Tangentially bi-degenerate submanifolds.

Goo Ishikawa*

1 Introduction.

Tangentially degenerate submanifolds in projective spaces are studied from various aspects; differential geometry, algebraic geometry, singularity theory and so on. In particular, P. Griffiths and J. Harris [19] and A. Akivis and V.V. Goldberg [2][3][4] gave the description of tangentially degenerate submanifolds in detail.

Looking at unit normal vectors or tangent planes to space surfaces is the most fundamental method in differential geometry initiated by C.F. Gauss [17]. He, in particular, considered the class of tangentially degenerate surfaces by means of his (Gauss) mappings.

Naturally we are led to consider tangentially degenerate submanifolds in Euclidean spaces, or more naturally in projective spaces by means of Gauss mappings. One of important classes of tangentially degenerate submanifolds, then, consists of submanifolds with degenerate Gauss mappings. Another important class consists of hypersurfaces with degenerate projective dual. The tangential degeneracy of a hypersurface can be described by the degeneracy of its projective dual; the variety, in the dual projective space, consisting of tangent hyperplanes to the hypersurface. Moreover we note that, also for submanifolds of codimension greater than one, the tangential degeneracy can be described by means of projective duality. This means that the Gauss mapping is degenerate, then the projective dual is necessarily degenerate [19].

Key words: bi-degenerate Legendre submanifold, projective duality, Ferus inequality.
2010 *Mathematics Subject Classification*: Primary 58K40; Secondly 53C15, 53C30, 53D10, 55Q25, 57R25, 57R45.
*This work was supported by JSPS KAKENHI Grant Number JP15H03615, and by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

Thus among tangentially degenerate submanifolds, we study, in this paper, submanifolds with degenerate projective duals, possibly with singularities.

The notions of projective duality and of incidence relation play the central role in projective geometry. We re-formulate the study on submanifolds with degenerate Gauss mappings using the incidence relation in projective geometry via contact geometry. In §2, we treat degenerate and bi-degenerate submanifolds. In §3, we formulate the symmetric Ferus inequalities and give several examples satisfying the symmetric Ferus inequalities. In §4, we recall a local classification of frontal mappings. Lastly in §5, we give, as examples of tangentially bi-degenerate submanifolds, one-developables of curves in the four space.

2 Degenerate and bi-degenerate submanifolds.

We denote by $\mathbf{R}P^{n+1} = P(\mathbf{R}^{n+2})$ the $(n+1)$-dimensional projective space and by $\mathbf{R}P^{n+1*} = P((\mathbf{R}^{n+2})^*)$ the $(n+1)$-dimensional dual projective space. Here $(\mathbf{R}^{n+2})^*$ means the dual vector space to $\mathbf{R}^{n+2}$.

Any submanifold $M^m \subseteq \mathbf{R}P^{n+1}$ lifts to a Legendre submanifold $\widetilde{M}$ of the manifold $P(T^*\mathbf{R}P^{n+1})$ of contact elements (tangent hyperplanes) of $\mathbf{R}P^{n+1}$. Actually $\widetilde{M}$ is defined to be the projective conormal bundle $P(T^*_M\mathbf{R}P^{n+1})$ of M. Here $T^*_M\mathbf{R}P^{n+1} \subseteq T^*\mathbf{R}P^{n+1}$ is the conormal bundle of M in $\mathbf{R}P^{n+1}$. Note that, independently of $m = \dim M$, the dimension of the Legendre lifting $\widetilde{M}$ is equal to n. In general the image of a Legendre submanifold by the projection $\pi : P(T^*\mathbf{R}P^{n+1}) \to \mathbf{R}P^{n+1}$ is called a *wave front* or simply a *front*. Therefore any submanifold of $\mathbf{R}P^{n+1}$ can be regarded as a front. It is not the case just only for $\mathbf{R}P^{n+1}$: any submanifold M of any manifold X lifts to a Legendre submanifold $P(T^*_M X)$ of $P(T^*X)$.

The special feature of $\mathbf{R}P^{n+1}$ is that $P(T^*\mathbf{R}P^{n+1})$ has the natural double Legendre fibrations:

$$\mathbf{R}P^{n+1} \longleftarrow P(T^*\mathbf{R}P^{n+1}) \longrightarrow \mathbf{R}P^{n+1*},$$

to $\mathbf{R}P^{n+1}$ and to the dual projective space $\mathbf{R}P^{n+1*}$.

Inverting the process, first we consider Legendre submanifolds in the manifold of contact elements $P(T^*\mathbf{R}P^{n+1})$, the projective cotangent bundle, then second we study their projections by $\pi : P(T^*\mathbf{R}P^{n+1}) \to \mathbf{R}P^{n+1}$ and by $\pi^* : P(T^*\mathbf{R}P^{n+1}) \to \mathbf{R}P^{n+1*}$.

The above constructions is described in term of projective duality. Set

$$\widetilde{I} = \{(x, y) \in \mathbf{R}^{n+2} \times (\mathbf{R}^{n+2})^* \mid x \cdot y = 0\},$$

where $x \cdot y$ denotes the canonical pairing of elements $x \in \mathbf{R}^{n+2}$ and $y \in (\mathbf{R}^{n+2})^*$.

