A note on the number of cusps of perturbations of complex polynomials

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1. Introduction

A smooth map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is called an *excellent map* if for any $p \in \mathbb{R}^2$, there exist local coordinates (x, y) centered at p and local coordinates centered at $f(p)$ such that f is locally described in one of the following forms:

(1) $(x, y) \mapsto (x, y)$,
(2) $(x, y) \mapsto (x, y^2)$,
(3) $(x, y) \mapsto (x, y^3 + xy)$.

A point in case (1) is a regular point. Points in cases (2) and (3) are called a *fold* and a *cusp*, respectively. Denote by $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ the set of all smooth maps $\mathbb{R}^2 \to \mathbb{R}^2$ equipped with the C^∞-topology. In [7], Whitney showed that the set of excellent maps is dense in $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$. It's known that there is a relation between the topology of surfaces and the topology of the critical locus of a map, see [5, 1]. Fukuda and Ishikawa also studied the number of cusps of stable perturbations of generic map germs [1]. They showed the number of cusps modulo 2 is a topological invariant of generic map germs. Moreover, the number of cusps modulo 2 depends only on the topology of surfaces.

Let $f(z)$ be a complex polynomial such that $f(0) = 0$. Then there exist a positive integer k and a complex polynomial g such that $f(z) = z^k g(z)$ and $g(0) \neq 0$. We call k *the multiplicity of* f *at the origin.* We consider certain perturbations of complex polynomials and calculate explicitly the number of cusps of perturbations by using multiplicities of singularities of complex polynomials.

We identify $\mathbb{C}$ with $\mathbb{R}^2$. Then $f(z)$ defines a real polynomial map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (\Re f(x, y), \Im f(x, y)),$$

where $z = x + \sqrt{-1}y$. Assume that the origin 0 of $\mathbb{C}$ is a singularity of f. We define a *linear perturbation* f_t *of* f as follows:

$$f_t(z) := f(z) + t(a + ib)\bar{z},$$

where $a, b, t \in \mathbb{R}, i = \sqrt{-1}$ and $0 < |t| \ll 1$. Note that a linear perturbation f_t of f is not a complex polynomial, but is a 1-variable mixed polynomial in the sense of Oka [4]. We now regard a mixed polynomial map $f_t : \mathbb{C} \to \mathbb{C}$ as a real polynomial map $(\Re f_t, \Im f_t) : \mathbb{R}^2 \to \mathbb{R}^2$. If $f(z) = z^n$, Fukuda and Ishikawa showed that the number of cusps of a linear perturbation of f is congruent to $n + 1$ modulo 2, see [1, Example 2.3]. If a and b lie outside the union of zero sets of analytic functions determined by a, b and f, f_t is an excellent map for $0 < |t| \ll 1$, see Lemma 2. The main theorem is the following [2].

Theorem 1. *Let $f(z)$ be a complex polynomial and k be the multiplicity of f the origin. Suppose that $k \geq 2$. If a linear perturbation f_t of f is an excellent map for $0 < |t| \ll 1$, then the number of cusps of $f_t|_U$ is equal to $k + 1$, where U is a sufficiently small neighborhood of the origin.*

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2. Singularities of polynomial maps

Let $g = (g_1, g_2) : U \to \mathbb{R}^2$ be a polynomial map, where U is an open set. Set $J = \frac{\partial(g_1,g_2)}{\partial(x,y)}, G_i = \frac{\partial(g_i,J)}{\partial(x,y)}$ for $i = 1, 2$. We define the algebraic set G' as follows:

$$G' := \Big\{(x, y) \in U \mid J(x, y) = G_1(x, y) = G_2(x, y) = \frac{\partial(G_1, J)}{\partial(x, y)} = \frac{\partial(G_2, J)}{\partial(x, y)} = 0\Big\}.$$

In [3, Proposition 2] and [6, Proposition 2.2], Krzyżanowska and Szafraniec showed the following proposition:

