ON UNIFORM K-STABILITY OF PAIRS FOR ALGEBRAIC CURVES

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Abstract. In this note, we shall consider uniform K-stability of pairs, which is recently detected by G. Tian [Ti18] and is originally studied by S. Paul [Pau12]. As a consequence, we show that the rational normal curve of degree $d \geqslant 2$ is either uniformly K-stable or strictly K-semistable with respect to the standard torus action.

1. Introduction

The concept of *uniform K-stability* originates the paper by Székelyhidi [Szé06] in order to deal with the existence problem on a constant scalar curvature Kähler (cscK) metric in a given Kähler class $[\omega]$ of a compact Kähler manifold (X, ω). This notion was deeply investigated by many researchers as in [Der16, BHJ17]. Meanwhile, it is crucial to see the asymptotic behavior of the K-energy map $\nu_\omega : \mathcal{H}_\omega \to \mathbb{R}$ when we discuss about existence of cscK metric, say the coercivity or boundedness of ν_ω [Hisa16, Pau12, Pau13]. The current philosophy is that one can restrict attention to the subspace of Bergman metrics (which is a Hermitian metric that can be defined as the pull back of Fubini-Study metric associated to the projective embedding $X \hookrightarrow \mathbb{P}^N$) in $\mathcal{H}_\omega$ to detect the coercivity and boundedness of ν_ω because of Tian's density theorem and the partial C^0-estimate.

Eventually it was conjectured that the K-energy bounds and coercivity along Bergman potential could be controlled by an appropriate notion of Mumford's GIT stability. This speculation was justified by S.Paul in his paper [Pau12, Pau13] building upon work of Tian [Ti97] and Gelfand-Kapranov-Zelevinsky [GKZ94]. Paul's formulation fits better with Mumford's GIT stability than (original) K-stability in [Ti97, Dona02], so to called *stability of pairs*. Very recently, this notion was developed by Tian in [Ti18] and he introduced the concept of *uniform K-stability of pairs*. In particular, it was shown that *K-stability of pairs implies that uniform K-stability of pairs* (cf. Theorem 2.5). This guarantees that K-stability of pairs is stronger concept than uniform K-stability of pairs. However, we don't know which smooth projective variety would be uniformly K-stable of pairs so far, by direct computation even in 1-dimensional algebraic curve case.

The aim of this note is to study the uniform K-stability of pairs on the rational normal curve of degree $d \geqslant 2$. Namely the main theorem in this article is the following.

Theorem 1.1. *Let $\mathbb{P}^1 \to X_d \subset \mathbb{P}^d$ be the rational normal curve of degree $d \geqslant 2$, which is given by the d-th Veronese embedding (See Section 4). Let R_X and Δ_X be the X-resultant and the X-discriminant respectively. Setting $v = R_X^{\deg(\Delta_X)}$ and $w = \Delta_X^{\deg(R_X)}$, we consider the weight polytopes $\mathcal{N}_H(v)$ and $\mathcal{N}_H(w)$ with respect to the standard torus*

$$H \cong \begin{pmatrix} t_0 & & & & \\ & t_1 & & & \mathit{0} \\ & & \ddots & & \\ & & & t_{d-1} & \\ & \mathit{0} & & & (t_0 \cdots t_{d-1})^{-1} \end{pmatrix} \leq \ \mathrm{SL}(\mathrm{d}+1, \mathbb{C}).$$

Then the inclusion $\mathcal{N}_H(v) \subseteq \mathcal{N}_H(w)$ always holds (Proposition 4.1). In particular

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- *If* $\operatorname{dist}(\partial\mathcal{N}_H(v), \partial\mathcal{N}_H(w)) > 0$, *then* $X \to \mathbb{P}^d$ *is uniformly* K*-stable with respect to* H*-action.*
- *If* $\operatorname{dist}(\partial\mathcal{N}_H(v), \partial\mathcal{N}_H(w)) = 0$, *then* $X \to \mathbb{P}^d$ *is strictly* K*-semistable with respect to* H*-action.*

This article is organized as follows. In Section 2, we define (uniform) K-stability of pairs for given nonzero vectors $v \in V \setminus \{0\}$ and $w \in W \setminus \{0\}$ where V and W are finite dimensional rational representations of a reductive group G. We also introduce some key theorems due to Paul and Tian which we will use later on. In Section 3, we define the X-resultant and the X-(hyper)discriminant which will play a crucial role to discuss on asymptotic of the K-energy. Section 4 is devoted to a brief review of X-resultants/discriminants for polynomials in one variable (see [GKZ94, Chapter 12]). Section 5 gives the proof of Theorem 1.1. A concrete example is dealt in the final section. We compute explicitly the X-resultant/discriminant and their weight polytopes for the quadric curve in $\mathbb{P}^2$ by using the secondary polytope.

2. Stability of Pairs

2.1. **Representation Theory.** Let G be the special linear group $\mathrm{SL}(N+1, \mathbb{C})$ and (V, ρ) be a rational G-representation with $v \in V \setminus \{0\}$. Recall that V is said to be *rational* if for any $v \in V \setminus \{0\}$ and $\alpha \in V^\vee$ (the dual vector space),

$$\varphi_{\alpha,v} : G \longrightarrow \mathbb{C} \qquad \varphi_{\alpha,v}(\sigma) := \alpha(\rho(\sigma)\cdot v)$$

is a regular function on G, that is,

$$\varphi_{\alpha,v} \in \mathbb{C}[G] := \text{ affine coordinate ring of } G.$$

Let T be a maximal algebraic torus of G. We denote the character lattice of T by

$$M_\mathbb{Z} := \mathrm{Hom}_\mathbb{Z}(T, \mathbb{C}^\times) \cong \mathbb{Z}^N.$$

Then the dual lattice $N_\mathbb{Z} = \mathrm{Hom}_\mathbb{Z}(\mathbb{C}^\times, T)$ is identified with the set of one parameter subgroups $\lambda : \mathbb{C}^\times \to T$. Note that the duality is given by

$$\langle\,\cdot\,,\,\cdot\,\rangle : N_\mathbb{Z} \times M_\mathbb{Z} \longrightarrow \mathbb{Z}, \qquad \chi(\lambda(t)) = t^{\langle\lambda,\chi\rangle}.$$

As usual, we set

$$M_\mathbb{R} := M_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^N \qquad \text{and} \qquad N_\mathbb{R} := N_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}.$$