On $\widetilde{I}$, we have $0 = d(x \cdot y) = dx \cdot y + x \cdot dy$. Moreover we set

$$I = \{([x], [y]) \in \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*} \mid x \cdot y = 0\},$$

the manifold of incident pairs or *the incidence manifold.* Then I is of dimension $2n + 1$ and I has the contact structure

$$D := \{dx \cdot y = 0\} = \{x \cdot dy = 0\} \subset TI.$$

Namely, a tangent vector $(u, v) \in T_{([x],[y])}I$ belongs to the contact distribution D if and only if $u \cdot y = 0$ and, if and only if $x \cdot v = 0$.

The projection $\pi : I \to \mathbf{R}P^{n+1}$ (resp. $\pi^* : I \to \mathbf{R}P^{n+1*}$) identify I, as contact manifolds, with the fiber-wise projectivisation $P(T^*\mathbf{R}P^{n+1})$ of $T^*\mathbf{R}P^{n+1}$ (resp. $P(T^*\mathbf{R}P^{n+1*})$ of $T^*\mathbf{R}P^{n+1*}$).

A submanifold $L \subset I$ is called *a Legendre submanifold* if L is an integral submanifold of the contact distribution D of dimension n. The integrality condition means that $TL \subset D|_L$.

Now, to any submanifold M of $\mathbf{R}P^{n+1}$ of any codimension m, there corresponds the Legendre submanifold in I:

$$\widetilde{M} := \{([x], [y]) \in I \mid [x] \in M, (T_x\widehat{M}) \cdot y = 0\},$$

which is called *the Legendre lifting* of M. Here $\widehat{M} \subseteq \mathbf{R}^{n+2} \setminus \{0\}$ is the corresponding $(m + 1)$-dimensional submanifold to $M \subseteq \mathbf{R}P^{n+1}$.

Also to any submanifold N of $\mathbf{R}P^{n+1*}$ of any codimension m^*, there corresponds the Legendre submanifold in I:

$$\widetilde{N} := \{([x], [y]) \in I \mid [y] \in N, (x \cdot T_y\widehat{N}) = 0\},$$

which is also called *the Legendre lifting* of N. Here $\widehat{N} \subseteq \mathbf{R}^{n+2*} \setminus \{0\}$ is the corresponding $(m^* + 1)$-dimensional submanifold to $N \subseteq \mathbf{R}P^{n+1*}$.

A *front* of L in $\mathbf{R}P^{n+1}$ (resp. in $\mathbf{R}P^{n+1*}$) is, by definition, the image of L by π (resp. π^*).

Thus any submanifold of $\mathbf{R}P^{n+1}$ (resp. $\mathbf{R}P^{n+1*}$) can be regarded as a front in $\mathbf{R}P^{n+1}$ (resp. in $\mathbf{R}P^{n+1*}$) of a Legendre submanifold of I. However a front may have singularities and we are interested in singularities as well.

Let $L \subset I$ be a Legendre submanifold in the manifold I of incident pairs. Set

$$m = \sup\{\mathrm{rank}_q(d(\pi|_L) : T_qL \to T_{\pi(q)}\mathbf{R}P^{n+1}) \mid q \in L\}.$$

Moreover set

$$m^* = \sup\{\mathrm{rank}_q(d(\pi^*|_L) : T_qL \to T_{\pi^*(q)}\mathbf{R}P^{n+1*}) \mid q \in L\}.$$

We call L *degenerate* if $m^* < n$. Moreover we call L *bi-degenerate* if $m < n$ and $m^* < n$.

Now we call a font $\pi(I)$ in $\mathbf{R}P^{n+1}$ (resp. $\pi^*(I)$ in $\mathbf{R}P^{n+1*}$) *tangentially degenerate* or briefly *degenerate* if $m^* < n$ (resp. $m < n$). Moreover we call a front $\pi(I)$ in $\mathbf{R}P^{n+1}$ (resp. $\pi^*(I)$ in $\mathbf{R}P^{n+1*}$) *tangentially bi-degenerate* or briefly *bi-degenerate* if both $m^* < n$ and $m < n$.

Example 2.1 Let n, m be integers with $0 \leq m \leq n$ Let $M = \mathbf{R}P^m \subset \mathbf{R}P^{n+1}$ be a projective subspace of dimension m. We denote by $M^\vee \subset \mathbf{R}P^{n+1*}$ the projective dual to M; $M^\vee$ consists of hyperplanes containing M, and $M^\vee$ is a projective subspace of $\mathbf{R}P^{n+1*}$ of dimension $n - m$. Set $L = M \times M^\vee \subset I$. Then L is the Legendre lifting of M. Then L is degenerate if and only if $0 < m \leq n$. Moreover L is bi-degenerate if and only if $0 < m < n$.