Proposition 1. *The algebraic set G' is empty if and only if the set of singularities of g consists of either fold singularities or cusps. Moreover, the number of cusps of g is equal to the number of $\{(x, y) \in U \mid J(x, y) = G_1(x, y) = G_2(x, y) = 0\}$.*

3. Multiplicity with sign

Set $z = x + iy$. Then a pair of real polynomials (g_1, g_2) defines a mixed polynomial $g(z, \bar{z})$ as follows:

$$\begin{aligned} g(z, \bar{z}) &= g_1(x, y) + ig_2(x, y) \\ &= g_1\Big(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\Big) + ig_2\Big(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\Big). \end{aligned}$$

Suppose that w is a mixed singularity of a mixed polynomial g, i.e., the gradient vectors of g_1 and g_2 at w are linearly dependent over $\mathbb{R}$. Then we have

$$\Big|\frac{\partial g}{\partial z}(w)\Big| = \Big|\frac{\partial g}{\partial \bar{z}}(w)\Big|,$$

see [4]. Let $\alpha \in \mathbb{C}$ be an isolated root of $g(z, \bar{z}) = 0$. Put

$$S^1_\varepsilon(\alpha) := \{z \in \mathbb{C} \mid |z - \alpha| = \varepsilon\},$$

where ε is a sufficiently small positive real number. We define the *multiplicity with the sign of the root α* by the mapping degree of the normalized function

$$\frac{g}{|g|} : S^1_\varepsilon(\alpha) \to S^1.$$

We denote the multiplicity with the sign of the root α by $m_s(g, \alpha)$.

We say that α is a *positive simple root* if α satisfies

$$\Big|\frac{\partial g}{\partial z}(\alpha)\Big| > \Big|\frac{\partial g}{\partial \bar{z}}(\alpha)\Big|.$$

Similarly, α is a *negative simple root* if α satisfies

$$\Big|\frac{\partial g}{\partial z}(\alpha)\Big| < \Big|\frac{\partial g}{\partial \bar{z}}(\alpha)\Big|.$$

In [4, Proposition 15], α is a positive (resp. negative) simple root if and only if $m_s(g, \alpha) = 1$ (resp. $m_s(g, \alpha) = -1$).

Consider a family of mixed polynomials $g_t(z, \bar{z}) = 0$ for $g_0 = g$ and $t \in \mathbb{R}$. Oka showed the following proposition, see [4, Proposition 16].

Proposition 2. *Let $\{P_1(t), \ldots, P_\nu(t)\}$ be the roots of $g_t(z, \bar{z}) = 0$ which are bifurcating from $z = \alpha$. Then we have*

$$\textstyle\sum_{j=1}^{\nu} m_s(g_t, P_j(t)) = m_s(g, \alpha).$$

4. The existence of linear perturbations which are excellent maps

Let $f(z)$ be a complex polynomial. Assume that $f(0) = 0$ and the origin of $\mathbb{C}$ is a singularity of f. Set $f_1 = \Re f$ and $f_2 = \Im f$. We take $a, b \in \mathbb{R}$. Then a linear perturbation f_t of f is defined by $f_t(z) = f(z) + t(a+ib)\bar{z}$, where $0 < |t| \ll 1$. Note that f_t is equal to

$$\begin{aligned} f_t(z) &= f(z) + t(a+ib)\bar{z} \\ &= f_1(z) + t(ax+by) + i\{f_2(z) + t(bx-ay)\}. \end{aligned}$$

Then f_t defines a real polynomial map from $\mathbb{R}^2$ to $\mathbb{R}^2$ as follows:

$$f_t : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto (f_1(x,y) + t(ax+by), f_2(x,y) + t(bx-ay)).$$

We calculate J, G_1 and G_2 of f_t. By the Cauchy–Riemann equations $\frac{\partial f_2}{\partial x} = -\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial y} = \frac{\partial f_1}{\partial x}$, J is modified as

$$\begin{aligned} J &= \det\begin{pmatrix} \frac{\partial f_1}{\partial x} + ta & \frac{\partial f_1}{\partial y} + tb \\ \frac{\partial f_2}{\partial x} + tb & \frac{\partial f_2}{\partial y} - ta \end{pmatrix} \\ &= \left|\frac{\partial f}{\partial z}\right|^2 - t^2(a^2+b^2). \end{aligned}$$