Denoting the image of λ in $N_\mathbb{R}$ by l_λ, we see that l_λ is an integral linear functional on $M_\mathbb{R}$. Furthermore, V decomposes into eigenvalue subspaces under the action of T such as:

$$V = \bigoplus_{\chi\in\mathrm{Supp}(V)} V_\chi, \qquad V_\chi := \{\, v \in V \mid \rho(t)\cdot v = \chi(t)\cdot v,\ \ t \in T \,\}$$

where $\mathrm{Supp}(V)$ denotes the support of V which is defined by

$$\mathrm{Supp}(V) := \{\, \chi \in M_\mathbb{Z} \mid V_\chi \neq 0 \,\}.$$

Definition 2.1. Let $v \in V \setminus \{0\}$ be a nonzero vector in V with

$$v = \sum_{\chi\in M_\mathbb{Z}} v_\chi, \qquad v_\chi \in V_\chi.$$

The *weight polytope* of v (with respect to T-action) is the lattice polytope in $M_\mathbb{R}$ defined by

$$\mathcal{N}_T(v) := \mathrm{Conv}\,\{\, \chi \in M_\mathbb{Z} \mid v_\chi \neq 0 \,\},$$

where $\mathrm{Conv}A$ denotes the convex hull of a finite set of points A.

Let $\mathcal{GL}(N+1,\mathbb{C})$ be the vector space of square matrices of size $N+1$. By the action of matrix multiplication, $\mathcal{GL}(N+1,\mathbb{C})$ can be regarded as a G-representation:

$$G \times \mathcal{GL}(N+1,\mathbb{C}) \longrightarrow \mathcal{GL}(N+1,\mathbb{C}) \qquad (\sigma, A) \longmapsto \sigma \cdot A.$$

Denoting $\mathbb{I} \in \mathcal{GL}(N+1,\mathbb{C})$ to be the identity operator, we define the *standard simplex* Q_N by

$$Q_N := \mathcal{N}_T(\mathbb{I}) \subset M_{\mathbb{R}} \cong \mathbb{R}^N.$$

Note that Q_N is full-dimensional convex polytope containing the origin in its interior. Using the standard simplex Q_N, we define the *degree* of V as

$$\deg(V) := \min\{\, k \in \mathbb{Z}_{>0} \mid \mathcal{N}(v) \subseteq kQ_N \text{ for all } v \in V \setminus \{0\}\,\}.$$

Definition 2.2. Let V be a rational representation of G, and let λ be a one parameter subgroup in T which is a maximal algebraic torus of G. We define the *weight* $w_\lambda(v)$ of λ on $v \in V \setminus \{0\}$ by

$$w_\lambda(v) := \min_{\chi \in \mathcal{N}(v)} l_\lambda(x) = \min\{\, \langle \chi, \lambda \rangle \mid \chi \in \operatorname{supp}(v)\,\}.$$

Alternatively $w_\lambda(v)$ is determined as the unique integer satisfying

$$\lim_{|t|\to 0} t^{-w_\lambda(v)} \lambda(t) v = v_0 \neq 0$$

where v_0 is the nonzero limit in V.

Definition 2.3. Let V and W be (finite dimensional) complex rational representations of G with nonzero vectors $v \in V \setminus \{0\}$ and $w \in W \setminus \{0\}$.

(1) The pair (v, w) is *K-semistable* if $w_\lambda(w) \leqslant w_\lambda(v)$ for any one parameter subgroup λ in G.

(2) (v, w) is *K-stable* if it is K-semistable and $w_\lambda(w) < w_\lambda(v)$ whenever the one parameter subgroup λ satisfying $\deg(V) w_\lambda(\mathbb{I}) < w_\lambda(v)$.

(3) (v, w) is said to be *uniformly K-stable* if there is an integer $m > 0$ such that for any one parameter subgroup λ in G, we have the inequality

$$m(w_\lambda(v) - w_\lambda(w)) \geqslant w_\lambda(v) - \deg(V) w_\lambda(\mathbb{I}).$$

The following Hilbert-Mumford criterion for stability of pairs was discovered by Paul [Pau13] and Tian [Ti18].

Theorem 2.4 (Hilbert-Mumford Criterion)**.** *The relationship between stability of pairs and weight polytopes is described as follows. In the table below, $P + Q$ denotes the Minkowski summation of two polytopes P and Q.*

Stability of pairs (v, w)	**Weight polytopes**
K-semistable	$\mathcal{N}_T(v) \subseteq \mathcal{N}_T(w)$ for all maximal torus $T \leq G$
uniformly K-stable	$^\exists m \in \mathbb{Z}_{>0}$ such that $\left(1 - \frac{1}{m}\right)\mathcal{N}_T(v) + \frac{1}{m}\deg(V) Q_N \subseteq \mathcal{N}_T(w)$ for all maximal torus $T \leq G$

The main theorem in [Ti18] is the following:

Theorem 2.5 (Tian)**.** *If (v, w) is K-stable, then it is uniformly K-stable.*

2.2. **A Kempf-Ness type functional.** For any complex vector space V with $v \in V \setminus \{0\}$, let us denote the line through v by $[v] \in \mathbb{P}(V)$. Then if V and W are two finite dimensional rational complex representations of G with $v \in V \setminus \{0\}$ and $w \in W \setminus \{0\}$ respectively, we consider the projective orbits given by

$$\mathcal{O}_{vw} := G \cdot [(v, w)] \subset \mathbb{P}(V \oplus W), \qquad \mathcal{O}_v := G \cdot [(v, 0)] \subset \mathbb{P}(V \oplus \{0\}).$$

We equip V and W with Hermitian norms $\| \ \|$. The *energy of pair* $p_{v,w}$ is a Kempf-Ness type functional defined by

$$p_{v,w}(\sigma) := \log \|\sigma \cdot w\|^2 - \log \|\sigma \cdot v\|^2, \qquad \sigma \in G.$$

Then we recall the following fact:

Proposition 2.6. [Pau13, Proposition 4.4] *$p_{v,w}$ is bounded from below if and only if*

$$\overline{\mathcal{O}}_{vw} \cap \overline{\mathcal{O}}_v = \emptyset \tag{2.1}$$

where $\overline{\mathcal{O}}_{vw}$, $\overline{\mathcal{O}}_v$ denote the Zariski closures of each orbit.

The definition of K-semistability due to Paul [Pau12, Pau12a, Pau13] originates from the following fact.

Theorem 2.7 (Paul)**.** *The pair (v, w) is K-semistable if and only if $\overline{\mathcal{O}}_{vw} \cap \overline{\mathcal{O}}_v = \emptyset$.*

Proof. Note that "only if " part is clear. Hence it suffices to show that if (v, w) is K-semistable, then (2.1) holds. In order to prove this, suppose (v, w) is K-semistable. Then (v, w) is either

(a) K-stable, or
(b) strictly K-semistable.