Example 2.2 Let $M^m \subset \mathbf{R}P^{n+1}$, $0 \leq m \leq n$, be a submanifold with degenerate Gauss mapping. Recall that the Gauss mapping $\gamma : M \to \mathrm{Gr}(m+1, \mathbf{R}^{n+2})$ is defined by $\gamma([x]) = T_x\widehat{M}, ([x] \in M)$. Then the required condition is that $\mathrm{rank}\gamma < m$. Thus we are assuming $0 < m \leq n$. Lots of examples have been found of submanifolds with degenerate Gauss mappings ([29]). Let L be the Legendre lifting of M. We have $M = \pi(L)$ and $\pi^*(L) = M^\vee \subset \mathbf{R}P^{n+1*}$ is the projective dual of M. Then L is degenerate. Moreover L is bi-degenerate if $m < n$. In other words, a submanifold with degenerate Gauss mapping is a degenerate front. Moreover if it is of codimension ≥ 2, then it is a bi-degenerate front.

Example 2.3 Let $W \subset \mathbf{C}P^n$ be a complex submanifold of complex dimension $\ell \leq n$. Consider the Hopf fibration $h : \mathbf{R}P^{2n+1} \to \mathbf{C}P^n$. Set

$M := h^{-1}W \subset \mathbf{R}P^{2n+1}$. Then M is a real submanifold of real dimension $2\ell+1$ with degenerate Gauss mapping. Let $L = \widetilde{M} \subset I \subset \mathbf{R}P^{2n+1} \times \mathbf{R}P^{2n+1*}$ be the Legendre lifting of M. Then L is bi-degenerate. In fact $\pi^*(L) = \overline{h^*}^{-1}W^\vee$, for the complex projective dual $W^\vee \subset \mathbf{C}P^{n*}$ and the Hopf fibration $h^* : \mathbf{R}P^{2n+1*} \to \mathbf{C}P^{n*}$. Now suppose W is a non-singular complex quadric hypersurface in $\mathbf{C}P^n$. Then $W^\vee$ is a non-singular complex quadric hypersurface in $\mathbf{C}P^{n*}$. Then both $\pi|_L$ and $\pi^*|_L$ are of constant rank $2n-1$. In this example $m = 2n-1 = m^*$ and $m + m^* - 2n = 2n-2$. If $n = 2$, then $m = 3 = m^*$, $\dim L = 4$ and $m + m^* - 4 = 2$.

In the last example, we observe the Legendre submanifold has the constant rank projections $\pi|_L$ and $\pi^*|_L$ so that $\pi(L)$ and $\pi^*(L)$ are both non-singular degenerate fronts.

3 Symmetric Ferus inequalities for degenerate Legendre submanifolds.

In this section, we give a formulation of Ferus inequality [15][16] in projective and symmetric form.

First we recall the Ferus inequality for submanifolds in a sphere or in a projective space with degenerate Gauss mappings [15][16]. See also [8][29].

Let $M^m \subset \mathbf{R}P^{n+1}$ be a submanifold with degenerate Gauss mapping. See Example 2.2. Set $r = \mathrm{rank}(\gamma)$, the rank of Gauss mapping γ of M.

First recall the *Adams number* $A(k)$ for $k \in \mathbf{N}$ from algebraic topology. The number $A(k)$ is, by definition, the maximal number of independent vector fields over the sphere S^{k-1}. For example, since Euler number of S^2 is not equal to zero, there does not exist nowhere vanishing vector field over S^2, so we have $A(3) = 0$. Since S^1 and S^3 are parallelisable, namely, TS^1 and TS^3 are trivial, we have $A(2) = 1$ and $A(4) = 3$. One of great results in algebraic topology or homotopy theory, is the following surprisingly simple formula due to Adams:

$$A((2b+1)2^{c+4d}) = 2^c + 8d - 1, (b, c, d \in \mathbf{N} \cup \{0\}, 0 \leq c \leq 3).$$

In particular $A(k)$ depends only on the exponent to 2 and the odd part in the primary decomposition of k.

Then define the *Ferus number* for $m \in \mathbf{N}$ by

$$F(m) = \min\{k \in \mathbf{N} \mid A(k) + k \geq m\}.$$

Then Ferus showed, in the framework of Riemannian geometry, the following crucial result:

Theorem 3.1 ([15][16]) *Let M^m be a closed and immersed submanifold of $\mathbf{R}P^{n+1}$ with $r = \mathrm{rank}(\gamma) < m$. Then $r < F(m)$ implies $r = 0$. In particular, if M is a closed and connected submanifold of $\mathbf{R}P^{n+1}$ and M is not a projective subspace, then $F(m) \leq r$.*

We write down $F(m)$, for smaller m:
$F(1) = 1, F(2) = 2, F(3) = 2, F(4) = 4, F(5) = 4, F(6) = 4, F(7) = 4,$
$F(8) = 8, F(9) = 8, F(10) = 8, F(11) = 8, F(12) = 8, F(13) = 8,$
$F(14) = 8, F(15) = 8, F(m) = 16, (16 \leq m \leq 24),$
$F(m) = 24, (25 \leq m \leq 31), F(m) = 32, (32 \leq m \leq 41),$
$F(m) = 40, (42 \leq m \leq 47), F(m) = 48, (48 \leq m \leq 56),$
$F(m) = 56, (57 \leq m \leq 63), F(m) = 64, (64 \leq m \leq 75),$
$F(m) = 72, (76 \leq m \leq 79), F(m) = 80, (80 \leq m \leq 88),$
$F(m) = 88, (89 \leq m \leq 95), F(m) = 96, (96 \leq m \leq 105)$
and so on. Moreover we have $F(m) = m$ if m is a power of 2.

