points of a special type of deformation of functions. Let us now present some of these results.

Let  $g : (X, 0) \rightarrow (\mathbb{C}, 0)$  be a complex analytic function which is tractable at the origin with respect to  $\mathcal{V}$  relative to  $f$ . Then  $\Gamma_{f,g}$  is a complex analytic curve and for  $0 < |\delta| \ll 1$  the critical points of  $g|_{f^{-1}(\delta) \cap X}$  in  $B_\epsilon$  lying outside  $\{g = 0\}$  are isolated. Let  $\tilde{g}$  be a partial Morsefication of  $g : f^{-1}(\delta) \cap X \cap B_\epsilon \rightarrow \mathbb{C}$  and, for each  $i \in \{1, \dots, q\}$ , let  $n_i$  be the number of stratified Morse critical points of  $\tilde{g}$  appearing on  $V_i \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$ .

**Theorem 2.11.** (Theorem 4.2 of [3]) Let  $\beta : X \rightarrow \mathbb{Z}$  be a constructible function with respect to the stratification  $\mathcal{V}$ . Suppose that  $g : (X, 0) \rightarrow (\mathbb{C}, 0)$  is a complex analytic function tractable at the origin with respect to  $\mathcal{V}$  relative to  $f$ . For  $0 < |\delta| \ll \epsilon \ll 1$ , we have

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) - \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \sum_{i=1}^q (-1)^{d_i-1} n_i \eta(V_i, \beta).$$

In the case that  $\beta = Eu_X$ , the last theorem implies the following.

**Corollary 2.12.** (Corollary 4.3 of [3]) Suppose that  $X$  is equidimensional and that  $g$  is tractable at the origin with respect to  $\mathcal{V}$  relative to  $f$ . For  $0 < |\delta| \ll \epsilon \ll 1$ , we have

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, Eu_X) - \chi(X \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\epsilon, Eu_X) = (-1)^{d-1} n_q.$$

If one supposes, in addition, that  $g$  is prepolar, a consequence of this result is a Lê-Greuel type formula for the Brasselet number.

**Theorem 2.13.** (Theorem 4.4 of [3]) Suppose that  $X$  is equidimensional and that  $g$  is prepolar with respect to  $\mathcal{V}$  at the origin. For  $0 < |\delta| \ll \epsilon \ll 1$ , we have

$$B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{d-1} n_q,$$

where  $n_q$  is the number of stratified Morse critical points on the top stratum  $V_q \cap f^{-1}(\delta) \cap B_\epsilon$  appearing in a Morsefication of  $g : X \cap f^{-1}(\delta) \cap B_\epsilon \rightarrow \mathbb{C}$ .

In [3], the authors also related the topology of the generalized Minor fibres of  $f$  and  $g$  and some number of Morse points.

**Theorem 2.14.** Suppose that  $g$  (resp.  $f$ ) is prepolar with respect to the good stratification induced by  $f$  (resp.  $g$ ) at the origin. Let  $\beta : X \rightarrow \mathbb{Z}$  be a constructible function with respect to the Whitney stratification  $\mathcal{V}$ . For  $0 < |\delta| \ll \epsilon \ll 1$ ,

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) - \chi(X \cap g^{-1}(\delta) \cap B_\epsilon, \beta) = \sum_{i=1}^q (-1)^{d_i-1} (n_i - m_i) \eta(V_i, \beta),$$

where  $n_i$  (resp.  $m_i$ ) is the number of stratified Morse critical points on the stratum  $V_i \cap f^{-1}(\delta) \cap B_\epsilon$  (resp.  $V_i \cap g^{-1}(\delta) \cap B_\epsilon$ ) appearing in a Morsefication of  $g : X \cap f^{-1}(\delta) \cap B_\epsilon \rightarrow \mathbb{C}$  (resp.  $f : X \cap g^{-1}(\delta) \cap B_\epsilon \rightarrow \mathbb{C}$ ).

In the case where  $\beta = Eu_X$ , the last theorem implies the following result.

**Corollary 2.15.** Suppose that  $X$  is equidimensional and that  $g$  (resp.  $f$ ) is prepolar with respect to the good stratification induced by  $f$  (resp.  $g$ ) at the origin. Then

$$B_{f,X}(0) - B_{g,X}(0) = (-1)^{d-1} (n_q - m_q),$$

where  $n_q$  (resp.  $m_q$ ) is the number of stratified Morse critical points on the top stratum  $V_q \cap f^{-1}(\delta) \cap B_\epsilon$  (resp.  $V_q \cap g^{-1}(\delta) \cap B_\epsilon$ ) appearing in a Morsefication of  $g : X \cap f^{-1}(\delta) \cap B_\epsilon \rightarrow \mathbb{C}$  (resp.  $f : X \cap g^{-1}(\delta) \cap B_\epsilon \rightarrow \mathbb{C}$ ).

