Surfaces of revolution of frontal curves

Keisuke Teramoto
Department of Mathematics, Kobe University

1. Introduction

A surface of revolution of plane curve is a classical object of differential geometry. Several geometric properties of surfaces of revolution are known (see [10, 15, 16]).

In this note, we consider geometric properties of surfaces of revolution of frontal curves from the viewpoint of framed surfaces introduced by Fukunaga and Takahashi [9]. We show relation between curvature of frontal curves and basic invariants of their surfaces of revolution. Moreover, we consider the classification problem for singularities of surfaces of revolution of frontals under the suitable equivalence relation, and give criteria for certain singularities by the curvature of a Legendre curve. Further, we give profile curves for given information of the curvatures, for instance, the Gauss curvature or the mean curvature. For the cases of constant Gauss and mean curvature surfaces of revolution, see [10].

This note is based on a joint work with Masatomo Takahashi [22]. The author would like to thank Minoru Yamamoto and Takahiro Yamamoto for giving him an opportunity to give a talk at the conference named Research on topology and differential geometry by singularity theory at Research Institute of Mathematical Science (RIMS). This work was supported by JSPS Research Fellows Number 17J02151.

2. Preliminaries

We review quickly the notions of Legendre curves in the Euclidean plane $\boldsymbol{R}^2$ and of framed surfaces in the Euclidean space $\boldsymbol{R}^3$. See [7–9] for detail.

2.1. **Legendre curves.** Let I be an open interval of $\boldsymbol{R}$ with coordinate t. Let $\gamma\colon I \to \boldsymbol{R}^2$ and $\nu\colon I \to S^1$ be C^∞ mappings. Then the pair $(\gamma,\nu)\colon I \to \boldsymbol{R}^2 \times S^1$ is said to be a *Legendre curve* if $\dot\gamma(t)\cdot\nu(t) = 0$ for all $t \in I$, where $\dot\gamma = d\gamma/dt$ and the dot '$\cdot$' is a canonical inner product of $\boldsymbol{R}^2$. A point $t_0 \in I$ is called a *singular point* of γ if $\dot\gamma(t_0) = 0$. If a Legendre curve $(\gamma,\nu)\colon I \to \boldsymbol{R}^2 \times S^1$ gives an immersion, it is called a *Legendre immersion*. We say that $\gamma\colon I \to \boldsymbol{R}^2$ is a *frontal* (resp. *front*) if there exists a map $\nu\colon I \to S^1$ such that $(\gamma,\nu)\colon I \to \boldsymbol{R}^2 \times S^1$ is a Legendre curve (resp. Legendre immersion) (cf. [1, 2]). Examples of Legendre curves see [11, 12].

For a frontal $\gamma\colon I \to \boldsymbol{R}^2$, we have the Frenet type formula as follows. We set $\boldsymbol{\mu}(t)$ as $\boldsymbol{\mu}(t) = J(\nu(t))$, where J is the anti-clockwise rotation by angle $\pi/2$. Then it is easy to see that the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ gives an orthonormal moving frame along $\gamma(t)$. Using this frame, we have

$$\begin{pmatrix}\dot\nu(t)\\ \dot{\boldsymbol{\mu}}(t)\end{pmatrix} = \begin{pmatrix}0 & \ell(t)\\ -\ell(t) & 0\end{pmatrix}\begin{pmatrix}\nu(t)\\ \boldsymbol{\mu}(t)\end{pmatrix}, \quad \dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t), \tag{2.1}$$

where $\ell(t) = \dot\nu(t)\cdot\boldsymbol{\mu}(t)$ and $\beta(t) = \dot\gamma(t)\cdot\boldsymbol{\mu}(t)$. We call the pair (ℓ,β) the *curvature of the Legendre curve*. By the Frenet type formula (2.1), we see that a point $t_0 \in I$

is a singular point of γ if and only if $\beta(t_0) = 0$. Moreover, it follows that (γ, ν) is a Legendre immersion if and only if the curvature (ℓ, β) does not vanish.

Definition 2.1. Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \boldsymbol{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $\boldsymbol{a}$ on $\boldsymbol{R}^2$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + \boldsymbol{a}$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$.

Theorem 2.2 (Existence Theorem for Legendre curves [7]). *Let $(\ell, \beta) : I \to \boldsymbol{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \boldsymbol{R}^2 \times S^1$ whose curvature of the Legendre curve is (ℓ, β).*

Actually, we have the following.

$$\gamma(t) = \left(-\int \beta(t) \sin\left(\int \ell(t)\, dt\right) dt,\ \int \beta(t) \cos\left(\int \ell(t)\, dt\right) dt\right),$$
$$\nu(t) = \left(\cos\left(\int \ell(t)\, dt\right),\ \sin\left(\int \ell(t)\, dt\right)\right).$$

Theorem 2.3 (Uniqueness Theorem for Legendre curves [7]). *Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \boldsymbol{R}^2 \times S^1$ be Legendre curves with the curvatures of Legendre curves (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$, respectively. Then (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are congruent as Legendre curves if and only if (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$ coincide.*

2.2. **Framed surfaces.** Let U be a simply connected domain in $\boldsymbol{R}^2$ and S^2 be the unit sphere in $\boldsymbol{R}^3$. Then we set 3-dimensional manifold Δ by

$$\Delta = \{(\boldsymbol{a}, \boldsymbol{b}) \in S^2 \times S^2 \mid \langle \boldsymbol{a}, \boldsymbol{b} \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\boldsymbol{R}^3$.