We call the inequality $F(m) \leq r$ *Ferus inequality.* Many examples satisfying in fact *Ferus equality* $F(m) = r$ have been found related to isoparametric submanifold, homogeneous submanifolds, austere submanifolds and so on ([29]).

However we may feel something missing, by the fact that, in Ferus inequality or Ferus equality, there appear just m and r, but, there does not appear the number n, or the dimension of the ambient space $\mathbf{R}P^{n+1}$.

Now we are going to formulate Ferus type inequality in term of Legendre submanifolds and in more symmetric form.

Theorem 3.2 *Let L be a closed (compact without boundary) immersed Legendre submanifold of the incidence relation $I \subset \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*}$. Suppose $\pi|_L$ and $\pi^*|_L$ are constant rank m and m^* respectively, and L is not the Legendre lifting of a projective subspace. Then we have*

$$F(m) \leq m + m^* - n, \quad F(m^*) \leq m^* + m - n.$$

Note that $n \leq m+m^*$. Moreover we see, if $m+m^* = n$ in the situation of Theorem 3.2, then L is the Legendre lifting of a projective subspace (Example 2.1).

Proof of Theorem 3.2: Set $M = \pi(L)$. Then M is a closed and immersed submanifold in $\mathbf{R}P^{n+1}$. It is easy to see that

$$\mathrm{rank}(\gamma) \leq m + m^* - n.$$

Thus we have $F(m) \leq m+m^* - n$ if M is not a projective subspace. By the symmetry, we also have $F(m^*) \leq m^* + m - n$. Thus we have Theorem 3.2. $\square$

Now we are led to the following fundamental question:

Question 1: For any positive integers n, m, m^* satisfying

$$F(m) = m + m^* - n, \quad F(m^*) = m^* + m - n,$$

the symmetric Ferus equalities, find examples of closed Legendre submanifolds $L^n \subset I^{2n+1} \subset \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*}$ such that $\pi|_L$ is of constant rank m and $\pi^*|_L$ is of constant rank m^*.

If the symmetric Ferus equalities are satisfied, then we have

$$F(m) = F(m^*) \ \textit{ and } \ n = m + m^* - F(m) (= m^* + m - F(m^*)).$$

Since $m \geq F(m)$ and $m^* \geq F(m^*)$, the inequalities $m \leq n, m^* \leq n$ are necessarily fulfilled. Thus Question 1 can be re-written as follows:

Question 2: For any positive integers m, m^* satisfying $F(m) = F(m^*)$, find examples of closed Legendre submanifolds $L^n \subset I^{2n+1} \subset \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*}$ with $n = m + m^* - F(m) (= m^* + m - F(m^*))$, such that $\pi|_L$ is of constant rank m and $\pi^*|_L$ is of constant rank m^*.

We give here some of known examples:

Example 3.3 By Example 2.3, we have examples for

$$(n, m, m^*) = (4,3,3), (6,5,5), (10,9,9), (18,17,17), (26,25,25), (34,33,33), \\ (50,49,49), (58,57,57), (66,65,65), (82,81,81), (90,89,89), (98,97,97),$$

and so on. Moreover, we have examples for the sequence : $(2^\ell+2, 2^\ell+1, 2^\ell+1)$, $\ell = 1, 2, 3, \ldots$.

Example 3.4 (Cartan hypersurfaces.)

(1) $(n, m, m^*) = (3, 3, 2)$. Let $M^3 \subset \mathbf{R}P^4$ be the Cartan hypersurface. Then $n = m = 3, m^* = 2$. Note that $F(3) = 2 = F(2)$. Thus we see the symmetric Ferus equalities hold.

(2) $(n, m, m^*) = (6, 6, 4)$. Let $M^6 \subset \mathbf{R}P^7$ be the Cartan hypersurface. Then $n = m = 6, m^* = 4$. Note that $F(6) = 4 = F(4)$. Thus we see the symmetric Ferus equalities hold.

(3) $(n, m, m^*) = (12, 12, 8)$. Let $M^{12} \subset \mathbf{R}P^{13}$ be the Cartan hypersurface. Then $n = m = 12, m^* = 8$. Note that $F(12) = 8 = F(8)$. Thus we see the symmetric Ferus equalities hold.

(4) $(n, m, m^*) = (24, 24, 16)$. Let $M^{24} \subset \mathbf{R}P^{25}$ be the Cartan hypersurface. Then $n = m = 24, m^* = 16$. Note that $F(24) = 16 = F(16)$. Thus we see the symmetric Ferus equalities hold.

4 Local classification.

Let us turn to the local description of a frontal mapping. Let $f : M, x \longrightarrow N, f(x)$ be a *frontal map-germ* at a point x of M. Take local coordinates $x_1, \dots, x_m$ of M around x, and $y_1, \dots, y_m; z_1, \dots, z_r$ of N around $f(x)$, respectively, such that T_0 is defined by $dz_1 = \cdots = dz_r = 0$. Then the frontal mapping f satisfies

$$dz_i = \sum_{1 \le j \le m} a_{ij} dy_j, \qquad 1 \le i \le r,$$

for some functions a_{ij}. Then $\widetilde{f} = (y, z, (a_{ij}))$.

