A note on "Another ordering of the ten cardinal characteristics in Cichoń's Diagram" and further remarks

Diego Alejandro Mejía

Faculty of Science, Shizuoka University.

Abstract

In this note, we relax the hypothesis of the main results in Kellner–Shelah–Tănasie's *Another ordering of the ten cardinal characteristics in Cichoń's diagram.*

1 Introduction

This work belongs to the framework of consistency results where several cardinal invariants are pairwise different, in particular those in Cichoń's diagram (Figure 1). Though we assume that the reader is familiar with this diagram, a characterization of its cardinal characteristics is presented in Section 2. It is well-known that this diagram is *complete* in the sense that no other inequality can be proved between two cardinal invariants there. See e.g. [BJ95] for details and original references.

Resent progress in this framework appears in [GMS16, FFMM18, Mej19, KTT18, GKS, KST19, BCM, GKMS]. In particular, [KTT18] deals with the method of Boolean ultrapowers, used in [GKS] to separate all cardinals in Cichoń's diagram using four strongly compact cardinals. This result was improved in [BCM] where it is shown that three strongly compact cardinals suffice. In [GKMS] more classical cardinal invariants are separated simultaneously.

Figure 1: Cichoń's diagram. The arrows represent $\leq$. The dashed arrows mean $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{M})\}$.

This survey focuses on Kellner–Shelah–Tănasie's work [KST19], where the consistency of an alternative order of the ten values of Cichoń's diagram is proved. Concretely, they show:

(A) Assuming GCH, if $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$ are regular $\aleph_1$-inaccessible cardinals,[1] then there is a ccc poset that forces $\mathrm{add}(\mathcal{N}) = \lambda_1$, $\mathfrak{b} = \lambda_2$, $\mathrm{cov}(\mathcal{N}) = \lambda_3$, $\mathrm{non}(\mathcal{M}) = \lambda_4$, and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda_5$.

(B) Assuming GCH, if $\kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ are regular cardinals such that λ_2 is a successor cardinal, λ_i is $\aleph_1$-inaccessible for $i = 1, \ldots, 5$ and κ_j is strongly compact for $j = 6, \ldots, 9$, then there is a ccc poset that forces

$$\mathrm{add}(\mathcal{N}) = \lambda_1, \ \mathfrak{b} = \lambda_2, \ \mathrm{cov}(\mathcal{N}) = \lambda_3, \ \mathrm{non}(\mathcal{M}) = \lambda_4,$$
$$\mathrm{cov}(\mathcal{M}) = \lambda_5, \ \mathrm{non}(\mathcal{N}) = \lambda_6, \ \mathfrak{d} = \lambda_7, \ \mathrm{cof}(\mathcal{N}) = \lambda_8, \ \text{and } \mathfrak{c} = \lambda_9.$$

In this work, we show how to relax many of the hypothesis on (A) and (B), for instance, replace GCH by very specific cardinal arithmetic conditions, show that the hypothesis "$\aleph_1$-inaccessible" is only required for λ_3, and that λ_2 could also be a limit (regular) cardinal in (B). To be precise, we prove:

Theorem A. *Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ be regular uncountable cardinals, and let $\lambda_5 = \lambda_5^{<\lambda_4}$ be a cardinal. Assume that either*

(i) $\lambda_2 = \lambda_3$, or

(ii) $\lambda_2 = \lambda_2^{<\lambda_2} < \lambda_3$, λ_3 is $\aleph_1$-inaccessible, and either $\lambda_4 = \lambda_5$ or $\lambda_4^{\aleph_0} < \lambda_5$.

Then there is a ccc poset forcing $\mathrm{add}(\mathcal{N}) = \lambda_1$, $\mathfrak{b} = \lambda_2$, $\mathrm{cov}(\mathcal{N}) = \lambda_3$, $\mathrm{non}(\mathcal{M}) = \lambda_4$, and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda_5$.

Theorem B. *Assume that $\kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 \leq \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ such that*

(i) $\lambda_1, \ldots, \lambda_5$ are as in the hypothesis of Theorem A;

(ii) for $j = 6, \ldots, 9$, κ_j is strongly compact and $\lambda_j^{\kappa_j} = \lambda_j$; and

(iii) for $j = 6, 7, 8$, λ_j is regular.

then there is a ccc poset that forces

$$\mathrm{add}(\mathcal{N}) = \lambda_1, \ \mathfrak{b} = \lambda_2, \ \mathrm{cov}(\mathcal{N}) = \lambda_3, \ \mathrm{non}(\mathcal{M}) = \lambda_4,$$
$$\mathrm{cov}(\mathcal{M}) = \lambda_5, \ \mathrm{non}(\mathcal{N}) = \lambda_6, \ \mathfrak{d} = \lambda_7, \ \mathrm{cof}(\mathcal{N}) = \lambda_8, \ \text{and } \mathfrak{c} = \lambda_9.$$

These improvements does not demand much changes in the original proofs, actually, the only dramatic modification is the construction of small posets, this to avoid the hypothesis "$\aleph_1$-inaccessible". In relation to Theorem B (or the method of Boolean ultrapowers) we show how to weaken the notion $\mathrm{COB}(\theta, \lambda)$ used in [KTT18, GKS, KST19] to the more natural "θ-dominating family of size λ" (see Definition 2.1(7)), i.e., that Boolean ultrapowers preserves such dominating families (by just modifying its size in some cases). Details are presented in Lemmas 5.1(g) and 5.2(c).

---

[1] A cardinal λ is θ-*inaccessible* if $\mu^\nu < \lambda$ for any $\mu < \lambda$ and $\nu < \theta$.

Remark 1.1. Any strict inequality after λ_5 can be replaced by $=$ arbitrarily, and in such case the corresponding strongly compact cardinal can be omitted. For example, if we let $\lambda_{j-1} = \lambda_j$ ($j = 6, 7, 8, 9$) then the strongly compact cardinal κ_j can be omitted, while $\lambda_{9-j} \leq \lambda_{10-j}$ is allowed (here $\lambda_0 = \aleph_1$).

On the other hand, in [GKMS] it is proved that the large gaps between the cardinals on the left side of Cichoń's diagram can be reduced, while separating on the right side. However, the strongly compact cardinals are still required for this purpose.

This paper is structured as follows. In Section 2 we review general notions related to preservation of unbounded families. In Section 3 we summarize some technicalities used in [KST19] without going too deep into details, e.g., we manage to avoid introducing FAM-limits, which is the core of the tools in that reference. In Section 4 we prove Theorem A and in Section 5 we present further remarks on the Boolean ultrapower method towards Theorem B.

2   Relational systems and preservation

In many cases, cardinal invariants of the continuum are defined through *relational systems.*

Definition 2.1. A *relational system* is a triplet $\mathbf{A} = \langle X, Y, \sqsubset \rangle$ where $\sqsubset$ is a relation contained in $X \times Y$. For $x \in X$ and $y \in Y$, $x \sqsubset y$ is often read $y \sqsubset$-*dominates* x.

Fix a cardinal θ.

(1) A family $D \subseteq Y$ is $\mathbf{A}$-*dominating* iff every member of X is $\sqsubset$-dominated by some member of D.

(2) A family $F \subseteq X$ is $\mathbf{A}$-*unbounded* if there is <u>no</u> real in Y that $\sqsubset$-dominates every member of F.

(3) The relational system $\mathbf{A}^{\perp} := \langle Y, X, \not\sqsupset \rangle$ is referred to as the *dual of* $\mathbf{A}$. Note that $(\mathbf{A}^{\perp})^{\perp} = \mathbf{A}$ and that $F \subseteq X$ is $\mathbf{A}$-unbounded iff it is $\mathbf{A}^{\perp}$-dominating.

(4) For a set M say that $y \in Y$ is $\mathbf{A}$-*dominating over* M if $x \sqsubset y$ for all $x \in X \cap M$.

(5) Say that $x \in X$ is $\mathbf{A}$-*unbounded over* M if it is $\mathbf{A}^{\perp}$-dominating over M, that is, $x \not\sqsubset y$ for all $y \in Y \cap M$.

(6) The cardinal $\mathfrak{b}(\mathbf{A})$ denotes the least size of an $\mathbf{A}$-unbounded family and $\mathfrak{d}(\mathbf{A})$ is the least size of an $\mathbf{A}$-dominating family. Note that $\mathfrak{b}(\mathbf{A}^{\perp}) = \mathfrak{d}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A}^{\perp}) = \mathfrak{b}(\mathbf{A})$.