We consider a map $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})\colon U \to \boldsymbol{R}^3 \times \Delta$. This map $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is called a *framed surface* if $\langle \boldsymbol{x}_u(u,v), \boldsymbol{n}(u,v) \rangle = \langle \boldsymbol{x}_v(u,v), \boldsymbol{n}(u,v) \rangle = 0$ hold for any $(u,v) \in U$, where $\boldsymbol{x}_u = \partial \boldsymbol{x}/\partial u$ and $\boldsymbol{x}_v = \partial \boldsymbol{x}/\partial v$. A map $\boldsymbol{x}\colon U \to \boldsymbol{R}^3$ is said to be a *framed base surface* if there exists a map $(\boldsymbol{n}, \boldsymbol{s})\colon U \to \Delta$ such that the pair $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ gives a framed surface. Similarly to the case of Legendre curves, the pair $(\boldsymbol{x}, \boldsymbol{n})\colon U \to \boldsymbol{R}^3 \times S^2$ is said to be a *Legendre surface* if $\langle \boldsymbol{x}_u(u,v), \boldsymbol{n}(u,v) \rangle = \langle \boldsymbol{x}_v(u,v), \boldsymbol{n}(u,v) \rangle = 0$ for all $(u,v) \in U$. Moreover, when a Legendre surface $(\boldsymbol{x}, \boldsymbol{n})\colon U \to \boldsymbol{R}^3 \times S^2$ gives an immersion, this is called a *Legendre immersion*. We say that $\boldsymbol{x}\colon U \to \boldsymbol{R}^3$ be a *frontal* (resp. a *front*) if there exists a map $\boldsymbol{n}\colon U \to S^2$ such that the pair $(\boldsymbol{x}, \boldsymbol{n})\colon U \to \boldsymbol{R}^3 \times S^2$ is a Legendre surface (resp. a Legendre immersion). By definition, the framed base surface is a frontal. At least locally, the frontal is a framed base surface. For a framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, we say that a point $p \in U$ is a *singular point of $\boldsymbol{x}$* if $\boldsymbol{x}$ is not an immersion at p. We remark that there are several differential geometrical studies of frontals and fronts in 3-space with certain singularities. See [6, 13, 17, 19, 21, 23], for example.

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})\colon U \to \boldsymbol{R}^3 \times \Delta$ be a framed surface. Set $\boldsymbol{t}(u,v) = \boldsymbol{n}(u,v) \times \boldsymbol{s}(u,v)$, where $\times$ is the vector product of $\boldsymbol{R}^3$. Then the pair $\{\boldsymbol{n}(u,v), \boldsymbol{s}(u,v), \boldsymbol{t}(u,v)\}$ gives a moving frame along $\boldsymbol{x}(u,v)$. Therefore we have the following system of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_u \\ \boldsymbol{x}_v \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{2.2}$$

$$\begin{pmatrix} \boldsymbol{n}_u \\ \boldsymbol{s}_u \\ \boldsymbol{t}_u \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{n}_v \\ \boldsymbol{s}_v \\ \boldsymbol{t}_v \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{2.3}$$

where $a_i, b_i, e_i, f_i, g_i : U \to \boldsymbol{R}, i = 1, 2$ are smooth functions and we call the functions *basic invariants* of the framed surface. We denote the matrices (2.2) and (2.3) by $\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *basic invariants* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Since the integrability condition $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$ and $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$, the basic invariants should satisfy the following conditions ([9]):

$$\begin{cases} a_{1,v} - b_1 g_2 = a_{2,u} - b_2 g_1, \\ b_{1,v} - a_2 g_1 = b_{2,u} - a_1 g_2, \\ a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1, \end{cases} \qquad \begin{cases} e_{1,v} - f_1 g_2 = e_{2,u} - f_2 g_1, \\ f_{1,v} - e_2 g_1 = f_{2,u} - e_1 g_2, \\ g_{1,v} - e_1 f_2 = g_{2,u} - e_2 f_1. \end{cases} \tag{2.4}$$

Definition 2.4. We define a smooth mapping $C_F = (J_F, K_F, H_F) : U \to \boldsymbol{R}^3$ by

$$\begin{aligned} J_F &= \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \quad K_F = \det \begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix}, \\ H_F &= -\frac{1}{2} \left\{ \det \begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix} - \det \begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix} \right\}. \end{aligned} \tag{2.5}$$