Set $g = (y_1 \circ f, \dots, y_m \circ f) : \mathbf{R}^m, 0 \longrightarrow \mathbf{R}^m, 0$. Then we see that $z_1 \circ f, \dots, z_r \circ f$ belong to the *ramification module* [20][21][22][23]:

$$R_g := \{h : \mathbf{R}^m, 0 \longrightarrow \mathbf{R} \mid dh \text{ is a functional linear combination of } dg_1, \dots, dg_m\}.$$

Conversely, if we take a system $h_1, \dots, h_r$ of elements in R_g for a map-germ $g : \mathbf{R}^m, 0 \longrightarrow \mathbf{R}^m, 0$, we get a frontal map-germ $f : \mathbf{R}^m, 0 \longrightarrow \mathbf{R}^{m+r}$ by $f = (g; h_1, \dots, h_r)$. Thus we have the generic classification of singularities of frontal mappings of kernel rank one.

Theorem 4.1 *For a generic frontal mapping $f : M^m \longrightarrow N^n, m < n$, with kernel rank at most one, the (right-left) equivalence classes of germs $f_x, x \in$*

M, have normal forms labeled by integers $k, \ell_1, \ldots, \ell_r$, $r = n - m$, with $k \geq \ell_1 \geq \cdots \geq \ell_r$ and $k + \ell_1 + \cdots + \ell_r \leq m$: $f_{m,k,\ell_1,\ldots,\ell_r} : \mathbf{R}^m, 0 \longrightarrow \mathbf{R}^{m+r}, 0$ defined by $y_1 = x_1, \ldots, y_{m-1} = x_{m-1}$, and

$$\begin{aligned}
y_m &= x_m^{k+1}/(k+1)! + x_1 x_m^{k-1}/(k-1)! + \cdots + x_{k-1}x_m \ (=: u) \\
v_1 &= x_m^{\ell_1+1}/(\ell_1+1)! + x_k x_m^{\ell_1}/\ell_1! + \cdots + x_{k+\ell_1-1}x_m \\
& \cdots\cdots\cdots\cdots\cdots \\
v_r &= x_m^{\ell_r+1}/(\ell_r+1)! + x_{k+\ell_1+\cdots+\ell_{r-1}} x_m^{\ell_r}/\ell_r! + \cdots + x_{k+\ell_1+\cdots+\ell_r-1}x_m \\
z_j &= \int v_j(\partial u/\partial x_m)dx_m, \quad 1 \leq j \leq r,
\end{aligned}$$

The result for $r = 1$ is given in [22].

In the case $(m, m^*) = (n, 1)$, the diffeomorphism classification of the singularities of degenerate fronts are studied in detail in [21][23][25]. Note that, if $n \geq 2$, $F(n) > 1 = n + 1 - n$, so $\pi|_L$ is never of constant rank. For example, in the case $n = 2$, the typical singularities of degenerate fronts (or frontal surfaces in the sense of the next section) of dimension 2 in $\mathbf{R}P^3$ are a *cuspidal edge*, a *folded umbrella* and a *swallowtail*. These are singularities of *tangent developables* of space curves of types $(1, 2, 3)$, $(1, 2, 4)$ and $(2, 3, 4)$, respectively.

For the classification by a weaker equivalence relation, namely by the homeomorphism classification of tangent developables is given in [26].

5 Singularities of one-developables to curves in the four space.

In this section, we give the classification of singularities of bi-degenerate Legendre submanifold in case $n = 3, m = 2, m^* = 2$. Note that, in this case, $F(2) = 2 > 1 = 2 + 2 - 3$, so that $\pi|_L$ and $\pi^*|_L$ are never of constant rank.

Consider the flag manifold

$$\mathcal{F} := \{V : \{0\} \subset V_1 \subset V_2 \subset V_3 \subset V_4 \subset \mathbf{R}^5\}.$$

Then we see $\dim \mathcal{F} = 10$. On $\mathcal{F}$, we define the *canonical distribution* $D \subset T\mathcal{F}$ by the following: a curve

$$V(t) : \{0\} \subset V_1(t) \subset V_2(t) \subset V_3(t) \subset V_4(t) \subset \mathbf{R}^5$$

on $\mathcal{F}$ is tangent to D at $t = t_0$ if the infinitesimal deformation of $V_1(t)$ at t_0 belongs to $V_2(t_0)$, the infinitesimal deformation of $V_2(t)$ at t_0 belongs to $V_3(t_0)$, and the infinitesimal deformation of $V_3(t)$ at t_0 belongs to $V_4(t_0)$. Then we see $\mathrm{rank} D = 4$.

We define the projection $\pi_1 : \mathcal{F} \to \mathbf{R}P^4$ (resp. $\pi_4 : \mathcal{F} \to \mathbf{R}P^{4*}$) by $\pi_1(V) = V_1$ ($\pi_4(V) = V_4$). Also we define the projection $\pi_{1,4} : \mathcal{F} \to I \subset \mathbf{R}P^4 \times \mathbf{R}P^{4*}$ by $\pi_{1,4}(V) = (V_1, V_4)$. Then we have $\pi_1 = \pi \circ \pi_{1,4}$ and $\pi_4 = \pi^* \circ \pi_{1,4}$.