(7) A family $D \subseteq Y$ is θ-$\mathbf{A}$-*dominating* if, for every $E \in [X]^{<\theta}$, there is some $\mathbf{A}$-dominating $y \in D$ over E.

(8) A family $F \subseteq X$ is θ-$\mathbf{A}$-*unbounded* if it is θ-$\mathbf{A}^{\perp}$-dominating, that is, for any $H \in [Y]^{<\theta}$ there is some $\mathbf{A}$-unbounded $x \in F$ over H.

(9) A family $D \subseteq Y$ is *strongly* θ-$\mathbf{A}$-*dominating* if $|D| \geq \theta$ and, for every $x \in X$, $|\{y \in D : x \not\sqsubset y\}| < \theta$.

(10) A family $F \subseteq X$ is *strongly θ-$\mathbf{A}$-unbounded* if it is strongly-θ-$\mathbf{A}^{\perp}$-dominating, that is, $|F| \geq \theta$ and, for every $y \in Y$, $|\{x \in F : x \sqsubset y\}| < \theta$.

Remark 2.2. Fix a relational system $\mathbf{A} = \langle X, Y, \sqsubset \rangle$.

(1) The cardinal invariants $\mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A})$ may not always exist. Concretely, $\mathfrak{b}(\mathbf{A})$ does not exist iff $\mathfrak{d}(\mathbf{A}) = 1$. Dually, $\mathfrak{d}(\mathbf{A})$ does not exists iff $\mathfrak{b}(\mathbf{A}) = 1$.

(2) Any subset of Y is $\mathbf{A}$-dominating iff it is 2-$\mathbf{A}$-dominating. Likewise, $\mathbf{A}$-unbounded is equivalent to 2-$\mathbf{A}$-unbounded.

(3) If $\theta \leq \theta'$ are cardinals, then any θ'-$\mathbf{A}$-dominating family is θ-$\mathbf{A}$-dominating. Likewise for unbounded families.

(4) If $\theta \geq 2$, then any θ-$\mathbf{A}$-dominating family is $\mathbf{A}$-dominating.[2] Moreover, if $D \subseteq Y$ is θ-$\mathbf{A}$-dominating family then $\mathfrak{d}(\mathbf{A}) \leq |D|$ and $\theta \leq \mathfrak{b}(\mathbf{A})$. Similar statements hold for unbounded families, e.g., if $F \subseteq X$ is θ-$\mathbf{A}$-unbounded then $\mathfrak{b}(\mathbf{A}) \leq |F|$ and $\theta \leq \mathfrak{d}(\mathbf{A})$.

(5) Any strongly θ-$\mathbf{A}$-dominating family is $\mathbf{A}$-dominating. Likewise for unbounded.

(6) If θ is regular and $D \subseteq Y$ is a strongly θ-$\mathbf{A}$-dominating family, then D is $|D|$-$\mathbf{A}$-dominating, so $\mathfrak{d}(\mathbf{A}) \leq |D| \leq \mathfrak{b}(\mathbf{A})$. Similarly, if $F \subseteq X$ is strongly θ-$\mathbf{A}$-unbounded then it is $|F|$-$\mathbf{A}$-unbounded, which implies $\mathfrak{b}(\mathbf{A}) \leq |F| \leq \mathfrak{d}(\mathbf{A})$.

Inequalities between cardinal invariants are often proved using the *Tukey order* between relational systems. If $\mathbf{A} = \langle X, Y, \sqsubset \rangle$ and $\mathbf{A}' = \langle X', Y', \sqsubset' \rangle$ are relational systems, $\mathbf{A} \preceq_{\mathrm{T}} \mathbf{A}'$ means that there are two maps $\varphi : X \to X'$ and $\psi : Y' \to Y$ such that, for any $x \in X$ and $y' \in Y'$, $\varphi(x) \sqsubset' y'$ implies $x \sqsubset \psi(y')$. In this case, the ψ-image of any $\mathbf{A}'$-dominating set is $\mathbf{A}$-dominating, and the φ-image of any $\mathbf{A}$-unbounded set is $\mathbf{A}'$-unbounded, thus $\mathfrak{b}(\mathbf{A}') \leq \mathfrak{b}(\mathbf{A})$ and $\mathfrak{d}(\mathbf{A}) \leq \mathfrak{d}(\mathbf{A}')$. Say that $\mathbf{A}$ and $\mathbf{A}'$ are *Tukey equivalent*, denoted by $\mathbf{A} \cong_{\mathrm{T}} \mathbf{A}'$, if $\mathbf{A} \preceq_{\mathrm{T}} \mathbf{A}'$ and $\mathbf{A}' \preceq_{\mathrm{T}} \mathbf{A}$.

Before we present examples, we review the preservation theory of unbounded families presented in [CM19, Sect. 4]. This a generalization of Judah and Shelah's [JS90] and Brendle's [Bre91] preservation theory.

Definition 2.3. Say that $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ is a *generalized Polish relational system (gPrs)* if

(I) X is a Perfect Polish space,

(II) $Y = \bigcup_{e \in \Omega} Y_e$ where Ω is a non-empty set and, for some Polish space Z, Y_e is non-empty and analytic in Z for all $e \in \Omega$, and

(III) $\sqsubset = \bigcup_{n < \omega} \sqsubset_n$ where $\langle \sqsubset_n : n < \omega \rangle$ is some increasing sequence of closed subsets of $X \times Z$ such that, for any $n < \omega$ and for any $y \in Y$, $(\sqsubset_n)^y = \{x \in X : x \sqsubset_n y\}$ is closed nowhere dense.

If $|\Omega| = 1$, we just say that $\mathbf{R}$ is a *Polish relational system (Prs)*.

For the rest of this section, fix a gPrs $\mathbf{R} = \langle X, Y, \sqsubset \rangle$.

---

[2]Any subset of Y is 0-$\mathbf{A}$-dominating; and $D \subseteq Y$ is 1-$\mathbf{A}$-dominating iff $D \neq \emptyset$.

Remark 2.4. $\langle X, \mathcal{M}(X), \in \rangle \preceq_T \mathbf{R}$, so $\mathfrak{b}(\mathbf{R}) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}(\mathbf{R})$.

Definition 2.5. Let θ be a cardinal. A poset $\mathbb{P}$ is θ-$\mathbf{R}$-*good* if, for any $\mathbb{P}$-name $\dot{h}$ for a member of Y, there exists a non-empty $H \subseteq Y$ (in the ground model) of size $< \theta$ such that, for any $x \in X$, if x is $\mathbf{R}$-unbounded over H then $\Vdash x \not\sqsubseteq \dot{h}$.

Say that $\mathbb{P}$ is $\mathbf{R}$-*good* if it is $\aleph_1$-$\mathbf{R}$-good.

Note that $\theta < \theta'$ implies that any θ-$\mathbf{R}$-good poset is θ'-$\mathbf{R}$-good. Also, if $\mathbb{P} \lessdot \mathbb{Q}$ and $\mathbb{Q}$ is θ-$\mathbf{R}$-good, then $\mathbb{P}$ is θ-$\mathbf{R}$-good.

The notion of goodness is practical to preserve special types of unbounded families in generic extensions.

Lemma 2.6 ([CM19, Lemma 4.7]). *Let θ be a regular cardinal, $\lambda \geq \theta$ a cardinal and let $\mathbb{P}$ be a θ-$\mathbf{R}$-good poset.*

(a) *If $F \subseteq X$ is λ-$\mathbf{R}$-unbounded, then $\mathbb{P}$ forces that it is $\dot{\lambda}'$-$\mathbf{R}$-unbounded where, in the $\mathbb{P}$-extension, $\dot{\lambda}'$ is the smallest cardinal $\geq \lambda$.*

(b) *If $\mathrm{cf}(\lambda) \geq \theta$ and $F \subseteq X$ is strongly λ-$\mathbf{R}$-unbounded then $\Vdash$ "if λ is a cardinal then F is strongly λ-$\mathbf{R}$-unbounded".*

(c) *If $\mathfrak{d}(\mathbf{R}) \geq \lambda$ then $\mathbb{P}$ forces that $\mathfrak{d}(\mathbf{R}) \geq \dot{\lambda}'$.*

As a first general example, every small poset is always good.