We call $C_F = (J_F, K_F, H_F)$ a *curvature of the framed surface.*

We also define a map $I_F : U \to \boldsymbol{R}^8$ by

$$I_F = \left(C_F, \det \begin{pmatrix} a_1 & g_1 \\ a_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix}, \right. \left. \det \begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} a_1 & e_1 \\ a_2 & e_2 \end{pmatrix} \right). \tag{2.6}$$

We call the mapping $I_F : U \to \boldsymbol{R}^8$ a *concomitant mapping* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

We remark that if $\boldsymbol{x}$ is a regular framed base surface, then K_F/J_F and H_F/J_F coincide with the Gauss curvature K and the mean curvature H of $\boldsymbol{x}$, respectively (cf. [9]).

By using a curvature as in (2.5) and a concomitant mapping (2.6) of a framed surface, we have the following.

Proposition 2.5 ([9])**.** *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})\colon U \to \boldsymbol{R}^3 \times \Delta$ be a framed surface. Then*

(1) *$\boldsymbol{x}$ is an immersion around $p \in U$ if and only if $J_F(p) \neq 0$,*
(2) *$(\boldsymbol{x}, \boldsymbol{n})$ is a Legendre immersion around $p \in U$ if and only if $C_F(p) \neq 0$,*
(3) *$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed immersion around $p \in U$ if and only if $I_F(p) \neq 0$.*

3. Surfaces of revolution of frontals

We consider surfaces of revolution whose profile curves are frontals. Let $(\gamma, \nu)\colon I \to \boldsymbol{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β). We write $\gamma(t) = (x(t), z(t))$

and $\nu = (a(t), b(t))$. Then one can take $\boldsymbol{\mu}$ as $\boldsymbol{\mu} = (-b, a)$. By the Frenet type formula (2.1), we have

$$\begin{pmatrix} \dot{x}(t) \\ \dot{z}(t) \end{pmatrix} = \beta(t) \begin{pmatrix} -b(t) \\ a(t) \end{pmatrix}, \quad \begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix} = \ell(t) \begin{pmatrix} -b(t) \\ a(t) \end{pmatrix} \tag{3.1}$$

We define two surfaces $\boldsymbol{x}, \boldsymbol{z} \colon I \times [0, 2\pi) \to \boldsymbol{R}^3$ by

$$\boldsymbol{x}(t,\theta) = (x(t), z(t)\cos\theta, z(t)\sin\theta), \quad \boldsymbol{z}(t,\theta) = (x(t)\cos\theta, x(t)\sin\theta, z(t)). \tag{3.2}$$

We call $\boldsymbol{x}$ (resp. $\boldsymbol{z}$) a *surface of revolution of γ around the x-axis* (resp. *the z-axis*).

Proposition 3.1. (1) *Under the above notations, $(\boldsymbol{x}, \boldsymbol{n}^x, \boldsymbol{s}^x) \colon I \times [0, 2\pi) \to \boldsymbol{R}^3 \times \Delta$ is a framed surface with basic invariants*

$$\mathcal{G}^x = \begin{pmatrix} 0 & -\beta(t) \\ -z(t) & 0 \end{pmatrix}, \quad \mathcal{F}_1^x = \begin{pmatrix} 0 & 0 & \ell(t) \\ 0 & 0 & 0 \\ -\ell(t) & 0 & 0 \end{pmatrix}, \quad \mathcal{F}_2^x = \begin{pmatrix} 0 & b(t) & 0 \\ -b(t) & 0 & -a(t) \\ 0 & a(t) & 0 \end{pmatrix},$$

where $\boldsymbol{n}^x(t,\theta) = (-a(t), -b(t)\cos\theta, -b(t)\sin\theta)$, $\boldsymbol{s}^x(t,\theta) = (0, \sin\theta, -\cos\theta)$.

(2) *Under the above notation, $(\boldsymbol{z}, \boldsymbol{n}^z, \boldsymbol{s}^z) : I \times [0, 2\pi) \to \boldsymbol{R}^3 \times \Delta$ is a framed surface with basic invariants,*

$$\mathcal{G}^z = \begin{pmatrix} 0 & -\beta(t) \\ -x(t) & 0 \end{pmatrix}, \quad \mathcal{F}_1^z = \begin{pmatrix} 0 & 0 & -\ell(t) \\ 0 & 0 & 0 \\ \ell(t) & 0 & 0 \end{pmatrix}, \quad \mathcal{F}_2^z = \begin{pmatrix} 0 & -a(t) & 0 \\ a(t) & 0 & b(t) \\ 0 & -b(t) & 0 \end{pmatrix},$$

where $\boldsymbol{n}^z(t,\theta) = (a(t)\cos\theta, a(t)\sin\theta, b(t))$, $\boldsymbol{s}^z(t,\theta) = (\sin\theta, -\cos\theta, 0)$.

