YORIOKA'S CHARACTERIZATION OF THE COFINALITY OF THE STRONG MEASURE ZERO IDEAL AND ITS INDEPENDENCY FROM OF CONTINUUM

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ABSTRACT. In this paper we present a simpler proof of that no inequality between $\mathrm{cof}(\mathcal{SN})$ and $\mathfrak{c}$ can be decided in ZFC using tecniques and results well known.

1. INTRODUCTION

Borel [Bor19] introduced the new class of Lebesgue measure zero subsets of the real line called *strong measure zero* sets, which we denote by $\mathcal{SN}$. The cardinal invariants associated with strong measure zero have been investigated. To summarize some of the results:

Theorem A. The following holds in ZFC

(i) (Carlson [Car93]) $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{SN})$,
(ii) $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{SN}) \leq \mathfrak{c}$,
(iii) (Miller [Mil81]) $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{SN}) \leq \mathrm{cov}(\mathcal{N})$ and $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{SN})\}$,
(iv) (Osuga [Osu08]) $\mathrm{cof}(\mathcal{SN}) \leq 2^{\mathfrak{d}}$.

Moreover, each of the following staments is consistent with ZFC

(v) (Goldstern, Judah and Shelah [GJS93]) $\mathrm{cof}(\mathcal{M}) < \mathrm{add}(\mathcal{SN})$,
(vi) (Pawlikowski [Paw90]) $\mathrm{cov}(\mathcal{SN}) < \mathrm{add}(\mathcal{M})$,
(vii) (Yorioka [Yor02]) $\mathfrak{c} < \mathrm{cof}(\mathcal{SN})$ (from CH),
(viii) (Yorioka [Yor02]) $\mathrm{cof}(\mathcal{SN}) < \mathfrak{c}$,
(ix) (Laver [Lav76]) $\mathrm{cof}(\mathcal{SN}) = \mathfrak{c}$.

To prove (vii) and (viii) Yorioka give a characterization of $\mathcal{SN}$, to do this he introduced the σ-ideals $\mathcal{I}_f$ parametrized by increasing functions $f \in \omega^\omega$, which we call *Yorioka ideals* (see Definition 2.1). These ideals are subideals of the null ideal $\mathcal{N}$ and they include $\mathcal{SN}$ and $\mathcal{SN} = \bigcap\{\mathcal{I}_f : f \in \omega^\omega \text{ increasing}\}$. Even more, he proved that $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa$ (see Definition 2.2) whenever $\mathrm{add}(\mathcal{I}_f) = \mathrm{cof}(\mathcal{I}_f) = \kappa$ for all increasing f. But Yorioka's original proof assumes $\mathrm{add}(\mathcal{I}_f) = \mathrm{cof}(\mathcal{I}_f) = \mathfrak{d} = \mathrm{cov}(\mathcal{M}) = \kappa$ for all increasing f, but $\mathfrak{d}$ and $\mathrm{cov}(\mathcal{M})$ can be omitted since $\mathrm{add}(\mathcal{N}) \leq \mathrm{minadd} \leq \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}) \leq \mathrm{supcof} \leq \mathrm{cof}(\mathcal{N})$ (see [Osu08, CM19]).

In this work, we provide a simpler proof of the result.

Main Theorem (Yorioka [Yor02])**.** Let κ, ν be an infinite cardinals such that $\aleph_1 \leq \kappa = \kappa^{<\kappa} < \nu = \nu^\kappa$ and assume that λ is a cardinal such that $\kappa \leq \lambda = \lambda^{\aleph_0}$. Then there is some poset $\mathbb{Q}$ such that $\Vdash_{\mathbb{Q}} \mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \kappa$, $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa = \nu$ and $\mathfrak{c} = \lambda$.

This result give the consisteny that values value $\mathrm{cof}(\mathcal{SN})$ may be less than $\mathfrak{c}$.