Lemma 2.7 ([CM19, Lemma 2.7]). *If θ is a regular cardinal then any poset of size $< \theta$ is θ-$\mathbf{R}$-good. In particular, Cohen forcing $\mathbb{C}$ is $\mathbf{R}$-good.*

If θ is an uncountable regular cardinal then any FS support iteration of θ-$\mathbf{R}$-good θ-cc posets is again θ-$\mathbf{R}$-good (and θ-cc). Hence, according to the previous lemma, they preserve λ-$\mathbf{R}$-unbounded families for any $\lambda \geq \theta$, and strongly λ-$\mathbf{R}$-unbounded families for any λ with $\mathrm{cf}(\lambda) \geq \theta$. Such unbounded families can be added using Cohen reals.

Lemma 2.8 ([CM19, Lemma 4.15]). *If ν is a cardinal with uncountable cofinality and $\mathbb{P}_\nu = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha < \nu}$ is a FS iteration of non-trivial $\mathrm{cf}(\nu)$-cc posets, then $\mathbb{P}_\nu$ adds a strongly ν-$\mathbf{R}$-unbounded family of size ν.*

Theorem 2.9 ([CM19, Thm. 4.16]). *Let θ be an uncountable regular cardinal, $\delta \geq \theta$ an ordinal, and let $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha < \delta}$ be a FS iteration such that, for each $\alpha < \delta$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name of a non-trivial θ-$\mathbf{R}$-good θ-cc poset. Then:*

(a) *For any cardinal $\nu \in [\theta, \delta]$ with $\mathrm{cf}(\nu) \geq \theta$, $\mathbb{P}_\nu$ adds a strongly ν-$\mathbf{R}$-unbounded family of size ν which is still strongly ν-$\mathbf{R}$-unbounded in the $\mathbb{P}_\delta$-extension.*

(b) *For any cardinal $\lambda \in [\theta, \delta]$, $\mathbb{P}_\lambda$ adds a λ-$\mathbf{R}$-unbounded family of size λ which is still λ-$\mathbf{R}$-unbounded in the $\mathbb{P}_\delta$-extension.*

(c) *$\mathbb{P}_\delta$ forces that $\mathfrak{b}(\mathbf{R}) \leq \theta$ and $|\delta| \leq \mathfrak{d}(\mathbf{R})$.*

In the practice, the converse inequalities of the last item are obtained by constructing a θ-$\mathbf{R}$-dominating family of size $|\delta|$.

Example 2.10. We fix some notation. For any fuction $b : \omega \to V \smallsetminus \{\emptyset\}$ and $h \in \omega^\omega$, denote $\prod b := \prod_{i<\omega} b(i)$ and $\mathcal{S}(b,h) := \prod_{i<\omega} [b(i)]^{\leq h(i)}$. For two functions x, y with domain ω, $x \in^* y$ denotes $\forall^\infty i < \omega(x(i) \in y(i))$. Denote by id_ω the identity function on ω.

(1) The relational system $\mathbf{D} = \langle \omega^\omega, \omega^\omega, \leq^* \rangle$ is a Prs, where $x \leq^* y$ means $\forall^\infty i < \omega(x(i) \leq y(i))$. Note that $\mathfrak{b} = \mathfrak{b}(\mathbf{D})$ and $\mathfrak{d} = \mathfrak{d}(\mathbf{D})$. Clearly, any ω^ω-bounding poset is $\mathbf{D}$-good. Any μ-Fr-linked poset (see Definition 2.11) is μ^+-$\mathbf{D}$-good ([Mej19, Thm. 3.30]).

(2) For $\mathcal{H} \subseteq \omega^\omega$ denote $\mathbf{Lc}(\omega, \mathcal{H}) := \langle \omega^\omega, \mathcal{S}(\omega, \mathcal{H}), \in^* \rangle$ where $\mathcal{S}(\omega, \mathcal{H}) := \bigcup_{h \in \mathcal{H}} \mathcal{S}(\omega, h)$ (here, ω denotes the constant function with value ω). It is clear that $\mathbf{Lc}(\omega, \mathcal{H})$ is a gPrs. Any ν-centered poset is ν^+-$\mathbf{Lc}(\omega, \mathcal{H})$-good ([JS90], see also [Bre91, Lemma 6] and [BM14, Lemma 5.13]).

Whenever $\mathcal{H}$ is countable and non-empty, $\mathbf{Lc}(\omega, \mathcal{H})$ is a Prs because $\mathcal{S}(\omega, \mathcal{H})$ is F_σ in $([\omega]^{<\omega})^\omega$. In addition, if $\mathcal{H}$ contains a function that goes to infinity then $\mathfrak{b}(\mathbf{Lc}(\omega, \mathcal{H})) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Lc}(\omega, \mathcal{H})) = \mathrm{cof}(\mathcal{N})$ (see [BJ95, Thm. 2.3.9]). Moreover, if all the members of $\mathcal{H}$ go to infinity then any Boolean algebra with a strictly positive finitely additive measure is $\mathbf{Lc}(\omega, \mathcal{H})$-good ([Kam89]). In particular, any subalgebra of random forcing is $\mathbf{Lc}(\omega, \mathcal{H})$-good.

For the rest of this paper, fix $\mathcal{H}_* := \{\mathrm{id}_\omega^{n+1} : n < \omega\}$.

(3) (Kamo and Osuga [KO14]) Fix a family $\mathcal{E} \subseteq \omega^\omega$ of size $\aleph_1$ of non-decreasing functions which satisfies

   (i) $\forall e \in \mathcal{E}(e \leq \mathrm{id}_\omega)$,

   (ii) $\forall e \in \mathcal{E}(\ \lim_{n \to +\infty} e(n) = +\infty$ and $\lim_{n \to +\infty}(n - e(n)) = +\infty)$,

   (iii) $\forall e \in \mathcal{E} \exists e' \in \mathcal{E}(e + 1 \leq^* e')$ and

   (iv) $\forall \mathcal{E}' \in [\mathcal{E}]^{\leq \aleph_0} \exists e \in \mathcal{E} \forall e' \in \mathcal{E}'(e' \leq^* e)$.

For $b, h \in \omega^\omega$ such that $b > 0$ and $h \geq^* 1$, we define $\hat{\mathcal{S}}(b, h) = \hat{\mathcal{S}}_\mathcal{E}(b, h)$ by

$$\hat{\mathcal{S}}(b, h) := \bigcup_{e \in \mathcal{E}} \mathcal{S}(b, h^e) = \left\{ \varphi \in \prod_{n < \omega} \mathcal{P}(b(n)) : \exists e \in \mathcal{E} \forall n < \omega(|\varphi(n)| \leq h(n)^{e(n)}) \right\}$$

Let $n < \omega$. For $\psi, \varphi : \omega \to [\omega]^{<\omega}$, define the relation $\psi \blacktriangleright_n \varphi$ iff $\forall k \geq n(\psi(k) \not\supseteq \varphi(k))$, and define $\psi \blacktriangleright \varphi$ iff $\forall^\infty k < \omega(\psi(k) \not\supseteq \varphi(k))$, i.e., $\blacktriangleright = \bigcup_{n < \omega} \blacktriangleright_n$. Put $\mathbf{aLc}^*(b, h) := \langle \mathcal{S}(b, h^{\mathrm{id}_\omega}), \hat{\mathcal{S}}(b, h), \blacktriangleright \rangle$ which is a gPrs where $\Omega = \mathcal{E}$, $Z = \mathcal{S}(b, h^{\mathrm{id}_\omega})$ and $Y_e = \mathcal{S}(b, h^e)$ for each $e \in \mathcal{E}$. Note that Y_e is closed in Z.

Any ν-centered poset is ν^+-$\mathbf{aLc}^*(b, h)$-good ([CM19, Lemma 4.25]). On the other hand, if $b \not\leq^* 1$ then any $(h, b^{h^{\mathrm{id}_\omega}})$-linked poset (see [KO14]) is 2-$\mathbf{aLc}^*(b, h)$-good ([KO14, Lemma 10]).

If $\sum_{i < \omega} \frac{h(i)^i}{b(i)} < \infty$ then $\mathbf{aLc}^*(b, h) \preceq_\mathrm{T} \langle \mathcal{N}, 2^\omega, \not\ni \rangle$, so $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}(\mathbf{aLc}^*(b, h))$ and $\mathfrak{d}(\mathbf{aLc}^*(b, h)) \leq \mathrm{non}(\mathcal{N})$ (see e.g. [KM18, Lemma 2.3]).