Since f is a complex valued harmonic function, $\frac{\partial f_1}{\partial x\partial x} = -\frac{\partial f_1}{\partial y\partial y}$. Then we have

$$\begin{aligned} G_1 &= \det\begin{pmatrix} \frac{\partial f_1}{\partial x} + ta & \frac{\partial f_1}{\partial y} + tb \\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y} \end{pmatrix} \\ &= 2\Big(\Big(\frac{\partial f_1}{\partial x}\Big)^2 - \Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial^2 f_1}{\partial x\partial y} + 4\frac{\partial f_1}{\partial x}\frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y\partial y} \\ &\quad + 2t\Big\{a\Big(\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x\partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y\partial y}\Big) - b\Big(-\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial y\partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x\partial y}\Big)\Big\}, \\ G_2 &= \det\begin{pmatrix} -\frac{\partial f_1}{\partial y} + tb & \frac{\partial f_1}{\partial x} - ta \\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y} \end{pmatrix} \\ &= 2\Big(\Big(\frac{\partial f_1}{\partial x}\Big)^2 - \Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial^2 f_1}{\partial y\partial y} - 4\frac{\partial f_1}{\partial x}\frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x\partial y} \\ &\quad + 2t\Big\{a\Big(-\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial y\partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x\partial y}\Big) + b\Big(\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x\partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y\partial y}\Big)\Big\}. \end{aligned}$$

If G_1 and G_2 are equal to 0 at (x,y), then (x,y) satisfies the following equation:

$$\begin{aligned} & a\Big[\Big(-3\Big(\frac{\partial f_1}{\partial x}\Big)^2 + \Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial f_1}{\partial y}\Big\{\Big(\frac{\partial^2 f_1}{\partial y\partial y}\Big)^2 - \Big(\frac{\partial^2 f_1}{\partial x\partial y}\Big)^2\Big\} \\ &\quad + 2\Big(-\Big(\frac{\partial f_1}{\partial x}\Big)^2 + 3\Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x\partial y}\frac{\partial^2 f_1}{\partial y\partial y}\Big] \\ & + b\Big[\Big(-\Big(\frac{\partial f_1}{\partial x}\Big)^2 + 3\Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial f_1}{\partial x}\Big\{\Big(\frac{\partial^2 f_1}{\partial y\partial y}\Big)^2 - \Big(\frac{\partial^2 f_1}{\partial x\partial y}\Big)^2\Big\} \\ &\quad - 2\Big(-3\Big(\frac{\partial f_1}{\partial x}\Big)^2 + \Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x\partial y}\frac{\partial^2 f_1}{\partial y\partial y}\Big] \\ &\quad = 0. \end{aligned} \tag{1}$$

Set real polynomials ϕ_1, ϕ_2 and Φ as follows:

$$\begin{aligned}
\phi_1 :=&\Big(-3\Big(\frac{\partial f_1}{\partial x}\Big)^2+\Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial f_1}{\partial y}\Big\{\Big(\frac{\partial^2 f_1}{\partial y\partial y}\Big)^2-\Big(\frac{\partial^2 f_1}{\partial x\partial y}\Big)^2\Big\}\\
&+2\Big(-\Big(\frac{\partial f_1}{\partial x}\Big)^2+3\Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x\partial y}\frac{\partial^2 f_1}{\partial y\partial y},\\
\phi_2 :=&\Big(-\Big(\frac{\partial f_1}{\partial x}\Big)^2+3\Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial f_1}{\partial x}\Big\{\Big(\frac{\partial^2 f_1}{\partial y\partial y}\Big)^2-\Big(\frac{\partial^2 f_1}{\partial x\partial y}\Big)^2\Big\}\\
&-2\Big(-3\Big(\frac{\partial f_1}{\partial x}\Big)^2+\Big(\frac{\partial f_1}{\partial y}\Big)^2\Big)\frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x\partial y}\frac{\partial^2 f_1}{\partial y\partial y},\\
\Phi :=& a\phi_1+b\phi_2.
\end{aligned}$$

Suppose that G_1 and G_2 are equal to 0 at (x, y). By the equation (1) and the definitions of ϕ_1, ϕ_2 and Φ, $\Phi(x, y)$ is also equal to 0. To show the existence of linear perturbations which are excellent maps, we consider the intersection of $\phi_1^{-1}(0)$ and $\phi_2^{-1}(0)$.