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2. Proof the main theorem

We first start with basic definitions and facts:

Let κ be an infinite cardinal. Let $f, g \in \kappa^\kappa$. Set $f \leq^* g$ if $\exists\alpha < \kappa\forall\beta > \alpha(f(\beta) \leq g(\beta))$. Denote $\mathrm{pow}_k : \omega \to \omega$ the function defined by $\mathrm{pow}_k(i) := i^k$, and define the relation $\ll$ on ω^ω as follows: $f \ll g$ iff $\forall k < \omega(f \circ \mathrm{pow}_k \leq^* g)$.

Definition 2.1. For $\sigma \in (2^{<\omega})^\omega$ define

$$[\sigma]_\infty := \{x \in 2^\omega : \exists^\infty n < \omega(\sigma(n) \subseteq x)\} = \bigcap_{n<\omega} \bigcup_{m\geqslant n} [\sigma(m)]$$

and $\mathrm{ht}_\sigma \in \omega^\omega$ by $\mathrm{ht}_\sigma(i) := |\sigma(i)|$ for each $i < \omega$. Let $f \in \omega^\omega$ be a increasing function, set

$$\mathcal{I}_f := \{X \subseteq 2^\omega : \exists\sigma \in (2^{<\omega})^\omega(X \subseteq [\sigma]_\infty \text{ and } h_\sigma \gg f)\}.$$

Any family of the form $\mathcal{I}_f$ if f increasing is called a *Yorioka ideal*, since Yorioka [Yor02] has proved that $\mathcal{I}_f$ is a σ-ideal in this case, and $\mathcal{SN} = \bigcap\{\mathcal{I}_f : f \text{ increasing}\}$. Denote

$$\mathrm{minadd} = \min\{\mathrm{add}(\mathcal{I}_f) : f \text{ increasing}\}, \ \ \mathrm{supcof} = \sup\{\mathrm{cof}(\mathcal{I}_f) : f \text{ increasing}\}$$

Definition 2.2. Let κ be a regular cardinals. Define the cardinal numbers $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ as follows:

$$\mathfrak{b}_\kappa = \min\{|F| : F \subseteq \kappa^\kappa \ \&\ \forall g \in \kappa^\kappa\exists f \in F(f \not\leq^* g)\} \textit{ the (un)bounding number for } \kappa^\kappa$$

and

$$\mathfrak{d}_\kappa = \min\{|D| : D \subseteq \kappa^\kappa \ \&\ \forall g \in \kappa^\kappa\exists f \in D(g \leq^* f)\} \textit{ the dominating number for } \kappa^\kappa$$

In particular, when $\kappa = \omega$, $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ are $\mathfrak{b}$ and $\mathfrak{d}$ respectively, well known as the *(un)bounding number* and *the dominating number*.

Set $\mathrm{Fn}_{<\kappa}(I, J) := \{p \subseteq I \times J : |p| < \kappa \text{ and } p \text{ function}\}$ for sets I, J and an infinite cardinal κ.

Lemma 2.3. *Let ν, κ be uncountable cardinals such that $\kappa^{<\kappa} = \kappa$ and $\nu > \kappa$. Then* $\mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa) \Vdash \mathfrak{d}_\kappa \geq \nu$.

Proof. Let $\vartheta < \nu$ and let $\{\dot{x}_\alpha : \alpha < \vartheta\}$ be a set of $\mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$-names of functions in κ^κ. Since $\mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$ is $(\kappa^{<\kappa})^+ = \kappa^+$-cc we can find a subset S of ν of size $< \nu$ such that $\dot{x}_\alpha$ is a $\mathrm{Fn}(S \times \kappa, \kappa)$-name for each $\alpha < \vartheta$.