(4) Denote $\Xi := \{f : 2^{<\omega} \to 2^{<\omega} : \forall s \in 2^{<\omega}(s \subseteq f(s))\}$ and set $\mathbf{Mg} := \langle 2^\omega, \Xi, \in^\bullet \rangle$ where $x \in^\bullet f$ iff $|\{s \in 2^{<\omega} : x \supseteq f(s)\}| < \aleph_0$. This is a Prs and $\mathbf{Mg} \cong_\mathrm{T} \langle 2^\omega, \mathcal{M}, \in \rangle$, so $\mathfrak{b}(\mathbf{Mg}) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}(\mathbf{Mg}) = \mathrm{cov}(\mathcal{M})$.

(5) Denote $\mathbf{Id} := \langle 2^\omega, 2^\omega, = \rangle$, which is clearly a Prs. Note that $\mathfrak{b}(\mathbf{Id}) = 2$ and $\mathfrak{d}(\mathbf{Ct}) = \mathfrak{c} := 2^{\aleph_0}$. It is easy to see that any θ-cc poset is θ-$\mathbf{Ct}$-good.

To finish this section, we review a general notion to preserve $\mathbf{D}$-unbounded families.

Definition 2.11 ([Mej19])**.** Let $\mathbb{P}$ be a poset and let μ be an infinite cardinal.

(1) A set $Q \subseteq \mathbb{P}$ is Fr-*linked* if, for any sequence $\bar{p} = \langle p_n : n < \omega \rangle$ in Q, there exists a $q \in \mathbb{P}$ that forces $|\{n < \omega : p_n \in \dot{G}\}| = \aleph_0$.

(2) The poset $\mathbb{P}$ is μ-Fr-*linked* if $\mathbb{P} = \bigcup_{\alpha < \mu} P_\alpha$ for some sequence $\langle P_\alpha : \alpha < \mu \rangle$ of Fr-linked subsets of $\mathbb{P}$. When $\mu = \aleph_0$, we write σ-Fr-*linked*.

(3) The poset $\mathbb{P}$ is μ-Fr-*Knaster* if any subset of $\mathbb{P}$ of size μ contains an Fr-linked set of size μ.

Theorem 2.12 ([BCM])**.** *If κ is an uncountable regular cardinal then any κ-Fr-Knaster poset preserves all the strongly κ-$\mathbf{D}$-unbounded families from the ground model.*

3   Iteration candidates

The main issue in the construction of the FS iteration for Theorem A is to ensure that strongly $\mathbf{D}$-unbounded families are preserved. For this purpose, Kellner, Tănasie and Shelah [KST19] use the method of FAM (finite additive measure) limits, originally introduced by Shelah [She00].

In the construction of the iteration, it is expected to use $\mathbb{E}$, the standard σ-centered poset to add an eventually different real, to control non($\mathcal{M}$), but it seems that they do not have FAM limits. For this reason, Kellner, Tănasie and Shelah used an alternative poset that behaves well with FAM limits.

Definition 3.1 ([KST19, Def. 1.11 & 1.13])**.** Define the following functions by recursion on $n < \omega$.

$$\rho_*(n) = \max\left\{\prod_{k<n} M_*(k), n+2\right\},$$
$$\pi_*(n) = \left((n+1)^2 \rho_*(n)^{n+1}\right)^{\rho_*(n)^n},$$
$$a_*(n) = \pi_*(n)^{n+2},$$
$$M_*(n) = a_*(n)^2.$$

For each $n < \omega$ define the norm $\mu_n^* : \mathcal{P}(M_*(n)) \to [0, \infty]$ by $\mu_n^*(x) = \log_{a_*(n)}\left(\frac{M_*(n)}{M_*(n)-|x|}\right)$ ($\mu_n^*(M_*(n)) = \infty$). Set the tree $T^* := \bigcup_{n<\omega} \prod_{k<n} M_*(k)$.

Define $\tilde{\mathbb{E}}$, ordered by $\subseteq$, as the poset whose conditions are subtrees $p \subseteq T^*$ such that, for some $2 \le m < \omega$,

(i) $\operatorname{trk}(p) > 3m$ where $\operatorname{trk}(p)$ is the stem of p,

(ii) $\mu_t^*(p) \ge 1 + \frac{1}{m}$ for any $t \in p$ above $\operatorname{trk}(p)$, where $\mu_t^*(p) := \mu_{|t|}\left(\{i \in M(|t|) : t^\frown i \in p\}\right)$.

Denote $\mathrm{ls}(p) := \frac{1}{m}$ where m is the maximal number satisfying (i) and (ii).

Lemma 3.2 ([KST19, Lemma 1.17]). *The forcing $\tilde{\mathbb{E}}$ satisfies the following properties.*

(a) *(ρ_*, π_*)-linked. In particular, it is σ-linked and $\mathbf{aLc}^*(b_*, \rho_*)$-good, where $b_*(n) := (n+1)\rho_*(n)^{n+1}$.*

(b) *It adds an eventually different real in ω^ω over the ground model. In particular, it adds an $\mathbf{Mg}$-dominating real over the ground model.*

(c) *It is equivalent to some subalgebra of random forcing. In particular, it is $\mathbf{Lc}(\omega, \mathcal{H})$-good whenever $\mathcal{H}$ is a countable infinite set of functions that diverge to infinity.*

Now we present the basic structure of the FS iterations that are used to prove the main results.

Definition 3.3. Let θ be an uncountable regular cardinal. A *θ-iteration candidate* $\mathbf{t}$ is composed of

(I) an ordinal $\delta_\mathbf{t}$ partitioned into two sets $S_\mathbf{t}, R_\mathbf{t}$ (some could be empty);

(II) a FS iteration $\langle \mathbb{P}_{\mathbf{t},\alpha}, \dot{\mathbb{Q}}_{\mathbf{t},\alpha} : \alpha < \delta_\mathbf{t} \rangle$;

(III) whenever $\alpha \in S_\mathbf{t}$, $\dot{\mathbb{Q}}_{\mathbf{t},\alpha}$ is a $\mathbb{P}_{\mathbf{t},\alpha}$-name of a ccc poset of size $< \theta$;

(IV) whenever $\alpha \in R_\mathbf{t}$, there is some $\mathbb{P}'_{\mathbf{t},\alpha} \lessdot \mathbb{P}_{\mathbf{t},\alpha}$ such that $\dot{\mathbb{Q}}_{\mathbf{t},\alpha} = \mathbb{S}^{V^{\mathbb{P}'_{\mathbf{t},\alpha}}}$ where $\mathbb{S}$ is either random forcing or $\tilde{\mathbb{E}}$.

When the context is clear, the subindex $\mathbf{t}$ is omitted.

Note that, whenever $\mathbf{t}$ is a θ-iteration candidate and $\alpha \in S$, since $\mathbb{P}_\alpha$ is ccc, we can find a cardinal $\mu_\alpha = \mu_{\mathbf{t},\alpha} < \theta$ in the ground model such that $\mathbb{P}_\alpha$ forces $|\dot{\mathbb{Q}}_\alpha| \leq \mu_\alpha$, so we can put $\dot{\mathbb{Q}}_\alpha = \{\dot{q}_{\alpha,\zeta} : \zeta < \mu_\alpha\}$.

For the rest of this section, fix an ordinal δ_*, a partition $P := \langle S, R \rangle$ of δ_*, and a regular uncountable cardinal θ. Denote $\Psi_P^\theta := \prod_{\alpha < \delta_*} W_{P,\alpha}^\theta$ where

$$W_{P,\alpha}^\theta := \begin{cases} \theta & \text{if } \alpha \in S, \\ \omega^{<\omega} \times \{\frac{1}{n} : 1 \leq n < \omega\} & \text{if } \alpha \in R. \end{cases}$$

The following result is essential to construct iteration candidates that serve our purposes.

Theorem 3.4 (Engelking and Karłowicz [EK65]). *Assume that $\theta = \theta^{\aleph_0}$ and $\delta_* < (2^\theta)^+$. Then there is some $H^* \subseteq \Psi_P^\theta$ of size θ such that any countable partial function from Ψ_P^θ is extended by some member of H^*.*

From now on, assume that $\theta = \theta^{\aleph_0}$, $\delta_* < (2^\theta)^+$ and that H^* is as in the previous lemma.

Consider random forcing $\mathbb{B}$, ordered by $\subseteq$, whose conditions are trees $T \subseteq 2^{<\omega}$ such that $\mathrm{Leb}([T] \cap [t]) > 0$ for any $t \in T$, where Leb denotes the Lebesgue measure on 2^ω.