In the case where (a): Since (v, w) is K-stable, obviously (2.1) holds by definition. Hence our problem can be reduced to show the following statement: □

Claim 2.8. *If (v, w) is strictly K-semistable, then* (2.1) *holds*

Proof of Claim 2.8. We use the contradiction. Suppose that (v, w) is strictly K-semistable and

$$\overline{G \cdot [v, w]} \cap \overline{G \cdot [v, 0]} \neq \emptyset \tag{2.2}$$

holds. Let $Z := \mathbb{P}(V \oplus \{0\})$ be a G-invariant closed subset in $\mathbb{P}(V \oplus W)$. Since $\overline{G \cdot [v, 0]} \subset Z$, there is an element $z \in Z \cap \overline{G \cdot [v, w]}$ by our assumption (2.2). Setting $x := [v, w]$, we apply Theorem 5.6 in [BHJ17] to x. Then there is a one parameter subgroup λ in G such that

$$\lim_{|t| \to 0} \lambda(t) \cdot x = z \in Z.$$

On the other hand, by definition of $w_\lambda(\cdot)$, we have

$$\lim_{|t| \to 0} \lambda(t) \cdot x = \lim_{|t| \to 0} \lambda(t)([v, w]) = \lim_{|t| \to 0} t^{-(w_\lambda(v) - w_\lambda(w))} \lambda(t)([v, w]).$$

The last equality implies that $w_\lambda(v) - w_\lambda(w) < 0$ but this contradicts to strict K-semistablity. □

Recall that the *Hilbert-Schmidt norm* of a matrix $A = (a_{ij})$ is a matrix norm defined by

$$\|A\|_{\mathrm{HS}} := \sqrt{\sum_{ij} a_{ij}^2}.$$

Then we have

Theorem 2.9. [Ti18, Theorem 1.4] *If (v, w) is K-stable, then there is a positive integer $m \in \mathbb{Z}_{>0}$ and uniform constant C such that*

$$m p_{v,w}(\sigma) \geqslant \deg(V) \log \|\sigma\|_{\mathrm{HS}}^2 - \log \|\sigma(v)\|^2 - C, \qquad \textit{for all } \sigma \in G.$$

Example 2.10 (Relationship with the classical GIT stability)**.** In order to clarify the relationship between stability of pairs and Hilbert-Mumford stability, we consider the case where $V \cong \mathbb{C}$ is the one dimensional trivial representation and $v = 1$:

Hilbert-Mumford (GIT) stability	Stability of pairs
$0 \notin \overline{G \cdot w}$	$\overline{\mathcal{O}}_{vw} \cap \overline{\mathcal{O}}_v = \emptyset$
$w_\lambda(w) \leqslant 0$ for any degeneration λ of G	$w_\lambda(w) - w_\lambda(v) \leqslant 0$ for any degeneration λ of G
For each maximal torus $T \leq G,\ \ 0 \in \mathcal{N}_T(w)$	For each maximal torus $T \leq G,\ \ \mathcal{N}_T(v) \subseteq \mathcal{N}_T(w)$
$\overline{G \cdot w} = G \cdot w$ and G_w is finite	$^{\exists} m \in \mathbb{Z}_{>0}$ s.t $(\mathbb{I}^{\deg(V)} \otimes v^m, w^{m+1})$ is K-semistable

2.3. **Support functions on weight polytopes.** Finally we introduce the key lemma of stability of pairs due to Tian for later use. Let $v \in V$ be a nonzero vector of a finite dimensional rational G-representation V. We fix a maximal torus $T \leq G$. Recall that the *support function* $h_v : N_{\mathbb{R}} \to \mathbb{R}$ of the weight polytope $\mathcal{N}_T(v)$ is the convex function defined as

$$h_v(\lambda) = \max_{\chi \in \mathrm{supp}(v)} \langle \chi, \lambda \rangle. \tag{2.3}$$

Then by Definition 2.2, we readily see that

$$w_\lambda(v) = \min \{ \langle \chi, \lambda \rangle \mid \chi \in \mathrm{supp}(v) \} = -h_v(-\lambda).$$

Then we have the following characterization of K-stability.

Lemma 2.11. [Ti18, Proposition 8.1 (2)] *The pair (v, w) is K-stable with respect to T if and only if the following two conditions*

(1) $\mathcal{N}_T(v) \subseteq \mathcal{N}_T(w)$
(2) $\{ x \in N_{\mathbb{R}} \mid h_v(x) = h_w(x) \} \subseteq \{ x \in N_{\mathbb{R}} \mid h_v(x) = h_{\deg(V)\mathcal{N}(\mathbb{I})}(x) \}$

are satisfied.

3. K-energy asymptotics and stability

Let $X \to \mathbb{P}^N$ be a linearly normal algebraic variety of $\deg X \geqslant 2$. Recall that a projective variety $X \subset \mathbb{P}^N$ is called *linearly normal* if it is non-degenerate (i.e., X is not contained in a hyperplane) and cannot be represented as an isomorphic projection of a non-degenerate variety from a projective space of higher dimension. These conditions require an isomorphism

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \cong H^0(X, \mathcal{O}_X(1)).$$

Unless otherwise stated, we consider an irreducible n-dimensional linearly normal complex projective variety $X^n \to \mathbb{P}^N$ of $\deg X \geqslant 2$ throughout this section.

We denote the Grassmannian of k-dimensional projective linear subspaces of $\mathbb{P}^N$ by $\mathbb{G}(k, N)$.

Definition 3.1. The *Cayley-Chow form* (*X-resultant*) of $X \to \mathbb{P}^N$, denoted by R_X, is the defining polynomial of the irreducible divisor

$$Z_X := \{ L \in \mathbb{G}(N-n-1, N) \mid L \cap X \neq \emptyset \} = \{ L \in \mathbb{G}(N-n-1, N) \mid R_X(L) = 0 \} \tag{3.1}$$

which is uniquely determined up to scaling. Observe that $\deg(R_X) = d$ in the Plucker coordinate and

$$R_X \in \mathbb{C}_{d(n+1)}[M_{(n+1)\times(N+1)}].$$

Let $(\mathbb{P}^N)^\vee$ be the dual projective space of $\mathbb{P}^N$ so that points of $(\mathbb{P}^N)^\vee$ correspond to hyperplanes in $\mathbb{P}^N$. Denoting $\operatorname{Sm} X$ the smooth points of X (i.e, $\operatorname{Sm} X = X \setminus \operatorname{Sing} X$), we consider the Zariski tangent space $\mathbb{T}_p(X)$ to X at $p \in \operatorname{Sm} X$ which is an n-dimensional projective linear subspace of $\mathbb{P}^N$.