Typical singularities appearing in bi-degenerate fronts in this situation are *cones* and 1-*developables*.

Let $c : \mathbf{R} \to \mathbf{R}P^4$,

$$c(t) = [x(t)] = [x_0(t), x_1(t), x_2(t), x_3(t), x_4(t))]$$

be a smooth curve. Consider the surface ruled by tangent (projective) lines to the curve. We call it 1-*developable* of the curve*. Then the tangent planes to regular points of the 1-developable are constant along each ruling. In fact the tangent plane to the 1-developable at a point on a tangent line coincides with the osculating 2-plane at the tangent point of the tangent line to the curve.

Let a_1, a_2, a_3, a_4 be integers with $1 \leq a_1 < a_2 < a_3 < a_4$. The curve c is called of type (a_1, a_2, a_3, a_4) at $t_0 \in \mathbf{R}$ if there exist a smooth coordinate t of $\mathbf{R}$ centred at t_0 and an affine coordinate x_1, x_2, x_3, x_4 such that $c(t)$ is represented near t_0 in the form

$$x_1(t) = t^{a_1} + o(t^{a_1}),\ x_2(t) = t^{a_2} + o(t^{a_2}),\ x_3(t) = t^{a_3} + o(t^{a_3}),\ x_4(t) = t^{a_4} + o(t^{a_4}).$$

The curve c is of finite type at t_0 if there exist such integers a_1, a_2, a_3, a_4 so that c is of type (a_1, a_2, a_3, a_4). The curve itself is called of finite type if it is of finite type at every point. Any curve $c : \mathbf{R} \to \mathbf{R}P^4$ of finite type lifts to unique D-integral curve $\widetilde{c} : \mathbf{R} \to \mathcal{F}$, by using osculating subspaces of dimension 1 (the tangent line), of dimension 2, of dimension 3 and of dimension 4. Moreover $c^* := \pi_4 \circ \widetilde{c} : \mathbf{R} \to \mathbf{R}P^{4*}$ is of finite type. If the original c is of type (a_1, a_2, a_3, a_4) at $t_0 \in \mathbf{R}$, then c^* is of type $(a_4 - a_3, a_4 - a_2, a_4 - a_1, a_4)$ at $t_0 \in \mathbf{R}$. We call c^* the *dual curve* to c ([42]).

Then we have the following fundamental result:

*In [27], it is called a *tangent surface* in a general situation.

Theorem 5.1 *The* 1*-developable of a curve* c *in* $\mathbf{R}P^4$ *of type* (a_1, a_2, a_3, a_4) *is a bi-degenerate front (or a bi-degenerate frontal surface) with* $m = 2, m^* = 2$. *Its projective dual is the* 1*-developable of the dual curve* c^* *of type* $(a_4 - a_3, a_4 - a_2, a_4 - a_1, a_4)$.

To classify singularities of subsets in $\mathbf{R}P^{n+1}$ we must define, at least, a local equivalence relation: a subset $A \subseteq N$ of a manifold N at a point $p_0 \in N$ and a subset $A' \subseteq N'$ of a manifold N' at a point $p'_0 \in N'$ are called *diffeomorphic* if there exists a diffeomorphism $\varphi : U \to U'$ of an open neighbourhood U of p_0 in N and an open neighbourhood U' of p'_0 in N' which maps $A \cap U$ to $A' \cap U'$.

Since an open dense part of $\pi(L)$ is a submanifold of dimension m, it is natural to consider a parametrisation by an m dimensional manifold. Then smooth mappings $f : M \to N$ at a point $t_0 \in M$ and $f' : M' \to N'$ at a point $t'_0 \in M'$ are called *diffeomorphic* if there exist a diffeomorphism $\psi : V \to V'$ of of an open neighbourhood V of t_0 in M and an open neighbourhood V' of t'_0 in M' and a diffeomorphism $\varphi : U \to U'$ of of an open neighbourhood U of $p_0 = f(t_0)$ in M and an open neighbourhood V' of $p'_0 = f'(t')$ in M' such that $\varphi \circ f = f' \circ \psi$ on U.

Theorem 5.2 (cf. [23]) *Let* $c : \mathbf{R} \to \mathbf{R}P^4$ *be a smooth curve and* $t_0 \in \mathbf{R}$. *Suppose* c *at* t_0 *is of one of following types:*

(I)$_r$: $(1, 2, 3, 3 + r)$, $r = 1, 2, \ldots$,
(II)$_0$: $(2, 3, 4, 5)$,
(II)$_1$: $(1, 3, 4, 5)$,
(II)$_2$: $(1, 2, 4, 5)$,
(III) : $(3, 4, 5, 6)$.

Then the diffeomorphism class in $\mathbf{R}P^4$ *of the* 1*-developable of the curve* c *at the point* $c(t_0)$ *is determined only by its type. In other words, if two curves have the same type, then their* 1*-developables are locally diffeomorphic.*

For a generic curve in $\mathbf{R}P^4$, only points of types (I)$_1$: $(1, 2, 3, 4)$ and (I)$_2$: $(1, 2, 3, 5)$ appear. Moreover, for the dual curve of a generic curve, only points of types (I)$_1$: $(1, 2, 3, 4)$ and (II)$_0$: $(2, 3, 4, 5)$ appear.