For $T \in \mathbb{B}$, denote by $\mathrm{trk}(T)$ the stem of T and set $\mathrm{ls}(T) = \frac{1}{m}$ where m is the maximal natural number such that $\mathrm{Leb}([T]) > (1 - \frac{1}{m})\mathrm{Leb}([\mathrm{trk}(T)])$.

The main point for the main results is to construct an iteration candidate that produces a θ-Fr-Knaster poset, this to ensure the preservation of strongly **D**-unbounded families. In the context of iterations, we need to look at quite uniform Δ-systems of conditions to deal with such property. The corresponding technicalities are presented below.

Definition 3.5. Let $\mathbf{t}$ be a θ-iteration candidate with $\delta_\mathbf{t} \leq \delta_*$, $S_\mathbf{t} \subseteq S$ and $R_\mathbf{t} \subseteq R$.

(1) For $\alpha < \delta$, let $\mathbb{P}^*_\alpha$ be the set of conditions $p \in \mathbb{P}_\alpha$ such that, for any $\xi \in \mathrm{dom}p$, if $\xi \in S$ then $p(\xi) = \dot{q}_{\xi,\zeta}$ for some $\zeta < \mu_\xi$; if $\xi \in R$ then $p(\xi)$ is a $\mathbb{P}'_\alpha$-name (not just a $\mathbb{P}_\alpha$-name) and both $\mathrm{trk}(p(\xi))$ and $\mathrm{ls}(p(\xi))$ are already decided by $p{\restriction}\xi$; and furthermore

$$\sum_{\xi \in R \cap \mathrm{dom}p} \sqrt{\mathrm{ls}(p(\xi))} < \frac{1}{2}.$$

It is shown in [KST19, Lemma 2.34] that $\mathbb{P}^*_\alpha$ is dense in $\mathbb{P}_\alpha$.

(2) Say that $\bar{p} = \langle p_i : i < \gamma \rangle$ is a $\mathbf{t}$-*uniform Δ-system* if it satisfies:

  (i) $p_i \in \mathbb{P}^*_\delta$;

  (ii) there is some $m < \omega$ such that $\mathrm{dom}p_i = \{\alpha_{i,k} : k < m\}$ (increasing enumeration);

  (iii) there is some $v \subseteq m$ such that, for any $k \in v$, the sequence $\langle \alpha_{i,k} : i < \gamma \rangle$ is constant with value α_k;

  (iv) if $k \in m \setminus v$ then the sequence $\langle \alpha_{i,k} : i < \gamma \rangle$ is increasing (hence, $\langle \mathrm{dom}p_i : i < \gamma \rangle$ forms a Δ-system with root $\{\alpha_k : k \in v\}$);

  (v) there is a partition $m = s \cup r \cup e$ such that, for any $i < \gamma$ and $k < m$: $\alpha_{i,k} \in S$ iff $k \in s$; and $k \in r$ iff $\dot{\mathbb{Q}}_{\alpha_{i,k}} = \mathbb{B}^{V^{\mathbb{P}'_{\alpha_{i,k}}}}$;

  (vi) for any $k \in r \cup e$, the sequence $\langle (\mathrm{trk}(p_i(\alpha_{i,k})), \mathrm{ls}(p_i(\alpha_{i,k}))) : i < \gamma \rangle$ has constant value (t_k, ε_k);

  (vii) for any $k \in s \cap v$ there is some ζ_k such that $p_i(\alpha_k) = \dot{q}_{\alpha_k,\zeta_k}$ for any $i < \gamma$.

It is not hard to see that a $\mathbf{t}$-uniform Δ-system $\bar{p}$ determines a partial function $h_{\bar{p}}$ in Ψ^θ_P with domain $\bigcup_{i<\gamma} \mathrm{dom}p_i$ such that $h(\alpha_{i,k}) = (t_k, \varepsilon_k)$ when $k \in r \cup e$, and $p_i(\alpha_{i,k}) = \dot{q}_{\alpha_{i,k},\zeta_{h(\alpha_{i,k})}}$ when $k \in s$. Given $h \in H^*$, say that $\bar{p}$ *follows* h if $h_{\bar{p}} \subseteq h$.

In the following result we consider the existence of a class Λ^θ_P of *good* iteration candidates, whose elements are iterations that match our goals. This class is not formally defined in [KST19] but instead it is clarified how such good iterations are constructed using FAM limits. In the result below, corresponding to the main technical lemma in [KST19], we try our best to summarize the important features of the good iterations without referring to the central notions of FAM limits at all.

Theorem 3.6 ([KST19, Lemma 2.39]). *There is a class Λ^θ_P of θ-iteration candidates such that*

(a) the trivial candidate $\mathbf{t}_0$ with $\delta_{\mathbf{t}_0} = 0$ is in Λ^θ_P;

(b) if $\mathbf{t} \in \Lambda_P^\theta$ then $\delta_{\mathbf{t}} \leq \delta_*$, $S_{\mathbf{t}} \subseteq S$ and $R_{\mathbf{t}} \subseteq R$;

(c) if $\mathbf{t} \in \Lambda_P^\theta$, $\delta = \delta_{\mathbf{t}} \in S$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\delta$-name of a ccc poset of size $< \theta$, then $\mathbf{t}^+ \in \Lambda_P^\theta$ where $\mathbf{t}^+$ is the natural θ-iteration candidate that extends $\mathbf{t}$ such that $\delta_{\mathbf{t}^+} = \delta + 1$ and $\dot{\mathbb{Q}}_{\mathbf{t}^+,\delta} = \dot{\mathbb{Q}}$;

(d) if $\mathbf{t} \in \Lambda_P^\theta$, $\delta = \delta_{\mathbf{t}} \in R$, $A \subseteq \mathbb{P}_\delta$ and $\mathbb{S}$ is either $\mathbb{B}$ or $\tilde{\mathbb{E}}$, then there is some $\mathbf{t}^+ \in \Lambda_P^\theta$ extending $\mathbf{t}$ such that $A \subseteq \mathbb{P}'_{\mathbf{t}^+,\delta}$, $|\mathbb{P}'_{\mathbf{t}^+,\delta}| \leq \max\{|A|, \theta\}^{\aleph_0}$, and $\dot{\mathbb{Q}}_{\mathbf{t}^+,\delta} = \mathbb{S}^{V^{\mathbb{P}'_{\mathbf{t}^+,\delta}}}$;

(e) in addition to the above, if $\langle \mathbf{t}_\eta^+ : \eta < \omega_1 \rangle$ is a sequence form Λ_P^θ such that $\delta_{\mathbf{t}_\eta^+} = \delta + 1$, $\mathbf{t}_\eta$ extends $\mathbf{t}$ (using the same $\mathbb{S}$ at δ) and $\mathbb{P}'_{\mathbf{t}_\xi^+,\delta} \subseteq \mathbb{P}'_{\mathbf{t}_\eta^+,\delta}$ whenever $\xi < \eta < \omega_1$, then $\mathbf{t}^+ \in \Lambda_P^\theta$ where $\mathbf{t}^+$ is the iteration candidate extending $\mathbf{t}$ such that $\delta_{\mathbf{t}^+} = \delta + 1$, $\mathbb{P}'_{\mathbf{t}^+,\delta} = \bigcup_{\eta < \omega_1} \mathbb{P}'_{\mathbf{t}_\eta^+,\delta}$ and $\dot{\mathbb{Q}}_{\mathbf{t}^+,\delta}$ is defined using the same $\mathbb{S}$.

(f) if $\delta \leq \delta_*$ is limit and $\langle \mathbf{t}_\alpha : \alpha < \delta \rangle$ is a sequence of iteration candidates in Λ_P^θ such that $\delta_{\mathbf{t}_\alpha} = \alpha$ and $\mathbf{t}_\beta$ extends $\mathbf{t}_\alpha$ whenever $\alpha < \beta < \delta$, then the natural direct limit of $\langle \mathbf{t}_\alpha : \alpha < \delta \rangle$ belongs to Λ_P^θ;

(g) if $\mathbf{t} \in \Lambda_P^\theta$ and $\bar{p} = \langle p_n : n < \omega \rangle$ is a $\mathbf{t}$-uniform Δ-system, then there is some $q \in \mathbb{P}_{\delta_{\mathbf{t}}}$ that forces $|\{n < \omega : p_n \in \dot{G}\}| = \aleph_0$.

As a consequence, good iteration candidates are θ-Fr-Knaster, as desired.