Definition 3.2. The *dual variety* $X^* \subset (\mathbb{P}^N)^\vee$ of $X \to \mathbb{P}^N$ is the Zariski closure of the set of tangent hyperplanes to X:

$$X^* := \text{ Zariski Closure} \left\{ f \in (\mathbb{P}^N)^\vee \;\middle|\; \mathbb{T}_p(X) \subset \ker(f) \text{ for some } p \in \operatorname{Sm} X \right\}.$$

When X^* has codimension one in $(\mathbb{P}^N)^\vee$, i.e, the *dual defect*

$$\delta(X) := N - \dim(X^*) - 1$$

is zero, then there exists an irreducible homogeneous defining polynomial

$$\Delta_X \in \mathbb{C}[(\mathbb{P}^N)^\vee] \cong \mathbb{C}[M_{1\times(N+1)}]$$

of X^* which we shall call X*-discriminant*:

$$X^* = \left\{ f \in (\mathbb{P}^N)^\vee \;\middle|\; \Delta_X(f) = 0 \right\}.$$

We further consider the following Segre embedding

$$X \times \mathbb{P}^{n-1} \longrightarrow \mathbb{P}\left(M^\vee_{n\times(N+1)}\right) \tag{3.2}$$

for a given projective embedding $X^n \to \mathbb{P}^N$. Then it follows that the dual defect $\delta(X \times \mathbb{P}^{n-1})$ of the Segre image of (3.2) is zero (see [Pau13, Proposition 3.1]). Hence there exists a non-constant homogeneous polynomial

$$\Delta_{X\times\mathbb{P}^{n-1}} \in \mathbb{C}\left[M_{n\times(N+1)}\right]$$

such that

$$\left(X \times \mathbb{P}^{n-1}\right)^* = \left\{ f \in \mathbb{P}\left(M_{n\times(N+1)}\right) \;\middle|\; \Delta_{X\times\mathbb{P}^{n-1}}(f) = 0 \right\}.$$

We call $\Delta_{X\times\mathbb{P}^{n-1}}$ the X*-hyperdiscriminant*

In order to state the main result in [Pau12], we introduce the standard notation of Kähler geometry. For a smooth complex projective variety $X \to \mathbb{P}^N$ of degree $d \geqslant 2$, we set $\omega := \omega_{\mathrm{FS}}|_X$ where ω_{FS} is the standard Fubini-Study Kähler form on $\mathbb{P}^N$. For each $\sigma \in G$, we consider the *Bergman potential* $\varphi_\sigma \in C^\infty(X)$ given by

$$\sigma^*\omega = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi_\sigma > 0.$$

We readily see that φ_σ is given by the formula

$$\varphi_\sigma = \log\left(\frac{\|\sigma z\|^2}{\|z\|^2}\right) \in \mathcal{H}_\omega$$

where $\|\cdot\|$ denotes the standard norm on polynomials defined as

$$\|f\|^2 = \sum \frac{|C_{\alpha_0\cdots\alpha_N}|^2}{\alpha_0!\alpha_1!\cdots\alpha_N!} \qquad \text{for} \quad f(z) = \sum C_{\alpha_0\cdots\alpha_N} z_0^{\alpha_0}\cdots z_N^{\alpha_N}.$$

Theorem 3.3 (Theorem A in [Pau12])**.** *Let $X^n \hookrightarrow \mathbb{P}^N$ be a smooth, linearly normal complex projective variety of degree $d \geqslant 2$. Let R_X and $\Delta_{X\times\mathbb{P}^{n-1}}$ be the X-resultant and the X-hyperdiscriminant respectively. Then under a suitable normalization of norms, the K-energy ν_ω of (X,ω) restricted to the Bergman metrics is*

$$\mu_\omega(\varphi_\sigma) = \deg(R_X)\log\frac{\|\sigma\cdot\Delta_{X\times\mathbb{P}^{n-1}}\|^2}{\|\Delta_{X\times\mathbb{P}^{n-1}}\|^2} - \deg\left(\Delta_{X\times\mathbb{P}^{n-1}}\right)\log\frac{\|\sigma\cdot R_X\|^2}{\|R_X\|^2}.$$

Combining Thereom 3.3, Proposition 2.6 and Theorem 2.4, we have the following.

Theorem 3.4 (Theorem C in [Pau12])**.** *The K-energy of (X,ω) restricted to the Bergman metrics is bounded from below along any one parameter subgroup λ in G if and only if the inclusion between the weight polytopes*

$$\deg(\Delta_{X\times\mathbb{P}^{n-1}})\mathcal{N}_T(R_X)\subseteq\deg(R_X)\mathcal{N}_T(\Delta_{X\times\mathbb{P}^{n-1}}) \tag{3.3}$$

holds for each maximal torus $T\leq G$.

Hence it is natural to define the following.

Definition 3.5. Let $X^n\to\mathbb{P}^N$ be a linearly normal complex projective (not necessarily smooth) variety of $\deg X\geqslant 2$. Then X is said to be *K-semistable* (resp. *K-stable, uniformly K-stable*) if the pair

$$\left(R_X^{\deg(\Delta_{X\times\mathbb{P}^{n-1}})},\ \Delta_{X\times\mathbb{P}^{n-1}}^{\deg(R_X)}\right)$$

is K-semistable (resp. K-stable, uniformly K-stable) in the sense of Definition 2.3 along all one parameter subgroup λ in G.

Remark 3.6. From the view point of the Hilbert-Mumford criterion (Theorem 2.4), K-semistablity of $X\to\mathbb{P}^N$ is equivalent to the condition of (3.3) for each maximal tori $T\leq G$.

As a special case of Theorem 3.4, we conclude that the following fact.

Corollary 3.7 (Theorem 4.2 in [Pau08])**.** *The restriction of the K-energy of a projective algebraic curve $X\hookrightarrow\mathbb{P}^N$ is bounded from below along all one parameter subgroup λ in G if and only if the following inclusion holds for any maximal torus $T\leq G$:*

$$\frac{\deg(\Delta_X)}{\deg(R_X)}\mathcal{N}_T(R_X)\subseteq\mathcal{N}_T(\Delta_X). \tag{3.4}$$

4. The rational normal curves and GKZ theory

One of the interesting case study about (3.4) is where X is the rational normal curve of $\deg X\geqslant 2$. Recall that the *rational normal curve* of degree d is given by the image of the d-th Veronese embedding of $\mathbb{P}^1$ such that

$$\nu_d:\ \mathbb{P}^1\longrightarrow\mathbb{P}^d,\qquad [z_0:z_1]\longmapsto[z_0^d:z_0^{d-1}z_1:\cdots:z_1^d].$$

The image $X_d:=\nu_d(\mathbb{P}^1)\subset\mathbb{P}^d$ is an algebraic curve of the genus $g(X)=0$ with degree $d>0$. In general, it is known that

$$\deg(\Delta_X)=2d-2+2g\qquad\text{and}\qquad\deg R_X=2d,$$

where g denotes the genus of the curve. In [Pau08, Proposition 4.3], it was observed that the following inclusion holds for the standard torus action of $H\leq G$.