We call the 1-developable surface *cuspidal edge* in the case of type $(1, 2, 3, 4)$, and *open swallowtail* in the case of type $(2, 3, 4, 5)$.

Example 5.3 (Cuspidal edge.) The 1-developable surface of a curve of type $(1, 2, 3, 4)$ has the normal form under the diffeomorphisms:

$$(x, t) \mapsto (x,\ 3t^2 + 2xt,\ 2t^3 + xt^2,\ \frac{3}{4}t^4 + \frac{1}{3}xt^3).$$

Moreover it is diffeomorphic to

$$(x, t) \mapsto (x,\ t^2,\ t^3, 0).$$

Example 5.4 The 1-developable surface of a curve of type $(1, 2, 3, 5)$ has the normal form under the diffeomorphisms:

$$(x, t) \mapsto (x,\ 3t^2 + 2xt,\ 2t^3 + xt^2,\ \frac{2}{5}t^5 + \frac{1}{6}xt^4).$$

However it is actually diffeomorphic to

$$(x, t) \mapsto (x,\ t^2,\ t^3, 0),$$

namely, diffeomorphic to the cuspidal edge.

Actually we can prove the following:

Theorem 5.5 *The* 1*-developable of a curve of type* $(\mathrm{I})_r : (1, 2, 3, 3 + r), (r = 1, 2, 3, \dots)$ *is diffeomorphic to the cuspidal edge.*

Also we observe that the dual of 1-developable of a curve of type $(1, 2, 3, 4)$ and the dual of 1-developable of a curve of type $(1, 2, 3, 5)$ are not diffeomorphic:

Example 5.6 (Open swallowtail.) The 1-developable surface of a curve of type $(2, 3, 4, 5)$ has the normal form under the diffeomorphisms:

$$(x, t) \mapsto (x,\ 3t^3 + 2xt,\ \frac{9}{4}t^4 + xt^2,\ \frac{9}{10}t^5 + \frac{1}{3}xt^3).$$

This is not diffeomorphic to the cuspidal edge.

References

[1] M. A. Akivis, *On multidimensional strongly parabolic surfaces,* Izv. Vyssh. Uchebn. Zaved. Mat., **5** (1987), 3–10 = Soviet Math., **31–5** (1987), 1–11.

[2] M. A. Akivis and V. V. Goldberg, *Projective differential geometry of submanifold*, North-Holland Math. Lib., **49**, North-Holland, Amsterdam, 1993.

[3] M. A. Akivis and V. V. Goldberg, *On the structure of submanifolds with degenerate Gauss maps,* Geometriae Dedicata **86** (2001), 205–226.

[4] M. A. Akivis and V. V. Goldberg, *Differential geometry of varieties with degenerate Gauss maps,* CMS books in Math., **18**, Springer-Verlag, New York (2004).

[5] V.I. Arnold, *Singularities of Caustics and Wave Fronts,* Kluwer Academic Publishers, 1990.

[6] V.I. Arnold, *The geometry of spherical curves and algebra of quaternions,* Uspekhi Mat. Nauk., **50–1** (1995), 3–68 = Russian Math. Surveys, **50–1** (1995), 1– 68.

[7] A. C. Asperti, *Immersions of surfaces into 4-dimensional spaces with nonzero normal curvature,* Ann. Mat. Pura Appl., **125** (1980), 313–328.

[8] A. A. Borisenko, *Extrinsic geometry of strongly parabolic multidimensional submanifolds*, Uspekhi Mat. Nauk **52** (1997), no.6(318); translation in Russian Math. Surveys **52** (1997), no. 6, 1141-1190.

[9] A. A. Borisenko, *Extrinsic geometry of parabolic and saddle multidimensional submanifolds,* Russ. Math. Surveys, **53–6** (1998), 1111–1158.

[10] É. Cartan, *Sur quelques familles remarquables d'hypersurfaces* C. R. Congrès Math. Liège, (1939), 30–41.

[11] É. Cartan, *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques,* Math. Z., **45** (1939), 335–367.

[12] A. Cayley, *Mémoire sur les coubes à double courbure et les surfaces développables,* Journal de Mathématique Pure et Appliquées (Liouville), **10** (1845), 245–250. = The Collected Mathematical Papers vol.I, pp. 20 7–211.

[13] T. Cecil and P. Ryan, *Tight and Taut Immersions of Manifolds,* Research Notes in Math. **107**, Pitman, London (1985).

[14] M. Dajczer, D. Gromoll, *Gauss parametrizations and rigidity aspects of submanifolds,* J. Diff. Geom., **22** (1985), 1–12.

[15] D. Ferus, *On the type number of hypersurfaces in spaces of constant curvature,* Math. Ann. **187** (1970), 310–316.

[16] D. Ferus, *Totally geodesic foliations*, Math. Ann. **188** (1970), 313-316.

[17] C.F. Gauss, Disquisitiones genetales circa superficies curveas, (Allgemeine Flächentheorie), 1827.

[18] A.B. Givental, *Lagrangian imbeddings of surfaces and unfolded Whitney umbrella,* Funksional Anal. i Prilozhen., **20–3** (1986), 35–41.