Corollary 3.7. *If $\mathbf{t} \in \Lambda_P^\theta$ then $\mathbb{P}_\delta$ is θ-Fr-Knaster.*

Proof. Assume that $\{p_i : i < \theta\} \subseteq \mathbb{P}_\delta^*$. It is possible to find an increasing function $g : \theta \to \theta$ such that $\{p_{g(i)} : i < \theta\}$ forms a $\mathbf{t}$-uniform Δ-system. Hence, by Theorem 3.6(g), this family is Fr-linked. $\qquad\square$

4 The main results

Now we are ready to prove Theorem A. Like in [KST19], we require the additional hypothesis $\lambda_5 \leq 2^{\lambda_2}$ to make sense of Theorem 3.6 (because it relies on Theorem 3.4), but afterwards it is shown how to get rid of this requirement.

The case $\lambda_2 = \lambda_3$ of Theorem A can be solved by very standard methods from [Bre91] without good iteration candidates. The argument below can be imitated to prove this simpler case, just ignore (3) and (3') and, in (4) and (4'), use small models as in (1),(1') and (2),(2').

Theorem 4.1. *Assume that $\lambda_1 \leq \lambda_2 = \lambda_2^{\aleph_0} < \lambda_3 \leq \lambda_4$ are uncountable regular cardinals, let λ_5 be a cardinal such that $\lambda_5 = \lambda_5^{<\lambda_4} \leq 2^{\lambda_2}$ and either $\lambda_4 = \lambda_5$ or $\lambda_4^{\aleph_0} < \lambda_5$, and assume that λ_3 is $\aleph_1$-inaccessible. Then there is a FS iteration of ccc posets that adds*

(a) *a strongly θ-$\mathbf{R}_i$-unbounded family of size θ for any regular $\lambda_i \leq \theta \leq \lambda_5$ and $i < 4$, where $\mathbf{R}_0 = \mathbf{Id}$, $\lambda_0 = 2$, $\mathbf{R}_1 = \mathbf{Lc}(\omega, \mathcal{H}_*)$, $\mathbf{R}_2 = \mathbf{aLc}^*(b_*, \rho_*)$, and $\mathbf{R}_3 = \mathbf{D}$;*

(b) *a strongly θ-$\mathbf{R}_i$-unbounded family of size θ for any regular $\theta \in \{\lambda_4\} \cup (\lambda_4^{\aleph_0}, \lambda_5]$, where $\mathbf{R}_4 = \mathbf{Mg}$;*

(c) a λ_i-$\mathbf{R}_i$-dominating family of size λ_5 for any $i < 5$.

In particular, this poset forces $\mathrm{add}(\mathcal{N}) = \lambda_1$, $\mathfrak{b} = \lambda_2$, $\mathrm{cov}(\mathcal{N}) = \lambda_3$, $\mathrm{non}(\mathcal{M}) = \lambda_4$, *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda_5$.

Proof. Let W be the model obtained after adding λ_5-many Cohen reals. Fix $\delta_* = \lambda_5\lambda_4$ (ordinal product) and a partition $\langle S_k : 1 \leq k \leq 4\rangle$ of λ_5. Put $S := \{\lambda\beta + \rho : \beta < \lambda_4, \ \rho \in S_1 \cup S_2\}$, $R := \{\lambda\beta + \rho : \beta < \lambda_4, \ \rho \in S_3 \cup S_4\}$ and $P := \langle S, R\rangle$. Using Theorem 3.6, we construct an iteration candidate $\mathbf{t} \in \Lambda_P^{\lambda_2}$ of length δ_* such that, for $\xi = \lambda_5\beta + \rho$ with $\beta < \lambda_4$ and $\rho < \lambda_5$:

(1) if $\rho \in S_1$ then $\dot{\mathbb{Q}}_\xi = \mathbb{LOC}^{N_\xi}$ (localization forcing) where N_ξ, in $W_{\lambda_5\beta} = W^{\mathbb{P}_{\lambda_5\beta}}$, is a transitive model of (a large fragment of) ZFC of size $<\lambda_1$;

(2) if $\rho \in S_2$ then $\dot{\mathbb{Q}}_\xi = \mathbb{D}^{N_\xi}$ (Hechler forcing) where N_ξ, in $W_{\lambda_5\beta}$, is a transitive model of ZFC of size $<\lambda_2$;

(3) if $\rho \in S_3$ then $\mathbb{P}'_\xi \lessdot \mathbb{P}_\xi$ has size $<\lambda_3$ and $\dot{\mathbb{Q}}_\xi = \mathbb{B}^{W^{\mathbb{P}'_\xi}}$; and

(4) if $\rho \in S_4$ then $\mathbb{P}'_\xi \lessdot \mathbb{P}_\xi$ has size $\leq\lambda_4^{\aleph_0}$ and $\dot{\mathbb{Q}}_\xi = \tilde{\mathbb{E}}^{W^{\mathbb{P}'_\xi}}$.

Furthermore, the models N_ξ and the subposets $\mathbb{P}'_\xi$ are constructed such that, for any $\beta < \lambda_4$:

(1') if $F \in W_{\lambda_5\beta}$ is a subset of ω^ω of size $<\lambda_1$, then there is some N_ξ with $\xi = \lambda_5\beta + \rho$ (as above) such that $\rho \in S_1$ and $F \subseteq N_\xi$;

(2') if $F \in W_{\lambda_5\beta}$ is a subset of ω^ω of size $<\lambda_2$, then there is some N_ξ with $\xi = \lambda_5\beta + \rho$ such that $\rho \in S_2$ and $F \subseteq N_\xi$;

(3') if $\mathcal{A} \in W_{\lambda_5\beta}$ is a family of Borel null sets of size $<\lambda_3$, then there is some $\xi = \lambda_5\beta + \rho$ such that $\rho \in S_3$ and all the members of $\mathcal{A}$ are coded in $W^{\mathbb{P}'_\xi}$;

(4') if $F \in W_{\lambda_5\beta}$ is a subset of ω^ω of size $<\lambda_4$, then there is some $\xi = \lambda_5\beta + \rho$ such that $\rho \in S_4$ and $F \subseteq W^{\mathbb{P}'_\xi}$.

Using standard counting arguments and Theorem 3.6, we can construct such a $\mathbf{t}$. We prove that $\mathbb{P} := \mathbb{C}_{\lambda_5} * \mathbb{P}_{\mathbf{t},\delta_*}$ is as required. For (a), we show that the first θ-many Cohen reals form a strongly θ-$\mathbf{R}_i$-unbounded family, in the $\mathbb{P}$-extension, for any regular $\lambda_i \leq \theta \leq \lambda_5$. The case $i = 0$ is quite obvious; when $i = 1, 3$, $\mathbb{P}$ is λ_i-$\mathbf{R}_i$-good (see Lemmas 2.7 and 3.2, and Example 2.10), so the desired family is strongly unbounded by Theorem 2.9; when $i = 2$, see Theorems 3.6 and 2.12, and Corollary 3.7.

For (b), since the iteration has cofinality λ_4, it adds a cofinal sequence of Cohen reals of length λ_4, which is a strongly λ_4-$\mathbf{R}_4$-unbounded family. Now, if $\lambda_4^{\aleph_0} < \lambda_5$, then all the iterands are posets of size $\leq \lambda_4^{\aleph_0}$, hence $\mathbb{P}$ is $(\lambda_4^{\aleph_0})^+$-$\mathbf{R}_4$-good (see Lemma 2.7). Thus, for any regular $\lambda_4^{\aleph_0} < \theta \leq \lambda_5$, the first θ-many Cohen reals form a strongly θ-$\mathbf{R}_4$-unbounded family by Theorem 2.9.

We finish with (c). The case $i = 0$ is again trivial (witnessed by the family of all the reals in the final extension). The other cases are similar, we show the case $i = 2$ as an example. By (2) and (2'), it is easy to see that $\{\dot{d}_{\beta,\rho} : \beta < \lambda_4, \ \rho \in S_2\}$ forms a λ_2-$\mathbf{R}_2$-dominating family where each $\dot{d}_{\beta,\rho}$ is the $\mathbf{D}$-dominating real over N_ξ (with $\xi = \lambda_5\beta + \rho$) added by $\dot{\mathbb{Q}}_\xi$. $\square$

Theorem 4.2. *If $\lambda_2^{<\lambda_2} = \lambda_2$ then the assumption $\lambda_5 \leq 2^{\lambda_2}$ in the previous theorem can be omitted.*

Proof. Assume $\lambda_2^{<\lambda_2} = \lambda_2$ and $2^{\lambda_2} < \lambda_5$. Let $\mathbb{R} := \mathrm{Fn}_{<\lambda_2}(\lambda_5 \times \lambda_2, 2)$. Note that $\mathbb{R}$ is $<\lambda_2$-closed and has λ_2-cc, so it preserves cofinalities and forces that the hypothesis of the previous theorem holds.