By Proposition 3.1, we have

$$J_F^x(t,\theta) = -\beta(t)z(t), \quad K_F^x(t,\theta) = -b(t)\ell(t), \quad H_F^x(t,\theta) = -\frac{1}{2}(z(t)\ell(t) + \beta(t)b(t)) \tag{3.3}$$

for $\boldsymbol{x}(t,\theta)$ and

$$J_F^z(t,\theta) = -\beta(t)x(t), K_F^z(t,\theta) = -a(t)\ell(t), H_F^z(t,\theta) = \frac{1}{2}(x(t)\ell(t) + \beta(t)a(t)) \tag{3.4}$$

for $\boldsymbol{z}(t,\theta)$. By (3.3) and (3.4), a surface $\boldsymbol{x}$ (resp. $\boldsymbol{z}$) is singular at (t_0, θ_0) if and only if $\beta(t_0) = 0$ or $z(t_0) = 0$ (resp. $\beta(t_0) = 0$ or $x(t_0) = 0$). In particular, we have the following.

Proposition 3.2. *Under the above notation, the following assertions hold.*

(1) *Both $\boldsymbol{x}$ and $\boldsymbol{z}$ as in* (3.2) *are frontals if and only if the profile curve γ is a frontal.*

(2) *Either $\boldsymbol{x}$ or $\boldsymbol{z}$ is a front if and only if γ is a front.*

In the following, we mainly focus on the surface $\boldsymbol{z}$, and we use notations $\boldsymbol{z}_0 = \boldsymbol{z}(t_0, \theta_0)$, $x_0 = x(t_0)$ and $z_0 = z(t_0)$.

3.1. **Singularities of surfaces of revolution.** In this subsection, we consider the singularities. To do this, we give definitions for some notions which we need (cf. [14]).

Definition 3.3. (1) Let f and $g : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ be smooth map-germs. Then f is $\mathcal{A}$-*equivalent* to g if there exist diffeomorphism germs $\phi : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ and $\Phi : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ such that $g = \Phi \circ f \circ \phi^{-1}$ holds. If the diffeomorphism germ Φ (respectively, ϕ) appeared in above is the identity map, we say that f is $\mathcal{R}$-*equivalent* (respectively, $\mathcal{L}$-*equivalent*) to g.

(2) Let $\gamma : (I, t_0) \to (\boldsymbol{R}^2, 0)$ be a smooth curve. We say that γ at t_0 is a j/i-*cusp*, where $(i, j) = (2, 3), (2, 5), (3, 4), (3, 5)$ if γ is $\mathcal{A}$-equivalent to the germ $t \mapsto (t^i, t^j)$ at the origin.

(3) Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a smooth map. We say that f at 0 is a j/i-*cuspidal edge*, where $(i, j) = (2, 3), (2, 5), (3, 4), (3, 5)$ if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^i, v^j)$ at the origin.

We note that curves with j/i-cusps are frontal (curves). Moreover, surfaces with j/i-cuspidal edges are not only frontal (surfaces), but also framed base surfaces (see [7–9]). In particular, 3/2-cusps and 4/3-cusps (respectively, 3/2-cuspidal edges and 4/3-cuspidal edges) are front singularities.

We consider the classification problem. First we consider the case of $x_0 > 0$.

Theorem 3.4. *Let* $\gamma\colon (I, t_0) \to (\boldsymbol{R}^2, (x_0, z_0))$ *and* $\widetilde{\gamma}\colon (\widetilde{I}, \widetilde{t}_0) \to (\boldsymbol{R}^2, (\widetilde{x}_0, \widetilde{z}_0))$ *be smooth curves. Let* $\boldsymbol{z}\colon (I \times [0, 2\pi), (t_0, \theta_0)) \to (\boldsymbol{R}^2 \setminus \{0\} \times \boldsymbol{R}, \boldsymbol{z}_0)$ *and* $\widetilde{\boldsymbol{z}}\colon (\widetilde{I} \times [0, 2\pi), (\widetilde{t}_0, \theta_0)) \to (\boldsymbol{R}^2 \setminus \{0\} \times \boldsymbol{R}, \widetilde{\boldsymbol{z}}_0)$ *be surfaces of revolution of* γ *and* $\widetilde{\gamma}$, *respectively.*