Lemma 1. *Let U be a sufficiently small neighborhood of the origin 0 of $\mathbb{C}$. Assume that U satisfies $\{w \in U \mid \frac{\partial f}{\partial z}(w) = 0\} = \{0\}$ and $\{w \in U \mid \frac{\partial^2 f}{\partial z\partial z}(w) = 0\} \subset \{0\}$. Then the intersection of $\phi_1^{-1}(0), \phi_2^{-1}(0)$ and U is equal to $\{0\}$.*

To study singularities of f_t, we define the mixed polynomial G_t as follows:

$$\begin{aligned}
G_t :=& G_1+iG_2\\
=& \Big(\frac{\partial f}{\partial z}+t(a+ib)\Big)\frac{\partial J}{\partial y}-i\Big(\frac{\partial f}{\partial z}-t(a+ib)\Big)\frac{\partial J}{\partial x}.
\end{aligned}$$

Since $\frac{\partial J}{\partial z}$ is equal to $\frac{1}{2}\Big(\frac{\partial J}{\partial x}-i\frac{\partial J}{\partial y}\Big)$, $\frac{\partial J}{\partial x}$ and $\frac{\partial J}{\partial y}$ are equal to

$$\begin{aligned}
\frac{\partial J}{\partial x} &= 2\Re\frac{\partial J}{\partial z} = 2\Re\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}} = \frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}+\overline{\frac{\partial^2 f}{\partial z\partial z}}\frac{\partial f}{\partial z},\\
\frac{\partial J}{\partial y} &= -2\Im\frac{\partial J}{\partial z} = -2\Im\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}} = i\Big(\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}-\overline{\frac{\partial^2 f}{\partial z\partial z}}\frac{\partial f}{\partial z}\Big),
\end{aligned}$$

where $z = x + iy$. Thus G_t is equal to

$$-2i\Big(\frac{\partial f}{\partial z}\Big)^2\overline{\frac{\partial^2 f}{\partial z\partial z}}+2ti(a+ib)\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}.$$

Suppose that z satisfies $G_t(z) = 0$ and $\frac{\partial f}{\partial z}(z)\frac{\partial^2 f}{\partial z\partial z}(z) \neq 0$. By the above equation, z satisfies $J(z) = 0$. Since the multiplicity k of f at the origin is greater than 1, $G_t(0) = 0$ and $\frac{\partial f}{\partial z}(0)\frac{\partial^2 f}{\partial z\partial z}(0) = 0$. Thus we have

$$\begin{aligned}
&\Big\{z \in U \mid G_t(z) = 0, \frac{\partial f}{\partial z}(z) \neq 0, \frac{\partial^2 f}{\partial z\partial z}(z) \neq 0\Big\}\\
=&\{z \in U \setminus \{0\} \mid G_t(z) = 0\} \subset J^{-1}(0).
\end{aligned}$$

Similarly, we define the following mixed polynomial:

$$\begin{aligned}
H_t :=& \det\begin{pmatrix}\frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y}\\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y}\end{pmatrix}+i\det\begin{pmatrix}\frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y}\\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y}\end{pmatrix}\\
=& \Big(\frac{\partial G_1}{\partial x}+i\frac{\partial G_2}{\partial x}\Big)\frac{\partial J}{\partial y}-\Big(\frac{\partial G_1}{\partial y}+i\frac{\partial G_2}{\partial y}\Big)\frac{\partial J}{\partial x}.
\end{aligned}$$