### 3 Brasselet numbers and empty relative polar varieties

In this final section, we present relations between Brasselet numbers of two function-germs in the case where the relative polar variety associated to these germs is empty.

Let  $(X, 0)$  be a complex analytic space with ambient space  $U \subset \mathbb{C}^n$ . Let  $f, g : (X, 0) \rightarrow (\mathbb{C}, 0)$  be germs of holomorphic functions and let  $\mathcal{V}$  be a good stratification of  $U$  relative to  $f$ . Suppose that  $\Sigma_{\mathcal{V}}g \cap \{f = 0\} = \{0\}$ . We aim to obtain information about the Brasselet number of  $f$  and the Brasselet number of  $g$  in the case where the relative polar variety  $\Gamma_{f,g}(\mathcal{V})$  is empty.

We begin with a description of two relevant subsets of the relative polar variety  $\Gamma_{f,g}(\mathcal{V})$ .

**Proposition 3.1.** *The stratified critical set  $\Sigma_{\mathcal{V}}g$  of  $g$  and the symmetric relative polar variety  $\tilde{\Gamma}_{f,g}(\mathcal{V})$  are subsets of  $\Gamma_{f,g}(\mathcal{V})$ .*

**Proof.** If  $x \in \Sigma_{\mathcal{V}}g$ ,  $d_x\tilde{g}|_{V_\alpha} = 0$ , for a stratum  $V_\alpha \in \mathcal{V}$  containing  $x$  and an analytic extension  $\tilde{g}$  of  $g$  in a neighborhood of  $x$ . If  $V_\alpha \subset \{f = 0\}$ , then  $x = 0$ , since  $\Sigma_{\mathcal{V}}g \cap \{f = 0\} = \{0\}$ . If  $V_\alpha \subset X \setminus \{f = 0\}$ ,

$$rk(d_x\tilde{f}|_{V_\alpha}, d_x\tilde{g}|_{V_\alpha}) \leq 1,$$

where  $\tilde{f}$  is an analytic extension of  $f$  in a neighborhood of  $x$ , that is  $x \in \Sigma(f, g)|_{V_\alpha} = \Sigma(f, g)|_{V_\alpha \setminus \{f=0\}}$ . Therefore,  $x \in \Gamma_{f,g}(V_\alpha)$ .

Furthermore,  $\tilde{\Gamma}_{f,g}(V_\alpha)$  is given by components of  $\Gamma_{f,g}(V_\alpha)$  no contained in  $\{g = 0\}$ , that is,  $\tilde{\Gamma}_{f,g}(V_\alpha) = \overline{\Gamma_{f,g}(V_\alpha) \setminus \{g = 0\}} \subseteq \Gamma_{f,g}(V_\alpha)$ . Therefore,

$$\tilde{\Gamma}_{f,g}(\mathcal{V}) \cup \Sigma_{\mathcal{V}}g \subseteq \Gamma_{f,g}(\mathcal{V}).$$

□

Using this proposition, we obtain the following useful information about the behavior of  $g$  with respect to the stratification  $\mathcal{V}$ .

**Corollary 3.2.** *If  $\Gamma_{f,g}(\mathcal{V})$  is empty, then  $g$  is prepolar at the origin with respect to the good stratification  $\mathcal{V}$  of  $X$  relative by  $f$ .*

**Proof.** By Proposition 3.1, if  $\Gamma_{f,g}(\mathcal{V})$  is empty,  $\Sigma_{\mathcal{V}}g$  is empty, that is,  $g$  has no stratified critical point with respect to  $\mathcal{V}$ . □

Let  $n_i$  be the number of stratified Morse critical points of a Morsefication of  $g : X \cap f^{-1}(\delta) \cap B_\epsilon \rightarrow \mathbb{C}$  in  $V_i \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$ , for each  $i \in \{1, \dots, q\}$ . The next proposition uses the relative polar variety  $\Gamma_{f,g}(\mathcal{V})$  for counting the numbers  $n_i$ .