(1) *If there exist diffeomorphism-germs* $\phi\colon (I, t_0) \to (\widetilde{I}, \widetilde{t}_0)$ *and* $\psi\colon (\boldsymbol{R}^2, (x_0, z_0)) \ni (X, Z) \mapsto (\psi_1(X, Z), \psi_2(X, Z)) \in (\boldsymbol{R}^2, (\widetilde{x}_0, \widetilde{z}_0))$ *such that* $\psi \circ \gamma = \widetilde{\gamma} \circ \phi$, *then there exist diffeomorphism-germs* $\Phi\colon (I \times [0, 2\pi), (t_0, \theta_0)) \ni (t, \theta) \mapsto (\phi(t), \theta) \in (\widetilde{I} \times [0, 2\pi), (\widetilde{t}_0, \theta_0))$ *and* $\Psi\colon (\boldsymbol{R}^2 \setminus \{0\} \times \boldsymbol{R}, \boldsymbol{z}_0) \to (\boldsymbol{R}^2 \setminus \{0\} \times \boldsymbol{R}, \widetilde{\boldsymbol{z}}_0)$ *of the form*

$$\Psi(X, Y, Z) = \left(\frac{X\psi_1(\sqrt{X^2 + Y^2}, Z)}{\sqrt{X^2 + Y^2}}, \frac{Y\psi_1(\sqrt{X^2 + Y^2}, Z)}{\sqrt{X^2 + Y^2}}, \psi_2(\sqrt{X^2 + Y^2}, Z)\right)$$

such that $\Psi \circ \boldsymbol{z} = \widetilde{\boldsymbol{z}} \circ \Phi$ *holds.*

(2) *If there exists diffeomorphism-germs* $\Phi\colon (I\times[0, 2\pi), (t_0, \theta_0)) \ni (t, \theta) \mapsto (\phi(t), \theta) \in (\widetilde{I} \times [0, 2\pi), (\widetilde{t}_0, \theta_0))$ *and* $\Psi\colon (\boldsymbol{R}^2 \setminus \{0\} \times \boldsymbol{R}, \boldsymbol{z}_0) \to (\boldsymbol{R}^2 \setminus \{0\} \times \boldsymbol{R}, \widetilde{\boldsymbol{z}}_0)$ *of the form* $\Psi(X, Y, Z) = (\Psi_1(X, Y, Z), \Psi_2(X, Y, Z), \Psi_3(X, Y, Z))$ *such that* $\Psi \circ \boldsymbol{z} = \widetilde{\boldsymbol{z}} \circ \Phi$, *then there exists a diffeomorphism-germ* $\psi\colon (\boldsymbol{R}^2, (x_0, z_0)) \to (\boldsymbol{R}^2, (\widetilde{x}_0, \widetilde{z}_0))$ *of the form*

$$\psi_1(X, Z) = \Psi_1(X \cos\theta_0, X \sin\theta_0, Z) \cos\theta_0 + \Psi_2(X \cos\theta_0, X \sin\theta_0, Z) \sin\theta_0,$$
$$\psi_2(X, Z) = \Psi_3(X \cos\theta_0, X \sin\theta_0, Z)$$

such that $\psi \circ \gamma = \widetilde{\gamma} \circ \phi$.

For a precise proof of this theorem, see [22]. As a corollary of this theorem, we have the following.

Corollary 3.5. *Let* $\gamma\colon (I, t_0) \to (\boldsymbol{R}^2, (x_0, z_0))$ *be a smooth curve with* $x_0 > 0$. *Then* γ *at* t_0 *is a* j/i*-cusp if and only if* $\boldsymbol{z}$ *at* (t_0, θ_0) *is a* j/i*-cuspidal edge.*

Thus it is enough to consider the types of singularities of a frontal curve γ in stead of the surface of revolution $\boldsymbol{z}$ for singularities. In particular, we can characterize types of singularities of γ by its curvature (ℓ, β) as follows.

Theorem 3.6. *Let $(\gamma, \nu)\colon I \to \boldsymbol{R}^2 \times S^1$ be a Legendre curve with the curvature (ℓ, β). Suppose that a point $t_0 \in I$ is a singular point of γ, that is, $\beta(t_0) = 0$. Then we have the following.*

(1) *γ at t_0 is a 3/2-cusp if and only if $\dot\beta(t_0)\ell(t_0) \neq 0$.*
(2) *γ at t_0 is a 5/2-cusp if and only if $\ell(t_0) = 0$, $\dot\beta(t_0) = 0$ and $\ddot\beta(t_0)\dot\ell(t_0) - \dot\beta(t_0)\ddot\ell(t_0) \neq 0$.*
(3) *γ at t_0 is a 4/3-cusp if and only if $\dot\beta(t_0) = 0$ and $\ddot\beta(t_0)\ell(t_0) \neq 0$.*
(4) *γ at t_0 is a 5/3-cusp if and only if $\dot\beta(t_0) = \ell(t_0) = 0$ and $\ddot\beta(t_0)\dot\ell(t_0) \neq 0$.*

One can show this theorem by applying criteria for these singularities in general settings (cf. [3, 20]).

The next, we consider the case of $x_0 = 0$. In this case, the profile curve γ across the rotation axis. Thus if the curve γ is regular, a surface of revolution arising from γ has singularity (such a singularity is called a *cone-like singularity*).