Proposition 4.1 (Paul)**.** *Let $X\hookrightarrow\mathbb{P}^d$ be the d-th rational normal curve and let G be* $\mathrm{SL}(d+1,\mathbb{C})$*. We consider the standard torus $H\leq G$ which is given by*

$$H\cong\begin{pmatrix} t_0 & & & & \\ & t_1 & & & \mathit{0} \\ & & \ddots & & \\ & & & t_{d-1} & \\ \mathit{0} & & & & (t_0\cdots t_{d-1})^{-1}\end{pmatrix}. \tag{4.1}$$

Then $\left(R_X^{\deg\Delta_X},\ \Delta_X^{\deg R_X}\right)$ is K-semistable with respect to H-action, in particular, the following inclusion holds:

$$\frac{\deg\Delta_X}{\deg R_X}\mathcal{N}_H(R_X)\subseteq\mathcal{N}_H(\Delta_X).$$

For the reader's convenience, we provide a proof of Proposition 4.1 in the previous paper. Firstly we recall the following results on the classical Resultants/Discriminants from Gelfand-Kapranov-Zelevinsky theory [GKZ94, Chapter 12].

Here and hereafter, we always assume that $X \subset \mathbb{P}^d$ is the rational normal curve of degree $d \geqslant 2$. By definition, the associated hypersurface $Z_X \subset \mathbb{G}(d-2, d)$ in (3.1) consists of $(d-2)$-dimensional linear subspaces L in $\mathbb{P}^d$ which meet X. Since such an L arises as the kernel of a linear map $A : \mathbb{C}^{d+1} \to \mathbb{C}^2$ of maximal rank 2, we write the matrix A as

$$A = \begin{pmatrix} a_0 & \cdots & a_d \\ b_0 & \cdots & b_d \end{pmatrix}, \qquad \text{with} \qquad a_0 b_0 \neq 0.$$

Define $L := \ker A$ and two polynomials

$$f(x) := a_0 x^d + \cdots + a_{d-1} x + a_d = a_0 \prod_{i=1}^{d} (x - \lambda_i),$$

$$g(x) := b_0 x^d + \cdots + b_{d-1} x + b_d = b_0 \prod_{j=1}^{d} (x - \mu_j).$$

Then we observe that

$$L \cap X \neq \emptyset \qquad \Longleftrightarrow \qquad f(x) \text{ and } g(x) \text{ have a common root.}$$

Hence the Cayley-Chow form R_X is the classical resultant

$$R(f, g) := a_0^d b_0^d \prod_{i,j} (\lambda_i - \mu_j).$$

Lemma 4.2 (Lemma 1.18 and Lemma 1.19 in [Muk03]). *The resultant $R(f, g)$ is $2d \times 2d$ determinant*

$$R(f,g) = \begin{vmatrix} a_0 & \cdots & \cdots & a_d & & & \\ & \ddots & & & \ddots & & \\ & & a_0 & \cdots & \cdots & a_d \\ b_0 & \cdots & \cdots & b_d & & & \\ & \ddots & & & \ddots & & \\ & & b_0 & \cdots & \cdots & b_d \end{vmatrix}.$$

Moreover the discriminant of $f(x)$ is given by the resultant of $f(x)$ and its derivative $f'(x)$:

$$\Delta_X(f) = R(f, f').$$

Example 4.3 (The Qudric in $\mathbb{P}^2$). Let $\nu_2 : \mathbb{P}^1 \to X_2 \subset \mathbb{P}^2$ be the 2-nd Veronese embedding of $\mathbb{P}^1$. Then the image $X_2 = \nu_2(\mathbb{P}^1)$ is a smooth quadric curve in $\mathbb{P}^2$. Therefore by Lemma 4.2, we compute the X-resultant R_X as 4×4 determinant

$$\begin{aligned} R_X &= \det \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix} \\ &= a_0^2 b_2^2 + a_0 a_2 b_1^2 - 2 a_0 a_2 b_0 b_2 - a_0 a_1 b_1 b_2 - a_1 a_2 b_0 b_1 + a_1^2 b_0 b_2 + a_2^2 b_0^2. \end{aligned}$$

Moreover,

$$f(x) = a_0 x^2 + a_1 x + a_2, \qquad f'(x) = 2a_0 x + a_1$$

implies that

$$R(f, f') = \det \begin{pmatrix} a_0 & a_1 & a_2 \\ 2a_0 & a_1 & 0 \\ 0 & 2a_0 & a_1 \end{pmatrix} = a_0(4a_0 a_2 - a_1^2).$$

Therefore we conclude that the discriminant of f is

$$\Delta_X(f) = 4a_0a_2 - a_1^2.$$

For the later convenience, let us denote

$$R_X\begin{pmatrix} a_0 & \cdots & a_d \\ b_0 & \cdots & b_d \end{pmatrix} = \sum_{\substack{(p;q)\in\mathbb{Z}_{\geqslant 0}^{2d+2} \\ |p|=|q|=d}} c_{p,q}\,\mathbf{a}^p\,\mathbf{b}^q$$

where $\mathbf{a}^p = a_0^{p_0}\cdots a_d^{p_d}$, $\mathbf{b}^q = b_0^{q_0}\cdots b_d^{q_d}$. The Newton polytope of R_X in $\mathbb{R}^{2d+2}$ is defined by the weight polytope with respect to the algebraic torus $(\mathbb{C}^\times)^{d+1}$

$$\mathcal{N}(R_X) := \mathcal{N}_{(\mathbb{C}^\times)^{d+1}}(R_X) = \mathrm{Conv}\,\{\,(p;q)\in\mathbb{Z}_{\geqslant 0}^{2d+2} \mid c_{p,q}\neq 0\,\}\,.$$

Similarly we consider the discriminant

$$\Delta_X(a_0,\ldots,a_d) = \sum_{\substack{p\in\mathbb{Z}_{\geqslant 0}^{d+1} \\ |p|=d}} c_{p,q}\,\mathbf{a}^p$$

and define its Newton polytope $\mathcal{N}(\Delta_X)$ in $\mathbb{R}^{d+1}$ as

$$\mathcal{N}(\Delta_X) = \mathrm{Conv}\,\{\,p\in\mathbb{Z}_{\geqslant 0}^{d+1} \mid c_p\neq 0\,\}\,.$$

The following beautiful characterization of $\mathcal{N}(R_X)$ and $\mathcal{N}(\Delta_X)$ was detected by Gelfand, Kapranov and Zelevinsky.