The differentials of G_t satisfy the following equations:

$$\frac{\partial G_t}{\partial z}=\frac{1}{2}\Big(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}\Big)+\frac{i}{2}\Big(\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\Big),\quad \frac{\partial G_t}{\partial \bar z}=\frac{1}{2}\Big(\frac{\partial G_1}{\partial x}-\frac{\partial G_2}{\partial y}\Big)+\frac{i}{2}\Big(\frac{\partial G_2}{\partial x}+\frac{\partial G_1}{\partial y}\Big).$$

Then we have

$$\begin{aligned}H_t&=\Big(\frac{\partial G_t}{\partial z}+\frac{\partial G_t}{\partial \bar z}\Big)\frac{\partial J}{\partial y}-i\Big(\frac{\partial G_t}{\partial z}-\frac{\partial G_t}{\partial \bar z}\Big)\frac{\partial J}{\partial x}\\&=\frac{\partial G_t}{\partial z}\Big(\frac{\partial J}{\partial y}-i\frac{\partial J}{\partial x}\Big)+\frac{\partial G_t}{\partial \bar z}\Big(\frac{\partial J}{\partial y}+i\frac{\partial J}{\partial x}\Big).\end{aligned}$$

Since $\frac{\partial J}{\partial y}-i\frac{\partial J}{\partial x}=-2i\frac{\partial J}{\partial \bar z}$ and $\frac{\partial J}{\partial y}+i\frac{\partial J}{\partial x}=2i\frac{\partial J}{\partial z}$, H_t is equal to

$$\begin{aligned}H_t=&-4\Big(\frac{\partial f}{\partial z}\Big)^2\frac{\partial^2 f}{\partial z\partial z}\Big\{2\overline{\Big(\frac{\partial^2 f}{\partial z\partial z}\Big)^2-\frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z\partial z\partial z}}\Big\}\\&+4t(a+ib)\overline{\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial z\partial z}}\Big\{-\Big(\frac{\partial^2 f}{\partial z\partial z}\Big)^2+\frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z\partial z\partial z}\Big\}.\end{aligned}$$

Note that $J(0)=|\frac{\partial f}{\partial z}(0)|^2-t^2(a^2+b^2)\neq 0$ for $t\neq 0$ and $(a,b)\neq(0,0)$. By the definitions of G_t and H_t, we have

$$\begin{aligned}&\{z\in U\setminus\{0\}\mid G_t(z)=H_t(z)=0\}\\=&\Big\{z\in U\mid J(z)=G_1(z)=G_2(z)=\frac{\partial(G_1,J)}{\partial(x,y)}(z)=\frac{\partial(G_2,J)}{\partial(x,y)}(z)=0\Big\}.\end{aligned}$$

By using Lemma 1, we show the existence of a linear perturbation f_t of f which is an excellent map for generic (a,b).

Lemma 2. *For a generic choice of (a,b), $f_t|_U$ is an excellent map.*

Let w be a singularity of f and U_w be a sufficiently small neighborhood of w. By changing coordinates of U_w and $f(U_w)$, we may assume that $w=0$ and $f(w)=0$. So we can apply Lemma 2 to any singularity of f. Thus we can check that f_t is an excellent map for $0<|t|\ll 1$ if a and b are generic.

5. Calculation of the number of cusps

To calculate the number of cusps of f_t, we study zero points of G_t and differentials of G_t.

Lemma 3. *The set $\{z\in U\mid G_t(z)=0, z\neq 0\}$ is the set of positive simple roots of G_t for $(a,b)\neq(0,0)$ and $0<|t|\ll 1$.*

Assume that f_t is an excellent map for $0<|t|\ll 1$. We calculate the number of cusps of $F_t|_U$. By Proposition 1, the number of cusps of $f_t|_U$ is equal to

$$\sharp\{z\in U\mid G_t(z)=0, z\neq 0\}.$$

Set $\{z\in U\mid G_t(z)=0, z\neq 0\}=\{w_1,\dots,w_\nu\}$. We denote the multiplicity of sign by $m_s(G_t,w_j)$ for $j=1,\dots,\nu$. By Proposition 2 and Lemma 3, we have

$$\Big(\textstyle\sum_{j=1}^{\nu} m_s(G_t,w_j)\Big)+m_s(G_t,0)=\nu+m_s(G_t,0)=m_s(G_0,0).$$