Theorem 3.7. *Let $\gamma\colon (I, t_0) \to (\boldsymbol{R}^2, 0)$ and $\widetilde\gamma\colon (\widetilde I, \widetilde t_0) \to (\boldsymbol{R}^2, 0)$ be smooth curves. Let $\boldsymbol{z}\colon (I \times [0, 2\pi), (t_0, \theta_0)) \to (\boldsymbol{R}^3, 0)$ and $\widetilde{\boldsymbol{z}}\colon (\widetilde I \times [0, 2\pi), (\widetilde t_0, \theta_0)) \to (\boldsymbol{R}^3, 0)$ be surfaces of revolution of γ and $\widetilde\gamma$, respectively.*

(1) *If there exist diffeomorphism-germs $\phi\colon (I, t_0) \to (\widetilde I, \widetilde t_0)$ and $\psi\colon (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ of the form $\psi(X, Z) = (X, \varphi(X^2, Z))$ such that $\psi \circ \gamma = \widetilde\gamma \circ \phi$, then there exist diffeomorphism-germs $\Phi\colon (I \times [0, 2\pi), (t_0, \theta_0)) \to (\widetilde I \times [0, 2\pi), \widetilde t_0, \theta_0)$ given by $\Phi(t, \theta) = (\phi(t), \theta)$ and $\Psi\colon (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ by $\Psi(X, Y, Z) = (X, Y, \varphi(X^2 + Y^2, Z))$ such that $\Psi \circ \boldsymbol{z} = \widetilde{\boldsymbol{z}} \circ \Phi$ holds.*
(2) *If there exist diffeomorphism-germs $\Phi\colon (I \times [0, 2\pi), (t_0, \theta_0)) \to (\widetilde I \times [0, 2\pi), \widetilde t_0, \theta_0)$ given by $\Phi(t, \theta) = (\phi(t), \theta)$ and $\Psi\colon (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ defined by $\Psi(X, Y, Z) = (X, Y, \varphi(X^2 + Y^2, Z))$ such that $\Psi \circ \boldsymbol{z} = \widetilde{\boldsymbol{z}} \circ \Phi$ holds, then there exists a diffeomorphism-germ $\psi\colon (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ of the form $\psi(X, Z) = (X, \varphi(X^2, Z))$ such that $\psi \circ \gamma = \widetilde\gamma \circ \phi$ holds.*

By Theorem 3.7, the following assertion holds.

Proposition 3.8. *Let $\gamma = (x, z), \widetilde\gamma = (\widetilde x, \widetilde z)\colon (I, t_0) \to (\boldsymbol{R}^2, 0)$ be smooth curves. If x is $\mathcal{R}$-equivalent to $\widetilde x$ and z is $\mathcal{L}$-equivalent to $\widetilde z$, then γ is equivalent to $\widetilde\gamma$ in the sense of* (1) *in Theorem 3.7.*

In detail for $\mathcal{L}$-equivalence, see [4, 5].

To state our next result, we review the notion of the order of zeros for functions. Let $f\colon (I, t_0) \to \boldsymbol{R}$ be a smooth function-germ and k the non-negative integer. Then f has a *zero of order* $k+1$ at t_0 if f satisfies

$$f(t_0) = \dot f(t_0) = \cdots = f^{(k)}(t_0) = 0, \quad f^{(k+1)}(t_0) \neq 0,$$

where $f^{(m)} = d^m f/dt^m$. When a function f has a zero of order $k+1$ at t_0, we write $\mathrm{ord}(f)(t_0) = k + 1$.

Using this notion, we characterize cone-like singularities on surfaces of revolution arising from regular curves as follows.

Proposition 3.9. *Let* $\gamma = (x, z)\colon (I, t_0) \to (\boldsymbol{R}^2, 0)$ *be a smooth curve. Suppose that* $\mathrm{ord}(x)(t_0) = k + 1$ *and* z *is regular at* t_0*, namely* $\dot{z}(t_0) \neq 0$*. Then* γ *is equivalent to the germ* $t \mapsto (t^{k+1}, t)$ *in the sense of* (1) *in Theorem 3.7.*

3.2. **Surfaces of revolution with prescribed curvature.** We consider a surface $\boldsymbol{z}(t, \theta)$ and use notations $J(t) = J_F^{\boldsymbol{z}}(t, \theta)$, $K(t) = K_F^{\boldsymbol{z}}(t, \theta)$ and $H(t) = H_F^{\boldsymbol{z}}(t, \theta)$ as in (2.5). First we consider the case that $K(t) = 0$ for all (t, θ).

Proposition 3.10. *If the condition* $K(t) = 0$ *holds for all* $(t, \theta) \in I \times [0, 2\pi)$*, then the profile curve* γ *of* $\boldsymbol{z}$ *is a part of a line.*

By Proposition 3.10, a surface of revolution of a Legendre curve which satisfies $K(t) = 0$ for all (t, θ) is one of a part of a *cone*, a *cylinder*, a *plane*, a *circle* or a *point*.