Theorem 4.4. *The Newton polytopes $\mathcal{N}(\Delta_X)$ and $\mathcal{N}(R_X)$ consists of all points satisfying the following linear equations and linear inequalities:*

$$\mathcal{N}(\Delta_X) = \left\{ (p_0,\ldots,p_d)\in\mathbb{R}_{\geqslant 0}^{d+1} \;\middle|\; \begin{array}{l} \sum_{i=0}^{d}(d-i)p_i = d(d-1),\ \sum_{i=0}^{d} p_i = 2d-2, \\ \sum_{i=0}^{j}(j-i)p_i \geqslant j(j-1) \quad 0\leqslant j\leqslant d \end{array} \right\},$$

$$\mathcal{N}(R_X) = \Bigg\{ (p_0,\ldots,p_d;q_0,\ldots,q_d)\in\mathbb{R}_{\geqslant 0}^{2d+2} \;\Bigg|$$

$$\sum_{i=0}^{d}(d-i)p_i + \sum_{i=0}^{d}(d-i)q_i = d^2, \qquad \sum_{i=0}^{d} p_i = \sum_{i=0}^{d} q_i = d, \tag{4.2}$$

$$\sum_{i=0}^{j}(j-i)p_i + \sum_{k=0}^{\ell}(\ell-k)q_k \geqslant j\ell \quad 0\leqslant j,\ell\leqslant d \Bigg\}. \tag{4.3}$$

Now we define maps φ, pr_R and pr_Δ by

$$\begin{array}{lll} \Phi:\mathbb{R}^{2d+2}\longrightarrow\mathbb{R}^{d+1} & (p;q) = (p_0,\ldots,p_d\,;\,q_0,\ldots,q_d)\longmapsto(p_0+q_0,\ldots,p_d+q_d) \\ \mathrm{pr}_R:\mathbb{R}^{2d+2}\longrightarrow\mathbb{R}^{d} & (p;q)\longmapsto(p_0+q_0-(p_d+q_d),\ldots,p_{d-1}+q_{d-1}-(p_d+q_d)) \\ \mathrm{pr}_\Delta:\mathbb{R}^{d+1}\longrightarrow\mathbb{R}^{d} & (p_0,\ldots,p_d)\longmapsto(p_0-p_d,\ldots,p_{d-1}-p_d) \end{array} \tag{4.4}$$

respectively. Then we have the following

Claim 4.5. $\left(\dfrac{d-1}{d}\right)\Phi(\mathcal{N}(R_X))\subset\mathcal{N}(\Delta_X).$

Proof of Claim 4.5. Setting $r_i := p_i + q_i$, we readily see that for any point $(p\,;q) \in \mathcal{N}(R_X)$,

$$\left(\frac{d-1}{d}\right)\sum(p_i+q_i) = \left(\frac{d-1}{d}\right)\sum r_i = \frac{d-1}{d}\cdot 2d = 2(d-1),$$
$$\left(\frac{d-1}{d}\right)\sum(d-i)(p_i+q_i) = \left(\frac{d-1}{d}\right)\sum(d-i)r_i = \frac{d-1}{d}\cdot d^2 = d(d-1)$$

by (4.2). Furthermore, by taking $\ell = j$ and $k = i$, (4.3) becomes

$$\sum_{i=0}^{j}(j-i)(p_i+q_i) \geqslant j^2, \qquad 0 \leqslant j \leqslant d. \tag{4.5}$$

Since

$$\left(\frac{d-1}{d}\right)\Phi(\mathcal{N}(R_X)) \subset \Phi(\mathcal{N}(R_X)),$$

we see that for any $(r_0,\ldots,r_d) \in \left(\frac{d-1}{d}\right)\Phi(\mathcal{N}(R_X))$,

$$\sum_{i=0}^{j}(j-i)r_i \geqslant j^2 > j(j-1), \qquad 0 \leqslant j \leqslant d.$$

by (4.5). This implies that $(r_0,\ldots,r_d) \in \mathcal{N}(\Delta_X)$. □

Proof of Proposition 4.1. Let H be the standard torus defined by (4.1). We readily see that

$$\mathrm{pr}_R(\mathcal{N}(R_X)) = \mathcal{N}_H(R_X) \qquad \text{and} \qquad \mathrm{pr}_\Delta(\mathcal{N}(\Delta_X)) = \mathcal{N}_H(\Delta_X)$$

by the definition of maps (4.4). In the following commutative diagram, the upper horizontal map $\left(\frac{d-1}{d}\right)\Phi$ is injective by Claim 4.5.

$$\begin{array}{ccc} \mathcal{N}(R_X) & \xrightarrow{\left(\frac{d-1}{d}\right)\Phi} & \mathcal{N}(\Delta_X) \\ {\scriptstyle\left(\frac{d-1}{d}\right)\mathrm{pr}_R}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathrm{pr}_\Delta} \\ \left(\frac{d-1}{d}\right)\mathcal{N}_H(R_X) & \xrightarrow{\ i\ } & \mathcal{N}_H(\Delta_X) \end{array} \tag{4.6}$$

This induces the inclusion map

$$i : \frac{\deg \Delta_X}{\deg R_X}\mathcal{N}_H(R_X) \longrightarrow \mathcal{N}_H(\Delta_X)$$

because $\deg \Delta_X = 2(d-1)$ and $\deg R_X = 2d$. □

5. Proof of the main theorem

In this section, we provide the proof of Theorem 1.1.

Let $X \overset{\nu_d}{\hookrightarrow} \mathbb{P}^d$ be the rational normal curve of degree $d \geqslant 2$. Let v and w be $R_X^{\deg \Delta_X}$ and $\Delta_X^{\deg R_X}$ respectively. By Proposition 4.1, we already knew that

$$\mathcal{N}_H(v) \subseteq \mathcal{N}_H(w)$$

for the standard torus H in (4.1). Under consideration of Lemma 2.11, we conclude the first condition is satisfied. Hence, it is crucial to see the second condition in Lemma 2.11 holds or not. Then we shall consider the following two cases.