The multiplicity $m_s(G_0,0)$ is equal to

$$\begin{aligned}&\deg\Big(-2i\Big(\frac{\partial f}{\partial z}\Big)^2\overline{\frac{\partial^2 f}{\partial z\partial z}}\Big/\Big|-2i\Big(\frac{\partial f}{\partial z}\Big)^2\overline{\frac{\partial^2 f}{\partial z\partial z}}\Big|:S^1_\varepsilon(0)\to S^1\Big)\\&=2(k-1)-(k-2)=k,\end{aligned}$$

where $S^1_\varepsilon(0) = \{z \in U \mid |z| = \varepsilon\}$ and $0 < \varepsilon \ll 1$. By the definition of G_t, for any $t \neq 0$, $m_s(G_t, 0)$ is equal to

$$\begin{aligned}&\deg\left(2ti(a+ib)\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}\Bigg/\left|2ti(a+ib)\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}\right| : S^1_{\varepsilon_t}(0) \to S^1\right)\\ &= k-2-(k-1) = -1,\end{aligned}$$

where $0 < \varepsilon_t \ll \varepsilon$. Thus the number ν of cusps of $f_t|_U$ is equal to $k+1$.

We estimate the number of cusps of f_t in $\mathbb{R}^2$.

Corollary 1. *Let f_t be a liner perturbation of a complex polynomial f in Theorem 1 and $n = \deg f$. Assume that $n \geq 2$. Then the number of cusps of f_t belongs to $[n+1, 3n-3]$. In particular, the number of cusps of f_t is at least three.*

6. Examples

In this section, we construct a perturbation of a complex polynomial which has $(n+1)$-cusps and also a perturbation which has $(3n-3)$-cusps.

Example 1. *Let $f(z) = z^n$ and $f_t(z) = z^n + t(a+ib)\bar{z}$ be a perturbation of f which is an excellent map. Then $G_t(z)$ is equal to*

$$\begin{aligned}G_t(z) &= -2in^3(n-1)z^{2n-2}\bar{z}^{n-2} + 2tn^2(n-1)(a+ib)z^{n-2}\bar{z}^{n-1}\\ &= -2in^2(n-1)|z|^{2n-4}\{nz^n - t(a+ib)\bar{z}\}.\end{aligned}$$

Set $z = re^{i\theta}$ and $a+ib = \tau e^{i\iota}$, where $\tau > 0$. Then we have

$$-2in^2(n-1)r^{2n-4}\{nr^n e^{ni\theta} - t\tau r e^{i(\iota-\theta)}\}.$$

Assume that $z \neq 0$ and $G_t(z) = 0$. Then z satisfies

$$r = \left(\frac{t\tau}{n}\right)^{\frac{1}{n-1}}, \quad \theta = \frac{\iota + 2j\pi}{n+1},$$

for $j = 0, \ldots, n$. Thus the number of cusps of f_t is equal to $n+1$.

Example 2. *Let $f(z) = z^n + z$. Then the number of singularities of f is equal to $n-1$ and the multiplicity at each singularity of f is equal to 2. Let $f_t(z) = z^n + z + t(a+ib)\bar{z}$ be a perturbation of f which is an excellent map. By the same argument as in the proof of Corollary 1, the number of cusps of f_t is equal to $3n-3$.*

7. Non-linear perturbations

7.1. **Perturbations of f_t.** Let f_t be a linear perturbation of f which is an excellent map. We fix a, b and t. Let $g(z, \bar{z})$ be a mixed polynomial which satisfies $\frac{\partial g}{\partial z}(0) = \frac{\partial g}{\partial \bar{z}}(0) = 0$. In this subsection, we study a perturbation of f_t:

$$f_{t,s}(z) := f(z) + t(a+ib)\bar{z} + sg(z, \bar{z}),$$

where $0 < |s| \ll |t| \ll 1$. Since $|s|$ is sufficiently small, we can show the following theorem.