We next consider the case that prescribed Gauss curvature case.

Theorem 3.11. *Let* $(\gamma, \nu) = ((x, z), (a, b))\colon (I, t_0) \to \boldsymbol{R}^2 \times S^1$ *be a Legendre curve with the curvature* (ℓ, β)*. Suppose that* β *is real analytic around* t_0 *and there exists a function* $\alpha\colon (I, t_0) \to \boldsymbol{R}$ *such that* $K(t) = \alpha(t)J(t)$ *and* $\alpha(t)\beta(t)^2(t - t_0)^2$ *is real analytic around* t_0*. Then* $x(t)$ *is a solution of*

$$\beta(t)\ddot{x}(t) - \dot{\beta}(t)\dot{x}(t) + \alpha(t)\beta(t)^3 x(t) = 0$$

around t_0*,* $z(t)$ *is given by*

$$z(t) = \pm \int \beta(t) \left(1 - \left(\int \alpha(t)\beta(t)x(t)dt\right)^2\right)^{\frac{1}{2}} dt$$

and

$$a(t) = \pm \left(1 - \left(\int \alpha(t)\beta(t)x(t)dt\right)^2\right)^{\frac{1}{2}}, \; b(t) = \int \alpha(t)\beta(t)x(t)dt.$$

By Theorem 3.6, we have the following.

Proposition 3.12. *Let* $(\gamma, \nu) = ((x, z), (a, b))\colon (I, t_0) \to \boldsymbol{R}^2 \times S^1$ *be a Legendre curve which is given by Theorem 3.11. Let* t_0 *be a singular point of* γ*. Suppose that* $x(t_0) \neq 0$ *and* $\alpha(t_0) \neq 0$*. Then we have the following.*

(1) *If* $a(t_0) \neq 0$*, then* γ *is a frontal but not a front at* t_0*.*
(2) *Suppose that* $a(t_0) = 0$*,* $\mathrm{ord}(a)(t_0) = m + 1$ *and* $\mathrm{ord}(\beta)(t_0) = n + 1$ *for non-negative integers* m*,* n*. Then* γ *is a front at* t_0 *if and only if* $\mathrm{ord}(a)(t_0) = \mathrm{ord}(\beta)(t_0)$ *holds.*
(3) *Suppose that there exists a real constant* $c \neq 0$ *such that* $\alpha(t) = c$*, and* $\mathrm{ord}(a)(t_0) = m + 1 \leq \mathrm{ord}(\beta)(t_0) = n + 1$*. Then*
    (a) γ *at* t_0 *is a* $3/2$*-cusp if and only if* $\mathrm{ord}(a)(t_0) = ord(\beta)(t_0) = 1$*.*
    (b) γ *at* t_0 *is a* $4/3$*-cusp if and only if* $\mathrm{ord}(a)(t_0) = ord(\beta)(t_0) = 2$*.*
    (c) γ *at* t_0 *is a* $5/3$*-cusp if and only if* $\mathrm{ord}(a)(t_0) = 1$*,* $ord(\beta)(t_0) = 2$*.*
    (d) γ *cannot have a* $5/2$*-cusp at* t_0*.*

We next consider the prescribed mean curvature case. The following assertion holds.

Theorem 3.13. *Let $(\gamma,\nu)=((x,z),(a,b))\colon(I,t_0)\to\boldsymbol{R}^2\times S^1$ be a Legendre curve with the curvature (ℓ,β). Suppose that $x(t_0)>0$, we give β and there exists a smooth function $\alpha\colon(I,t_0)\to\boldsymbol{R}$ such that $H(t)=\alpha(t)J(t)$. Then (γ,ν) is given by*

$$x(t)=(F(t)^2+G(t)^2)^{\frac{1}{2}},\ z(t)=\int\frac{\beta(t)}{x(t)}\left(F(t)\sin\eta(t)-G(t)\cos\eta(t)\right)dt$$

and

$$a(t)=\frac{F(t)\sin\eta(t)-G(t)\cos\eta(t)}{x(t)},\ b(t)=\frac{F(t)\cos\eta(t)+G(t)\sin\eta(t)}{x(t)},$$

where

$$F(t)=-\int\beta(t)\cos\eta(t)dt,\ G(t)=-\int\beta(t)\sin\eta(t)dt,\ \eta(t)=2\int\alpha(t)\beta(t)dt.$$

We remark that this kind representations are given by Kenmotsu [15] for regular case and Martins, Saji, Santos and the author [18] for singular case admitting unbounded mean curvature.

For a Legendre curve which is given by Theorem 3.13, we have the following characterizations of singularities.