Proof of Theorem 1.1. Case 1) $\mathcal{N}_H(v) \subsetneq \mathcal{N}_H(w)$. In this case, there is a sufficiently small $0 < \varepsilon_0 \ll 1$ such that

$$\operatorname{dist}(\partial\mathcal{N}_H(v), \partial\mathcal{N}_H(w)) \geqslant \varepsilon_0 > 0.$$

Hence we can find $0 < \delta_0 \ll 1$ so that any $\delta \in (0, \delta_0)$ satisfies

$$(1-\delta)\mathcal{N}_H(v) + \delta \deg(V)\mathcal{N}_H(\mathbb{I}) \subseteq \mathcal{N}_H(w)$$

because the standard simplex $\mathcal{N}_H(\mathbb{I})$ is compact. Eventually, we may find an appropriate integer $m \gg 1$ satisfying

$$\left(1 - \frac{1}{m}\right)\mathcal{N}_H(v) + \frac{1}{m}\deg(V)\mathcal{N}_H(\mathbb{I}) \subseteq \mathcal{N}_H(w).$$

This concludes that $\left(R_X^{\deg \Delta_X},\ \Delta_X^{\deg R_X}\right)$ is uniformly K-stable.

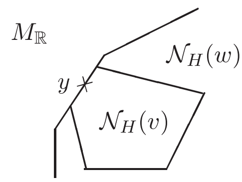
Case 2) $\operatorname{dist}(\partial\mathcal{N}_H(v), \partial\mathcal{N}_H(w)) = 0$. By the definition of $\deg(V)$, we have the inclusion

$$\mathcal{N}_H(v) \subseteq \deg(V)\mathcal{N}_H(\mathbb{I}). \tag{5.1}$$

From the definition of support functions (2.3), we observe that

$$\begin{aligned}&\{\, x \in N_{\mathbb{R}} \mid h_v(x) = h_w(x) \,\}\\ &\quad = \{\, x \in N_{\mathbb{R}} \mid h_v(x) = \langle x, y\rangle \ \text{ and } \ h_w(x) = \langle x, y\rangle \ \text{ for some } \ y \in \partial\mathcal{N}_H(v) \cap \partial\mathcal{N}_H(w) \,\}.\end{aligned}$$

Fixing this $y \in \mathcal{N}_H(v)$, we see that $y \in \deg(V)\mathcal{N}_H(\mathbb{I})$ by (5.1).



Now we *suppose* that the inclusion

$$\deg(V)\mathcal{N}_H(\mathbb{I}) \subseteq \mathcal{N}_H(w)$$

holds, so that $y \in \mathcal{N}_H(w)$. Then by definition of support functions,

$$h_{\deg(V)\mathcal{N}_H(\mathbb{I})}(x) := \max_{y' \in \operatorname{Supp}(\deg(V)\mathcal{N}_H(\mathbb{I}))} \langle x, y'\rangle \leqslant \max_{y' \in \operatorname{supp}(w)} \langle x, y'\rangle = h_w(x) = \langle x, y\rangle.$$

Since $h_{\deg(V)\mathcal{N}_H(\mathbb{I})}(x) = \langle x, y\rangle$ in the above, this implies that $h_v(x) = h_{\deg(V)\mathcal{N}_H(\mathbb{I})}(x)$ and hence, the second condition in Lemma 2.11 follows.

Thus it is crucial to see whether

$$\deg(V)\mathcal{N}_H(\mathbb{I}) \subseteq \mathcal{N}_H(\Delta_X^{\deg R_X}) = \deg(R_X)\mathcal{N}_H(\Delta_X) \tag{5.2}$$

holds or not for the rational normal curve X of degree $d \geqslant 2$. Since $\deg(V) = \deg R_X \cdot \deg \Delta_X$ in our case, (5.2) becomes

$$\deg(\Delta_X)\mathcal{N}_H(\mathbb{I}) \subseteq \mathcal{N}_H(\Delta_X).$$

Now we claim that

Claim 5.1. *For the rational normal curve $X \to \mathbb{P}^d$ of degree $d \geqslant 2$ with the standard torus actions of $H \leq G = \mathrm{SL}(d+1, \mathbb{C})$, we have*

$$\deg(\Delta_X)\mathcal{N}_H(\mathbb{I}) \supsetneq \mathcal{N}_H(\Delta_X).$$

If Claim 5.1 has been proved, the assertion would be verified by Lemma 2.11 and Proposition 4.1. □

Proof of Claim 5.1. We first consider the Newton polytope, namely the weight polytope with respect to $T = (\mathbb{C}^\times)^{d+1}$-action. Then we have

$$\begin{aligned}\deg(\Delta_X)\mathcal{N}_{(\mathbb{C}^\times)^{d+1}}(\mathbb{I}) &= \deg(\Delta_X)\operatorname{Conv}\{\, e_1, \ldots, e_{d+1} \,\}\\ &= \left\{ (p_0, \ldots, p_d) \in \mathbb{R}^{d+1}_{\geqslant 0} \;\middle|\; \sum_{i=0}^{d} p_i = 2d-2 \right\}\end{aligned}$$

because $\deg(\Delta_X) = 2d-2$. Meanwhile, Theorem 4.4 yields

$$\mathcal{N}_{(\mathbb{C}^\times)^{d+1}}(\Delta_X) = \left\{ (p_0,\ldots,p_d) \in \mathbb{R}^{d+1}_{\geqslant 0} \;\middle|\; \begin{array}{l} \sum_{i=0}^{d}(d-i)p_i = d(d-1),\ \sum_{i=0}^{d} p_i = 2d-2, \\ \sum_{i=0}^{j}(j-i)p_i \geqslant j(j-1) \quad 0 \leqslant j \leqslant d \end{array} \right\}.$$

Obviously

$$\mathcal{N}_{(\mathbb{C}^\times)^{d+1}}(\Delta_X) \subsetneq \deg(\Delta_X)\,\mathcal{N}_{(\mathbb{C}^\times)^{d+1}}(\mathbb{I}). \tag{5.3}$$

For a given point $(p_0,\ldots,p_d) \in \mathbb{R}^{d+1}$, we define the projection by

$$\pi_H : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d, \qquad (p_0,\ldots,p_d) \longmapsto (p_0-p_d,\ldots,p_{d-1}-p_d).$$

Then by (5.3), we conclude that

$$\begin{aligned} \deg(\Delta_X)\pi_H\left(\mathcal{N}_{(\mathbb{C}^\times)^{d+1}}(\mathbb{I})\right) &= \deg(\Delta_X)\mathcal{N}_H(\mathbb{I}) \\ &\supsetneq \pi_H\left(\mathcal{N}_{(\mathbb{C}^\times)^{d+1}}(\Delta_X)\right) = \mathcal{N}_H(\Delta_X). \end{aligned}$$