Denote the ground model by V and $V' := V^{\mathbb{R}}$. It is clear that any ccc poset in V is also ccc in V'. On the other hand, by Easton's Lemma, if $\mathbb{Q}$ is a ccc poset in V then $\mathbb{R}$ is $<\lambda_2$-distributive in $V^{\mathbb{Q}}$, hence $\omega^\omega \cap V^{\mathbb{R} \times \mathbb{Q}} = \omega^\omega \cap V^{\mathbb{Q}}$.

In V, define δ_*, $\langle S_k : 1 \leq k \leq 4 \rangle$ and $P = \langle S, R \rangle$ as in the previous proof. In $W := V^{\mathbb{C}_{\lambda_5}}$, by recursion, we define an iteration candidate $\mathbf{t}$ of length δ_* such that (1)–(4) and (1')–(4') of the previous proof hold, and such that $\Vdash_{\mathbb{R}^V}^W \mathbf{t} \in \Lambda_P^{\lambda_2}$. Though we construct $\mathbf{t}$ in W, Theorem 3.6 is used in $W' := V^{\mathbb{R} \times \mathbb{C}_{\lambda_5}}$ to ensure that $\mathbf{t} \in \Lambda_P^{\lambda_2}$. Let $\xi = \lambda_5 \beta + \rho$ where $\beta < \lambda_4$ and $\rho < \lambda_5$, and assume that $\mathbf{t}$ has been constructed up to ξ. Limit steps are easy since it is enough to take the direct limit; when $\rho \in S$, we can define $\dot{\mathbb{Q}}_\xi$ arbitrarily as in (1) and (2) of the previous proof; the interesting case is when $\rho \in R$.

We show the case $\rho \in S_3$ (the case $\rho \in S_4$ is similar). Start with $\mathbb{P}_\xi^0 \lessdot \mathbb{P}_\xi$ of size $<\lambda_3$ containing a small family of codes as in (3') of the previous proof. By applying Theorem 3.6(d) in W', there is some $\mathbb{P}_\xi^1 \lessdot \mathbb{P}_\xi$ of size $<\lambda_3$ such that $\mathbb{P}_\xi^0 \subseteq \mathbb{P}_\xi^1$ and that further iterating with $\mathbb{B}^{W'^{\mathbb{P}_\xi^1}}$ yields an iteration candidate in $\Lambda_P^{\lambda_2}$. Recall that the reals in $W'_\xi := W'^{\mathbb{P}_\xi}$ are the same as those in $W_\xi := W^{\mathbb{P}_\xi}$. Therefore, in W, since $\mathbb{R}^V$ has λ_3-cc, we can find $\mathbb{P}_\xi^2 \lessdot \mathbb{P}_\xi$ of size $<\lambda_3$ such that, in W', $\mathbb{P}_\xi^1 \subseteq \mathbb{P}_\xi^2$. Repeating this argument (and taking unions at limit stages), construct a $\subseteq$-increasing sequence $\langle \mathbb{P}_\xi^\eta : \eta < \omega_1 \rangle$ of complete subposets of $\mathbb{P}_\xi$ of size $<\lambda_3$ such that $\langle \mathbb{P}_\xi^{2\eta} : \eta < \omega_1 \rangle \in W$ and, in W', when the iteration so far is extended one step by $\mathbb{B}^{W'^{\mathbb{P}_\xi^{2\eta+1}}}$, it is in $\Lambda_P^{\lambda_2}$. Put $\mathbb{P}_\xi' := \bigcup_{\eta < \omega_1} \mathbb{P}_\xi^{2\eta}$. Thus, $\mathbb{P}_\xi' \lessdot \mathbb{P}_\xi$ belongs to W, it has size $<\lambda_3$, and by Theorem 3.6(e) the iteration extended by $\mathbb{B}^{W'^{\mathbb{P}_\xi'}} = \mathbb{B}^{W^{\mathbb{P}_\xi'}}$ (though in W) belongs to $\Lambda_P^{\lambda_2}$ in the model W'. This finishes the construction.

The same argument to prove (a), (b) and (c) in the previous theorem works, with the exception of the strongly θ-**D**-unbounded families for any regular $\theta \in [\lambda_2, \lambda_5]$. In W', like in the previous proof, we have that the first θ-many Cohen reals $\{c_\alpha : \alpha < \theta\}$ (which are in W) form a strongly θ-**D**-unbounded family in W_{δ_*}'. It remains to show that the same holds in W_{δ_*}. Let $y \in \omega^\omega \cap W_{\delta_*}$. In W_{δ_*}', we have that $|\{\alpha < \theta : c_\alpha \leq^* y\}| < \theta$, and since this set is in W_{δ_*}, the same holds in this model. $\qquad\square$

5 Further remarks on Boolean ultrapowers

The following result summarizes the properties of Boolean ultrapowers used in [KTT18, GKS, KST19] to separate cardinal characteristics, with the exception of property (g). The latter is an observation of the author.

Lemma 5.1. *Assume that κ is a strongly compact cardinal and that θ is a cardinal such that $\theta^\kappa = \theta$. Then there is a complete embedding $j : V \to M$ such that:*

(a) M is transitive and $M^{<\kappa} \subseteq M$.

(b) $\kappa = \min\{\alpha \in \mathbf{On} : j(\alpha) > \alpha\}$ *(the critical point of j).*

(c) *For any cardinal $\lambda \geq \kappa$ such that either $\lambda \leq \theta$ or $\lambda^\kappa = \lambda$, we have $\max\{\lambda, \theta\} \leq j(\lambda) < \max\{\lambda, \theta\}^+$.*

(d) *If a is a set of size $< \kappa$ then $j(a) = j[a]$.*

(e) *If $\lambda > \kappa$ and I is a $<\lambda$-directed partial order, then $j[I]$ is cofinal in $j(I)$.*

(f) *If $\mathrm{cf}(\alpha) \neq \kappa$ then $\mathrm{cf}(j(\alpha)) = \mathrm{cf}(\alpha)$.*

(g) *If $\mathbf{A} = \langle X, Y, \subseteq \rangle$ is a relational system and $D \subseteq Y$ is λ-$\mathbf{A}$-dominating, then*

    (i) *if $\lambda < \kappa$ then $j(D)$ is λ-$j(\mathbf{A})$-dominating;*

    (ii) *if $\kappa < \lambda$ then $j[D]$ is λ-$j(\mathbf{A})$-dominating.*

Proof. Let $\mathbb{A}$ be the Boolean completion of the poset $\mathrm{Fn}_{<\kappa}(\theta, \kappa)$. As in [KTT18], there is an ultrafilter on $\mathbb{A}$ such that its Boolean ultrapower yields a complete embedding $j : V \to M$ satisfying (a)–(f). We show (g) (though technicalities are referred to [KTT18]). Let $E \subseteq j(X)$ be a set of size $<\lambda$. If $\lambda < \kappa$ then $E \in M$ by (a), and since $M \models$ "$j(D)$ is λ-$j(\mathbf{A})$-dominating", we can find a $z \in j(Y)$ such that $w \, j(\sqsubseteq) \, z$ for any $w \in E$. This shows that $j(D)$ is λ-$j(\mathbf{A})$-dominating.

Now assume $\kappa < \lambda$. Any $w \in E$ is a mixture of κ-many possibilities $\langle x_\alpha^w : \alpha < \kappa \rangle \in X^\kappa$. Now, since $\{x_\alpha^w : w \in E, \ \alpha < \kappa\}$ has size $< \lambda$, there is some $y \in D$ $\mathbf{A}$-dominating over this set. This implies that $j(y)$ is $j(\mathbf{A})$-dominating over E, which shows that $j[D]$ is λ-$j(\mathbf{A})$-dominating. $\qquad\square$

This method of Boolean ultrapowers works to preserve and modify strong unbounded families and also certain type of dominating families. Though in the original framework from [KTT18, GKS, KST19] they preserve a very strong type of dominating family (the property COB), we show in item (c) that a weaker type of dominating families is preserved.