[19] P. Griffiths and J. Harris, *Algebraic geometry and local differentiable geometry,* Ann. sci. Éc. Norm. Sup. (4), **12** (1979), 355–432.

[20] G. Ishikawa, *The local model of an isotropic map-germ arising from one-dimensional symplectic reduction*, Math. Proc. Camb. Phil. Soc. **111** (1992), 103–112.

[21] G. Ishikawa, *Determinacy of envelope of the osculating hyperplanes to a curve,* Bull. London Math. Soc., **25** (1993), 603–610.

[22] G. Ishikawa, *Parametrized Legendre and Lagrange varieties*, Kodai Math. J., **17–3** (1994), 442–451.

[23] G. Ishikawa, *Developable of a curve and its determinacy relatively to the osculation-type*, Quarterly J. Math.Oxford, **46** (1995), 437–451.

[24] G. Ishikawa, *Developable hypersurfaces and homogeneous spaces in a real projective space*, Lobachevskii J. Math., **3** (1999), 113-125.

[25] G. Ishikawa, *Singularities of Developable Surfaces,* London Math. Soc. Lecture Note Series, Cambridge Univ. Press, vol. 263, pp. 403–418, 1999.

[26] G. Ishikawa, *Topological classification of the tangent developables of space curves,* J. London Math. Soc., **62** (2000), 583–598.

[27] G. Ishikawa, *Singularities of tangent varieties to curves and surfaces,* Journal of Singularities, **6** (2012), 54–83.

[28] G. Ishikawa and T. Morimoto, *Solution surfaces of Monge-Ampère equations,* Differential Geometry and its Applications, 14 (2001), 113–124.

[29] G. Ishikawa, M. Kimura, R. Miyaoka, *Submanifolds with degenerate Gauss mappings in spheres,* in "Lie Groups, Geometric Structures and Differential Equations — One Hundred Years after Sophus Lie —", Advanced Study in Pure Mathematics **37** (2002), pp. 115–149.

[30] M. Kimura, *A note for compact hypersurface with type number* 2 *in a sphere,* mimeographical note, 2001.

[31] M. Kokubu, M. Umehara, K. Yamada, *Flat fronts in hyperbolic 3-space,* Pacific J. Math., **216–1** (2004), 149–175.

[32] M. Kokubu, W. Rossman, K. Saji, M. Umehara, K. Yamada, *Singularities of flat fronts in hyperbolic space,* Pacific J. Math., **221–2** (2005), 303–351.

[33] M. Kossowski, *Prescribing invariants for integral surfaces in the Grassmann bundle of* 2*-planes in* 4*-space,* Topology **35–1** (1995), 1–12.

[34] J. M. Landsberg, *On second fundamental forms of projective varieties,* Invent. Math., **117** (1994), 303–315.

[35] J. M. Landsberg, *On degenerate secant and tangential varieties and local differential geometry,* Duke Math. Journal, **85–3** (1996), 605–634.

[36] R. Miyaoka, *The linear isotropy group of $G_2/SO(4)$, the Hopf fibering and isoparametric hypersurfaces,* Osaka J. of Math., **30**,(1993), 179–202.

[37] R. Miyaoka, *A new proof of the homogeneity of isoparametric hypersurfaces with $(g, m) = (6, 1)$,* in Proceedings of the Beijing conference 1999, Geometry and Topology of Submanifolds X, World Scientific (2000), pp. 178–199.

[38] R. Miyaoka, *Isoparametric hypersurfaces with $(g, m) = (6, 2)$*, Annals of Math., **177** (2013), 53–110.

[39] T. Morimoto, *La géométrie des équations de Monge-Ampère,* C. R. Acad. Sci. Paris, **289**, (1979), A-25 –A-28.

[40] T. Morimoto, *Monge-Ampère equations viewed from contact geometry,* Banach Center Publications **39**, "Symplectic Singularities and Geometry of Gauge Fields" 1998, pp. 105–120.

[41] V. Yu. Rovenskii, *Characterization of totally geodesic submanifolds of S^n and $\mathbf{RP}^n$,* Ukrain. Geom. Sb., **28** (1985), 106–116. = J. Soviet Math., **48** (1990), 87–94.

[42] O.P. Shcherbak, *Projectively dual space curves and Legendre singularities,* Trudy Tbiliss. Univ. **232–233** (1982), 280–336. = Sel. Math. Sov. **5-4** (1986), 391–421.

[43] N. Tanaka, *On affine symmetric spaces and the automorphism groups of product manifolds,* Hokkaido Math. J., **14** (1985), 277–351.

[44] H. Wu, *Complete developable submanifolds in real and complex Euclidean spaces*, International J. Math., **6-3** (1995), 461-489.

[45] K. Yamaguchi, *Geometrization of jet bundles*, Hokkaido Math. J. **12** (1983), 27–40.

[46] F. L. Zak, *Tangents and Secants of Algebraic Varieties,* Translations of Math. Monographs **127**, Amer. Math. Soc., Providence (1993).

ISHIKAWA Goo,
Department of Mathematics, Hokkaido University,
Sapporo 060-0810, JAPAN.
e-mail : ishikawa@math.sci.hokudai.ac.jp

北海道大学・理学研究院　石川剛郎