□

6. An example: The Quadric Curve

Again we consider the quadric curve X_2 in $\mathbb{P}^2$ which is dealt in Example 4.3. Recall that X_2 is the image of the 2-nd Veronese embedding

$$\nu_2 : \mathbb{P}^1 \longrightarrow X_2 \subset \mathbb{P}^2 \qquad [z_0 : z_1] \longmapsto [z_0^2 : z_0z_1 : z_1^2] = [X : Y : Z]$$

where $X = z_0^2,\ Y = z_0z_1$ and $Z = z_1^2$. Hence X_2 is isomorphic to the hypersurface of degree 2 in $\mathbb{P}^2$:

$$X_2 \cong \left\{ [X:Y:Z] \in \mathbb{P}^2 \;\middle|\; XZ - Y^2 = 0 \right\}.$$

In particular, the X-resultant R_X can be regarded as the defining equation of X_2 :

$$F(X,Y,Z) := XZ - Y^2 \ (= R_X)$$

whose degree is 4. Meanwhile one can also compute the discriminant Δ_{X_2} using the Gauss map as follows. Let $[a:b:c]$ be a homogenous (dual) coordinates on $(\mathbb{P}^2)^\vee$. Since $X_2 = V(F) \subset \mathbb{P}^2$ is a smooth irreducible hypersurface, the Gauss map is defined as

$$\mathcal{G} :\ X_2 \longrightarrow (\mathbb{P}^2)^\vee \qquad p \longmapsto \mathbb{T}_pX_2 = \left[\frac{\partial F}{\partial X}(p) : \frac{\partial F}{\partial Y}(p) : \frac{\partial F}{\partial Z}(p) \right]$$

where

$$\begin{aligned} \left[\frac{\partial F}{\partial X}(p) : \frac{\partial F}{\partial Y}(p) : \frac{\partial F}{\partial Z}(p) \right] &= [\, Z(p) : -2Y(p) : X(p) \,] \\ &= \left[\, z_1^2 : -2z_0z_1 : z_0^2 \,\right] = [\, a:b:c \,] \in \left(\mathbb{P}^2\right)^\vee. \end{aligned}$$

Since X_2 is a smooth projective variety,

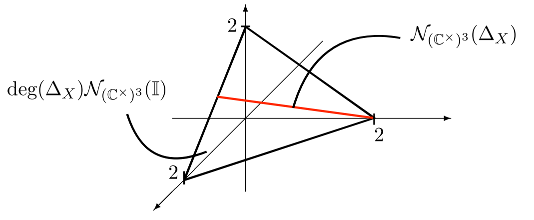
$$\begin{aligned} X_2^* &:= \left\{ f \in (\mathbb{P}^2)^\vee \;\middle|\; \ker f \supset \mathbb{T}_pX_2 \quad p \in X \right\} \\ &= \left\{ [a:b:c] \in (\mathbb{P}^2)^\vee \mid b^2 - 4ac = 0 \right\} \end{aligned}$$

which we conclude that $\Delta_X = b^2 - 4ac$. This agree with the formula $\deg \Delta_X = 2 \cdot 2 - 2 = 2$. Consequently we see that

$$\mathcal{N}_{(\mathbb{C}^\times)^3}(\Delta_X) = \mathrm{Conv}\left\{ \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\} \qquad \text{and}$$

$$\deg(\Delta_X)\mathcal{N}_{(\mathbb{C}^\times)^3}(\mathbb{I}) = \mathrm{Conv}\left\{ \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\2\end{pmatrix} \right\}.$$

Hence the plane $\deg(\Delta_X)\mathcal{N}_{(\mathbb{C}^\times)^3}(\mathbb{I})$ contains the line $\mathcal{N}_{(\mathbb{C}^\times)^3}(\Delta_X)$ as in the picture below.



Finally we compare $\mathcal{N}_{(\mathbb{C}^\times)^3}(v)$ with $\mathcal{N}_{(\mathbb{C}^\times)^3}(w)$ where $v = R_X^{\deg(\Delta_X)}$ and $w = \Delta_X^{\deg(R_X)}$ respectively. From the above argument, we already see that

$$\mathcal{N}_{(\mathbb{C}^\times)^3}(w) = \deg(R_X)\mathcal{N}_{(\mathbb{C}^\times)^3}(\Delta_X) = \mathrm{Conv}\left\{ \begin{pmatrix}0\\8\\0\end{pmatrix} \begin{pmatrix}4\\0\\4\end{pmatrix} \right\}.$$

Let us compute $\mathcal{N}_{(\mathbb{C}^\times)^3}(R_{X_2})$. We regard X_2 as a toric variety $\mathbb{P}^1$ with projective embedding

$$\mathbb{P}^1 \xrightarrow{|\mathcal{O}(2)|} X_2 \subset \mathbb{P}^2$$

whose corresponding moment polytope is the interval $P = [-1, 1] \subset \mathbb{R}$. Then there are 2 regular triangulations of P, namely

$T_1 = \{ [-1, 1] \}$ (the trivial triangulation) and

$T_2 = \{ [-1, 0], [0, 1] \}$ (a triangulation separated into 2 pieces).

Each corresponding GKZ vector ψ_{T_i} is

$$\psi_{T_1} = \begin{pmatrix}2\\0\\2\end{pmatrix} \qquad \text{and } \psi_{T_2} = \begin{pmatrix}1\\2\\1\end{pmatrix}.$$

Since $\mathcal{N}_{(\mathbb{C}^\times)^3}(R_{X_2})$ coincides with the secondary polytope of P (see [GKZ94, p. 260, Theorem 3.1] and [Yotsu16]), we find that

$$\mathcal{N}_{(\mathbb{C}^\times)^3}(R_{X_2}) = \mathrm{Conv}\left\{ \begin{pmatrix}2\\0\\2\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix} \right\}$$

Thus

$$Q_1 := \mathcal{N}_{(\mathbb{C}^\times)^3}(v) = \mathrm{Conv}\left\{ \begin{pmatrix}4\\0\\4\end{pmatrix}, \begin{pmatrix}2\\4\\2\end{pmatrix} \right\} = \mathrm{Conv}\{ v_1, v_2 \},$$

$$Q_2 := \mathcal{N}_{(\mathbb{C}^\times)^3}(w) = \mathrm{Conv}\left\{ \begin{pmatrix}4\\0\\4\end{pmatrix}, \begin{pmatrix}0\\8\\0\end{pmatrix} \right\} = \mathrm{Conv}\{ w_1, w_2 \}$$

with $v_1 = w_1$ and $v_2 = \dfrac{1}{2}(w_1 + w_2)$ which gives $Q_1 \subseteq Q_2$.

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