**Proposition 3.3.** *If  $\Gamma_{f,g}(\mathcal{V})$  is empty, then  $n_i = 0$ , for all  $i \in \{1, \dots, q\}$ .*

**Proof.** Let  $V_i$  be a stratum of  $\mathcal{V}$  and  $x$  be a critical point of  $g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon}$ . Then, if  $\tilde{f}$  and  $\tilde{g}$  are analytic extensions of  $f$  and  $g$  in a neighborhood of  $x$ , respectively,  $x \in V_i \cap f^{-1}(\delta) \cap B_\epsilon$  and  $rk(d_x\tilde{f}|_{V_i}, d_x\tilde{g}|_{V_i}) \leq 1$ , that is,

$$x \in (V_i \cap f^{-1}(\delta) \cap B_\epsilon) \cap (\Sigma_{\mathcal{V}}f \cup \Sigma_{\mathcal{V}}g \cup \tilde{\Gamma}_{f,g}(\mathcal{V})).$$

By Proposition 1.3 of [12],  $\Sigma_{\mathcal{V}}f \subset \{f = 0\}$ . Therefore,



$$\Sigma g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon} = V_i \cap f^{-1}(\delta) \cap B_\epsilon \cap (\tilde{\Gamma}_{f,g}(V_i) \cup \Sigma g|_{V_i}) \subseteq V_i \cap f^{-1}(\delta) \cap B_\epsilon \cap \Gamma_{f,g}(V_i).$$

Since  $\Gamma_{f,g}(\mathcal{V})$  is empty, by Proposition 3.1,  $\Sigma g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon}$  is empty. Therefore,  $n_i = 0$ , for all  $i \in \{1, \dots, q\}$ .  $\square$

In [3], Dutertre and Grulha proved a Lê-Greuel type formula for the Brasselet number, with which it is possible count the number of stratified Morse critical points using Brasselet numbers. We apply their result to obtain a relation between Brasselet number in the setting we already know the number of Morse points. First let us show a more general result.

**Corollary 3.4.** *If  $\beta : X \rightarrow \mathbb{Z}$  is a constructible function with respect to the good stratification  $\mathcal{V}$  of  $X$  relative to  $f$  and  $\Gamma_{f,g}(\mathcal{V})$  is empty, then*

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \chi(X \cap \{g = 0\} \cap f^{-1}(\delta) \cap B_\epsilon, \beta).$$

**Proof.** By Corollary 3.2, since  $\Gamma_{f,g}(\mathcal{V})$  is empty,  $g$  is prepoliar at the origin with respect to  $\mathcal{V}$  and, by Proposition 1.12 of [12], tractable at the origin with respect to  $\mathcal{V}$ . Then, by Theorem 4.2 of [3], we obtain

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \chi(X \cap \{g = 0\} \cap f^{-1}(\delta) \cap B_\epsilon, \beta) + \sum_{i=1}^q (-1)^{d-1} n_i \eta(V_i, \beta),$$

where  $n_i$  is the number of stratified Morse critical points of a Morsefication of  $g|_{V_i \cap f^{-1}(\delta) \cap B_\epsilon}$  appearing in  $V_i \cap f^{-1}(\delta) \cap B_\epsilon$ . Using again that  $\Gamma_{f,g}(\mathcal{V})$  is empty,  $n_i = 0$ , for all  $i \in \{1, \dots, q\}$  and we conclude the equality.  $\square$

If the constructible function  $\beta$  is giving by the local Euler obstruction, one obtains a relation between Brasselet numbers.

**Corollary 3.5.** *If  $X$  is equidimensional and  $\Gamma_{f,g}(\mathcal{V})$  is empty, then  $B_{f,X}(0) = B_{f,X^g}(0)$ .*

**Proof.** By Theorem 4.4 of [3],  $B_{f,X}(0) = B_{f,X^g}(0) + (-1)^{d-1} n_q$ , where  $n_q$  is the number of stratified Morse critical points of a Morsefication of  $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$  appearing on  $V_q \cap f^{-1}(\delta) \cap B_\epsilon$ . Since  $\Gamma_{f,g}(\mathcal{V})$  is empty,  $n_q$  is zero and the equality holds.  $\square$

When  $f$  is a generic linear form on  $\mathbb{C}^n$ ,  $B_{f,X}(0) = Eu_X(0)$  and  $B_{f,X^g}(0) = Eu_{X^g}(0)$ . Therefore, Corollary 3.5 implies the following consequence.