Proposition 3.14. *Let $(\gamma,\nu)=((x,z),(a,b))\colon(I,t_0)\to\boldsymbol{R}^2\times S^1$ be a Legendre curve which is given by Theorem 3.13. Suppose that t_0 is a singular point of γ. Then we have the following.*

(1) *The curve γ is a frontal but not a front near t_0.*
(2) *Suppose that α is not a constant function. Then*
  (a) *γ at t_0 is a 5/2-cusp if and only if $\dot\beta(t_0)\dot\alpha(t_0)\neq 0$.*
  (b) *γ at t_0 is a 5/3-cusp if and only if $\dot\beta(t_0)=0$ and $\ddot\beta(t_0)\dot\alpha(t_0)\neq 0$.*
(3) *Suppose that α is a constant function. Then γ does not have j/i-cusps $((i,j)=(2,3),(2,5),(3,4),(3,5))$.*

References

[1] Arnol'd, V. I.: Singularities of Caustics and Wave Fronts. Mathematics and Its Applications **62** Kluwer Academic Publishers (1990)
[2] Arnol'd, V. I., Gusein-Zade, S. M., Varchenko, A. N.: Singularities of Differentiable Maps. Vol. I, Birkhäuser (1986)
[3] Bruce, J. W., Gaffney, T. J.: Simple singularities of mappings $\mathbf{C},0\to\mathbf{C}^2,0$. J. London Math. Soc. (2) **26**, 465–474 (1982)
[4] Bruce, J. W., Gaffney, T. J., du Plessis, A. A.: On left equivalence of map germs. Bull. London Math. Soc. **16**, 303–306 (1984)
[5] Bruce, J. W., Giblin, P. J.: Curves and Singularities. A Geometrical Introduction to Singularity Theory. Second Edition. Cambridge University Press, Cambridge (1992)
[6] Fujimori, S., Saji, K., Umehara, M., Yamada, K.: Singularities of maximal surfaces. Math. Z. **259**, 827–848 (2008)
[7] Fukunaga, T., Takahashi, M.: Existence and uniqueness for Legendre curves. J. Geom. **104**, 297–307 (2013)
[8] Fukunaga, T., Takahashi, M.: Evolutes of fronts in the Euclidean plane. J. Singul. **10**, 92–107 (2014)

[9] Fukunaga, T., Takahashi, M.: Framed surface in the Euclidean space. Bull. Braz. Math. Soc. (N.S.) (2018) https://doi.org/10.1007/s00574-018-0090-z
[10] Gray, A., Abbena, E., Salamon, S.: Modern Differential Geometry of Curves and Surfaces with Mathematica. Third edition. Studies in Advanced Mathematics. Chapman and Hall/CRC, Boca Raton, FL (2006)
[11] Ishikawa, G.: Zariski's moduli problem for plane branches and the classification of Legendre curve singularities. Real and complex singularities, World Sci. Publ., Hackensack, NJ, 56–84 (2007)
[12] Ishikawa, G,: Singularities of Curves and Surfaces in Various Geometric Problems. CAS Lecture Notes 10, Exact Sciences (2015)
[13] Izumiya, S., Saji, K.: The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces. J. Singul. **2**, 92–127 (2010)
[14] Izumiya, S., Romero-Fuster, M. C., Ruas, M. A. S., Tari, F.: Differential Geometry from a Singularity Theory Viewpoint. World Scientific Pub. Co Inc. (2015)
[15] Kenmotsu, K.: Surfaces of revolution with prescribed mean curvature. Tôhoku Math. J. (2) **32**, 147–153 (1980)
[16] Kenmotsu, K.: Surfaces of revolution with periodic mean curvature. Osaka J. Math. **40**, 687–696 (2003)
[17] Kokubu, M., Rossman, W., Saji, K., Umehara, M., Yamada, K.: Singularities of flat fronts in hyperbolic space. Pacific J. Math. **221**, 303–351 (2005)
[18] Martins, L. F., Saji, K., dos Santos, S. P., Teramoto, K.: Singular surfaces of revolution with prescribed unbounded mean curvature. An. Acad. Brasil. Ciênc. (to appear). arXiv: 1712.07330
[19] Martins, L. F., Saji, K., Umehara, M., Yamada, K.: Behavior of Gaussian curvature and mean curvature near non-degenerate singular points on wave fronts. Geometry and topology of manifolds, Springer Proc. Math. Stat, 154, 247-281 (2016)
[20] Porteous, I. R.: Geometric differentiation. For the intelligence of curves and surfaces. Second edition. Cambridge University Press, Cambridge (2001)
[21] Saji, K., Umehara, M., Yamada, K.: The geometry of fronts. Ann. of Math. (2) **169**, 491–529 (2009)
[22] Takahashi, M., Teramoto, K.: Surfaces of revolution of frontal in the Euclidean space, preprint, arXiv: 1812.10207
[23] Teramoto, K.: Focal surfaces of wave fronts in the Euclidean 3-space. Glasgow Math. J. **61**, 425–440 (2019)

Department of Mathematics
Graduate School of Science
Kobe University
Kobe 657-8501
Japan
E-mail address: teramoto@math.kobe-u.ac.jp