Theorem 2. *The set of singularities of $f_{t,s}$ consists of either fold singularities or cusps and the number of cusps of $f_{t,s}$ is constant for $0 \leq |s| \ll |t| \ll 1$.*

7.2. **Lower bounds of the numbers of cusps of non-linear perturbations.** Let $h(z,\bar{z})$ be a mixed polynomial which satisfies $h(0)=0$ and $|\frac{\partial h}{\partial z}(0)| \neq |\frac{\partial h}{\partial \bar{z}}(0)|$. We define a perturbation $f_{t,h}$ of a complex polynomial f as follows:

$$f_{t,h}(z) := f(z) + th(z,\bar{z}),$$

where $0<|t|\ll 1$. Set $h_1 = \Re h, h_2 = \Im h$ and

$$J_{t,h} = \det \begin{pmatrix} \frac{\partial f_1}{\partial x} + t\frac{\partial h_1}{\partial x} & \frac{\partial f_1}{\partial y} + t\frac{\partial h_1}{\partial y} \\ -\frac{\partial f_1}{\partial y} + t\frac{\partial h_2}{\partial x} & \frac{\partial f_1}{\partial x} + t\frac{\partial h_2}{\partial y} \end{pmatrix}.$$

Then any singularity of $f_{t,h}$ belongs to $J_{t,h}^{-1}(0)$. Assume that $f_{t,h}$ satisfies the following conditions:

(i) $f_{t,h}$ is an excellent map for $0<|t|\ll 1$,

(ii) any cusp of $f_{t,h}$ is a simple root of $G_{t,h}$, where

$$\begin{aligned} G_{t,h} &:= \det \begin{pmatrix} \frac{\partial f_1}{\partial x} + t\frac{\partial h_1}{\partial x} & \frac{\partial f_1}{\partial y} + t\frac{\partial h_1}{\partial y} \\ \frac{\partial J_{t,h}}{\partial x} & \frac{\partial J_{t,h}}{\partial y} \end{pmatrix} + i\det \begin{pmatrix} -\frac{\partial f_1}{\partial y} + t\frac{\partial h_2}{\partial x} & \frac{\partial f_1}{\partial x} + t\frac{\partial h_2}{\partial y} \\ \frac{\partial J_{t,h}}{\partial x} & \frac{\partial J_{t,h}}{\partial y} \end{pmatrix} \\ &= -2i\Big(\frac{\partial f}{\partial z} + t\frac{\partial h}{\partial z}\Big)\overline{\frac{\partial J_{t,h}}{\partial z}} + 2ti\frac{\partial h}{\partial \bar{z}}\frac{\partial J_{t,h}}{\partial z}. \end{aligned}$$

Since $f_{t,h}$ is an excellent map, the intersection of $J_{t,h}^{-1}(0)$ and $(\frac{\partial J_{t,h}}{\partial z})^{-1}(0)$ is empty by Proposition 1. Let U be a sufficiently small neighborhood of the origin. Then the number of cusps of $f_{t,h}|_U$ is equal to the number of $\{z\in U \mid G_{t,h}(z)=0, \frac{\partial J_{t,h}}{\partial z}(z)\neq 0\}$. We define

$$\delta = \begin{cases} 1 & |\frac{\partial h}{\partial z}(0)| > |\frac{\partial h}{\partial \bar{z}}(0)| \\ -1 & |\frac{\partial h}{\partial z}(0)| < |\frac{\partial h}{\partial \bar{z}}(0)| \end{cases}.$$

Theorem 3. *Let $f_{t,h}$ a perturbation of a complex polynomial f which satisfies the condition (i) and the condition (ii). Then the number of cusps of $f_{t,h}|_U$ is greater than or equal to $k-\delta$, where k is the multiplicity of f at the origin.*

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