**Corollary 3.6.** *If  $X$  is equidimensional and  $\Gamma_{f,g}(\mathcal{V})$  is empty, then  $Eu_X(0) = Eu_{X^g}(0)$ .*

When both  $f$  and  $g$  has isolated singularity at the origin, Dutertre and Grulha proved several formulas about the Brasselet numbers of  $f$  and  $g$ . Using this formulas, one can obtain further information about these numbers if  $\Gamma_{f,g}(\mathcal{V})$  is empty. If that is the case, then  $g$  is prepoliar at the origin with respect to the good stratification  $\mathcal{V}$  of  $X$  induced by  $f$ , given as a refinement of a Whitney stratification  $\mathcal{W} = \{W_i\}_i$  of  $X$ . By Corollary 6.1 of [3],  $f$  is prepoliar at the origin with respect to the good stratification  $\bar{\mathcal{V}}$  induced by  $g$ , also given by a

refinement of  $\mathcal{W}$ . Applying Proposition 1.12 of [12], we obtain that  $\Gamma_{f,g}(\mathcal{V}) = \tilde{\Gamma}_{f,g}(\mathcal{V})$  and  $\Gamma_{g,f}(\overline{\mathcal{V}}) = \tilde{\Gamma}_{g,f}(\overline{\mathcal{V}})$ . But

$$\begin{aligned} \tilde{\Gamma}_{f,g}(\mathcal{V}) &= \bigcup_{V_i \in \mathcal{V}} \overline{\Sigma(f, g)|_{V_i \setminus (\{f=0\} \cup \{g=0\})}} \\ &= \bigcup_{W_i \in \mathcal{W}} \overline{\Sigma(f, g)|_{(W_i \setminus \{f=0\}) \setminus \{g=0\}}} \\ &= \bigcup_{W_i \in \mathcal{W}} \overline{\Sigma(f, g)|_{(W_i \setminus \{g=0\}) \setminus \{f=0\}}} \\ &= \bigcup_{\overline{V}_i \in \overline{\mathcal{V}}} \overline{\Sigma(f, g)|_{\overline{V}_i \setminus (\{g=0\} \cup \{f=0\})}} \\ &= \tilde{\Gamma}_{g,f}(\overline{\mathcal{V}}). \end{aligned}$$

Therefore, these four polar varieties are equal. Using this description, one concludes the following.

**Proposition 3.7.** *If  $\beta : X \rightarrow \mathbb{Z}$  is a constructible function with respect to the Whitney stratification  $\mathcal{W}$  and  $\Gamma_{f,g}(\mathcal{V})$  is empty, where  $\mathcal{V}$  is the good stratification of  $X$  induced by  $f$ , then*

$$\chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) = \chi(X \cap g^{-1}(\delta) \cap B_\epsilon, \beta).$$

**Proof.** Since  $\Gamma_{f,g}(\mathcal{V})$  is empty,  $g$  is prepolar at the origin with respect to  $\mathcal{V}$ . Then,  $f$  is prepolar at the origin with respect to the good stratification  $\overline{\mathcal{V}}$  induced by  $g$ . By Theorem 6.4 of [3],

$$\begin{aligned} \chi(X \cap f^{-1}(\delta) \cap B_\epsilon, \beta) &= \chi(X \cap g^{-1}(\delta) \cap B_\epsilon, \beta) \\ &\quad + \sum_{i=1}^q (-1)^{d_i-1} (n_i - m_i) \eta(W_i, \beta), \end{aligned}$$

where  $d_i$  denotes the dimension of  $W_i \in \mathcal{W}$ . By Proposition 3.4, if  $m_i$  is the number of stratified Morse critical points of a Morsefication of  $f|_{X \cap g^{-1}(\delta) \cap B_\epsilon}$  appearing on  $V_i \cap g^{-1}(\delta) \cap \{f \neq 0\} \cap B_\epsilon$ , for each  $i \in \{1, \dots, q\}$ , since  $\Gamma_{f,g}(\mathcal{V})$  is empty,  $\Gamma_{f,g}(\overline{\mathcal{V}})$  is empty and  $m_i = 0$ . Since the number  $n_i$  of stratified Morse critical points of a Morsefication of  $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$  in  $V_i \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$  is zero, for each  $i \in \{1, \dots, q\}$ , the equality is proved.  $\square$

**Corollary 3.8.** *If  $X$  is equidimensional and  $\Gamma_{f,g}(\mathcal{V})$  is empty, then  $B_{g,X}(0) = B_{f,X}(0)$ .*