Lemma 5.2. *Assume that $j : V \to M$ is a complete embedding satisfying (a), (b), (d)–(g) of the previous lemma. Let $\mathbb{P}$ be a ν-cc poset with $\nu < \kappa$ uncountable regular, and let $\mathbf{A} = \langle X, Y, \sqsubseteq \rangle$ be a relational system where X, Y are analytic (or co-analytic) subsets of some Polish space, and $\sqsubseteq \subseteq X \times Y$ is also analytic (or co-analytic). Then*

(a) [KTT18, GKS] *$j(\mathbb{P})$ is ν-cc.*

(b) [KTT18, GKS] *If μ is regular and $\mathbb{P}$ adds a strongly μ-$\mathbf{A}$-unbounded family of size μ, then so does $j(\mathbb{P})$.*

(c) *If $\mathbb{P}$ adds a λ-$\mathbf{A}$-dominating family of size $\leq \chi$ then*

    (a) *if $\lambda < \kappa$ then $j(\mathbb{P})$ adds a λ-$\mathbf{A}$-dominating family of size $\leq |j(\chi)|$;*

    (b) *if $\kappa < \lambda$ then $j(\mathbb{P})$ adds a λ-$\mathbf{A}$-dominating family of size $\leq |\chi|$.*

Proof. Property (a) is immediate from Lemma 5.1(a). For (b), see e.g. [GKMS]. We show (c). Let X^* be the set of nice $\mathbb{P}$-names of members of X, and set Y^* likewise. Note that $j(X^*)$ coincides with the set of nice $j(\mathbb{P})$-names of members of X, likewise for $j(Y^*)$ (by Lemma 5.1(a)). On the other hand, $j[X^*]$ is equal to the set of nice $j(\mathbb{P})$-names of members of $j[X]$, likewise for $j[Y^*]$. For $\dot{x} \in X^*$ and $\dot{y} \in Y^*$, define $\dot{x} \sqsubset^* \dot{y}$ iff $\Vdash_{\mathbb{P}} \dot{x} \sqsubset \dot{y}$, and set $\mathbf{A}^* := \langle X^*, Y^*, \sqsubset \rangle$. Note that, for $\dot{u} \in j(X^*)$ and $\dot{v} \in j(Y^*)$, $\dot{u} \, j(\sqsubset^*) \, \dot{v}$ iff $\Vdash_{j(\mathbb{P})} \dot{u} \sqsubset \dot{v}$.

Assume that $\dot{D} = \{\dot{y}_\eta : \eta < \chi\}$ is forced by $\mathbb{P}$ to be a λ-$\mathbf{A}$-dominating family. It is easy to see that $\dot{D}$ is λ-$\mathbf{A}^*$-dominating. Hence, Lemma 5.1(g) applies: if $\lambda < \kappa$ then $j(\dot{D})$ is λ-$j(\mathbf{A}^*)$-dominating, which means that $j(\mathbb{P})$ forces that $j(\dot{D})$ is λ-$\mathbf{A}$-dominating (of size $\leq |j(\chi)|$); if $\kappa < \lambda$ then $\{j(\dot{y}_\eta) : \eta < \chi\}$ is λ-$j(\mathbf{A}^*)$-dominating, which means that $j(\mathbb{P})$ forces that $\{j(\dot{y}_\eta) : \eta < \chi\}$ is λ-$\mathbf{A}$-dominating (of size $\leq |\chi|$). □

Theorem 5.3. *If* $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 \leq \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ *such that*

(i) *for* $j = 6, 7, 8, 9$, κ_j *is strongly compact and* $\lambda_j^{\kappa_j} = \lambda_j$,

(ii) λ_i *is regular for* $i \neq 5, 9$,

(iii) $\lambda_2^{<\lambda_2} = \lambda_2$, $\lambda_4^{\aleph_0} = \lambda_4$, $\lambda_5^{<\lambda_4} = \lambda_5$, *and*

(iv) λ_3 *is* $\aleph_1$*-inaccessible,*

then there is a ccc poset that forces

$$\mathrm{add}(\mathcal{N}) = \lambda_1, \ \mathfrak{b} = \lambda_2, \ \mathrm{cov}(\mathcal{N}) = \lambda_3, \ \mathrm{non}(\mathcal{M}) = \lambda_4,$$
$$\mathrm{cov}(\mathcal{M}) = \lambda_5, \ \mathrm{non}(\mathcal{N}) = \lambda_6, \ \mathfrak{d} = \lambda_7, \ \mathrm{cof}(\mathcal{N}) = \lambda_8, \ and \ \mathfrak{c} = \lambda_9.$$

Proof. Use the ccc poset constructed in Theorem 4.2 and apply Boolean ultrapowers accordingly as in [KST19]. We omit all the details, but we remark that, instead of property COB, we can just preserve or modify dominating families thanks to Lemma 5.2(c). □

Remark 5.4. In [KST19], also for the corresponding results in [KTT18, GKS, BCM], the authors put quite an effort to construct (strong) dominating families satisfying COB through the iteration. But thanks to Lemma 5.2(c), we can work with a weaker and more natural type of dominating family, so the construction becomes as simple as in Theorem 4.1.

Acknowledgements

References

[BCM]   Jörg Brendle, Miguel A. Cardona, and Diego A. Mejía. Filter-linkedness and its effect on the preservation of cardinal characteristics. `arXiv:1809.05004`.

[BJ95]   Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line*. A K Peters, Wellesley, Massachusetts, 1995.

[BM14]   Jörg Brendle and Diego Alejandro Mejía. Rothberger gaps in fragmented ideals. *Fund. Math.*, 227(1):35–68, 2014.

[Bre91]   Jörg Brendle. Larger cardinals in Cichoń's diagram. *J. Symbolic Logic*, 56(3):795–810, 1991.

[CM19]   Miguel A. Cardona and Diego A. Mejía. On cardinal characteristics of Yorioka ideals. *Math. Log. Q.*, 2019. In press, `arXiv:1703.08634`.

[EK65]   Ryszard Engelking and Monika Karłowicz. Some theorems of set theory and their topological consequences. *Fund. Math.*, 57:275–285, 1965.

[FFMM18]   Vera Fischer, Sy D. Friedman, Diego A. Mejía, and Diana C. Montoya. Coherent systems of finite support iterations. *J. Symbolic Logic*, 83(1):208–236, 2018.

[GKMS]   Martin Goldstern, Jakob Kellner, Diego Mejía, and Saharon Shelah. Controlling cardinal characteristics without adding reals. Preprint.

[GKS]   Martin Goldstern, Jakob Kellner, and Saharon Shelah. Cichoń's maximum. `arXiv:1708.03691`.

[GMS16]   Martin Goldstern, Diego Alejandro Mejía, and Saharon Shelah. The left side of Cichoń's diagram. *Proc. Amer. Math. Soc.*, 144(9):4025–4042, 2016.

[JS90]   Haim Judah and Saharon Shelah. The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing). *J. Symbolic Logic*, 55(3):909–927, 1990.

[Kam89]   Anastasis Kamburelis. Iterations of Boolean algebras with measure. *Arch. Math. Logic*, 29(1):21–28, 1989.

[KM18]   Lukas Daniel Klausner and Diego Alejandro Mejía. Many Different Uniformity Numbers of Yorioka Ideals. 2018. `arXiv:1805.11005`.

[KO14]   Shizuo Kamo and Noboru Osuga. Many different covering numbers of Yorioka's ideals. *Arch. Math. Logic*, 53(1-2):43–56, 2014.

[KST19]   Jakob Kellner, Saharon Shelah, and Anda Tănasie. Another ordering of the ten cardinal characteristics in Cichoń's diagram. 2019. To appear, `arXiv:1712.00778`.

[KTT18]   Jakob Kellner, Anda Ramona Tănasie, and Fabio Elio Tonti. Compact cardinals and eight values in Cichoń's diagram. *J. Symb. Log.*, 83(2):790–803, 2018.

[Mej19]   Diego A. Mejía. Matrix iterations with vertical support restrictions. In Byunghan Kim, Jörg Brendle, Gyesik Lee, Fenrong Liu, R Ramanujam, Shashi M Srivastava, Akito Tsuboi, and Liang Yu, editors, *Proceedings of the 14th and 15th Asian Logic Conferences*, pages 213–248. World Sci. Publ., 2019.

[She00]   Saharon Shelah. Covering of the null ideal may have countable cofinality. *Fund. Math.*, 166(1-2):109–136, 2000. Saharon Shelah's anniversary issue.

Faculty of Science
Shizuoka University
836 Ohya, Suruga-ku, Shizuoka 422-8529
JAPAN
E–mail address: `diego.mejia@shizuoka.ac.jp`