**Proof.** Since  $\Gamma_{f,g}(\mathcal{V})$  is empty,  $g$  (resp.  $f$ ) is prepolar with respect to the good stratification of  $X$  induced by  $f$  (resp.  $g$ ). Applying Corollary 6.5 of [3], we obtain  $B_{f,X}(0) = B_{g,X}(0) + (-1)^{d-1} (n_q - m_q)$ , where  $n_q$  (resp.  $m_q$ ) is the number of stratified Morse critical points of a Morsefication of  $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$  (resp.  $f|_{X \cap g^{-1}(\delta) \cap B_\epsilon}$ ) appearing on the top stratum  $V_q \cap f^{-1}(\delta) \cap B_\epsilon$  (resp.  $V_q \cap g^{-1}(\delta) \cap B_\epsilon$ ). Using again that  $\Gamma_{f,g}(\mathcal{V})$  is empty, we have that  $n_q = m_q = 0$ , what leads to the equality.  $\square$

## Acknowledgements

The author thank the financial support from FAPESP-Brazil, Grants 2015/25191-9 and 2017/18543-1. This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

## References

- [1] BRASSELET, J.P., LÊ, D.T., SEADE, J., *Euler obstruction and indices of vector fields*, *Topology*, v.39, (2000), p. 1193-1208.
- [2] BRASSELET, J-P., MASSEY, D., PARAMESWARAN, AJ. AND SEADE, J. *Euler obstruction and defects of functions on singular varieties*, *Journal of the London Mathematical Society*, Cambridge University Press, 70, n.1, (2004), p. 59-76.
- [3] DUTERTRE, N., GRULHA JR., N.G., *Lê-Greuel type formula for the Euler obstruction and applications*, *Adv. Math.* 251 (2014), p. 127-146.
- [4] GORESKEY, M. AND MACPHERSON, R. *Stratified Morse theory*, Springer, v. 40, n. 1, (1988).
- [5] GREUEL, G-M. *Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten*, *Mathematische Annalen*, Springer, v. 214, n. 3, (1975), p. 235-266.
- [6] HAMM, H. *Lokale topologische Eigenschaften komplexer Räume*, *Mathematische Annalen*, v.191, n.3, (1971), p. 235-252.
- [7] LÊ, D.T. *Calcul Du Nombre de Milnor D'und Singularité Isolée D'intersection Complète*, *Centre de Mathematiques de l'Ecole Polytechnique*,(1973).
- [8] LÊ D. T., TEISSIER, B. *Variétés polaires locales et classes de Chern des variétés singulieres* , *Annals of Mathematics*, 114, No 3, (1981), p.457-491.
- [9] LOESER, F., *Formules intégrales pour certains invariants locaux des espaces analytiques complexes*, *Comment. Math. Helv.* 59(2)(1984) p. 204-225.
- [10] MACPHERSON, R. *Chern classes for singular algebraic varieties*, *Annals of Mathematics*, JSTOR, (1974), p. 423-432.
- [11] MASSEY, D. *Enriched Relative Polar Curves and Discriminants*, (2006).
- [12] MASSEY, D., *Hypercohomology of Milnor Fibers*, *Topology*, 35 (1996), no. 4 p. 969-1003.
- [13] MASSEY, D. *IPA-deformations of functions on affine space*, *Hokkaido Math. J.* 47 (2018), no. 3, 655-676.
- [14] MASSEY, D. *Numerical control over complex analytic singularities*, *American Mathematical Soc.*, (2003).
- [15] MILNOR, W. J. , *Singular Points of Complex Hypersurfaces*, *Annals of Mathematics Studies*, **25**, New Jersey, (1968).
- [16] MILNOR, W. J. , *Morse Theory - Based on lecture notes by M. Spivak and R. Wells*, *Annals of Mathematics Studies*, **51**, New Jersey, (1963).

- [17] SEADE, J., TIBĀR, M. AND VERJOVSKY, A. *Milnor numbers and Euler obstruction*, *Bulletin of the Brazilian Mathematical Society*, Springer, v. 36, n. 2, (2005), p. 275-283.
- [18] SCHÜRMAN, J. AND TIBAR, M. *Index formula for MacPherson cycles of affine algebraic varieties*, *BTohoku Mathematical Journal, Second Series*, Mathematical Institute, Tohoku University, v. 62, n. 1, (2010), p. 29-44.

Hellen Monção de Carvalho Santana

Institute of Mathematics and Computer Science - University of São Paulo. São Carlos. Brazil  
email: [hellenmcarvalho@hotmail.com](mailto:hellenmcarvalho@hotmail.com)