Claim 2.4. $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$ *adds an unbounded function in κ^κ over the ground model.*

Proof. Let G be a $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$-generic set over V. Let $c := c_G = \bigcup G \in \kappa^\kappa$ be the real generic added by $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$. Assume that $f \in \kappa^\kappa \cap V$. We will prove that $f \not\leq^* c$. To see this, for $\alpha < \kappa$, define the sets $D_\alpha := \{p \in \mathrm{Fn}_{<\kappa}(\kappa, \kappa) : \exists\beta > \alpha(p(\beta) > f(\beta))\}$ which are dense, so G intersects all of these yielding $\forall\alpha < \kappa\exists\beta < \alpha(c(\beta) > f(\beta))$. □

By Claim 2.4, $\mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$ forces that the κ-Cohen real at some $\xi \in \nu \smallsetminus S$ is not dominated by any $\dot{x}_\alpha$. □

As mentioned in the introduction that $\mathrm{add}(\mathcal{N}) \leq \mathrm{minadd} \leq \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}) \leq \mathrm{supcof} \leq \mathrm{cof}(\mathcal{N})$ (see [Osu08, CM19]) we can reformulate Yorioka's characterization of $\mathrm{cof}(\mathcal{SN})$ as follows.

Theorem 2.5 (Yorioka [Yor02])**.** *Let κ be a regular uncountable cardinal. Assume that* $\kappa = \mathrm{minadd} = \mathrm{supcof}$. *Then* $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa$.

To prove our Main Thereom we need to preserve $\mathfrak{d}_\kappa$ for κ regular. The following result show one condition under it can be preserved.

Lemma 2.6. *Let κ be a regular uncountable cardinal. Suppose that $\mathbb{P}$ is a κ-cc. Then $\Vdash_{\mathbb{P}} \mathfrak{d}_\kappa^V = \mathfrak{d}_\kappa$.*

Proof. It is enough to show that $\mathbb{P}$ is κ^κ-bounding[1] because κ^κ-bounding posets preserve $\mathfrak{d}_\kappa$. Let $\dot{x}$ be a $\mathbb{P}$-name for a member of κ^κ. We prove that $\forall\alpha < \kappa \exists z(\alpha) < \kappa (\Vdash_{\mathbb{P}} \dot{x}(\alpha) < z(\alpha))$. Fix any $\alpha < \kappa$. Towards a contradiction, assume that $\forall \beta < \kappa \exists p_\beta \in \mathbb{P}(p_\beta \Vdash_{\mathbb{P}} \beta \leq \dot{x}(\alpha))$.

Claim 2.7. *Assume that $\mathbb{P}$ is κ-cc and $\{p_\alpha : \alpha < \kappa\} \subseteq \mathbb{P}$. Then there is a $q \in \mathbb{P}$ such that $q \Vdash |\{\alpha < \kappa : p_\alpha \in \dot{G}\}| = \kappa$.*

Proof. To reason by contradiction assume that $\Vdash_{\mathbb{P}} |\{\alpha < \kappa : p_\alpha \in \dot{G}\}| < \kappa$. Let $\dot{\beta}$ be a $\mathbb{P}$-name such that $\Vdash \dot{\beta} \in \kappa$ and $\{\alpha < \kappa : p_\alpha \in \dot{G}\} \subseteq \dot{\beta}$. Fix a maximal antichain A deciding $\dot{\beta}$ and a function $h : A \to \kappa$ such that $p \Vdash h(p) = \dot{\beta}$ for all $p \in A$. Set $\gamma := \sup_{p \in A} h(p) < \kappa$. since κ is regular and $\mathbb{P}$ is κ-cc, $\gamma < \kappa$, so $\Vdash_{\mathbb{P}} \{\alpha < \kappa : p_\alpha \in \dot{G}\} \subseteq \gamma$. But $p_{\gamma+1} \Vdash \gamma + 1 \in \{\alpha < \kappa : p_\alpha \in \dot{G}\} \subseteq \gamma$, which is a contradiction. □

By Claim 2.7, we can find a condition $q \in \mathbb{P}$ such that $q \Vdash |\{\beta < \kappa : p_\beta \in \dot{G}\}| = \kappa$, so there are a $r \leq q$ and $\vartheta < \kappa$ such that $r \Vdash \dot{x}(\alpha) = \vartheta$, even more, we can find $s \leq r$ and $\varepsilon > \vartheta$ such that $s \Vdash p_\varepsilon \in \dot{G}$. Hence $s \Vdash \dot{x}(\alpha) = \vartheta < \varepsilon \leq \dot{x}(\alpha)$ because $p_\varepsilon \Vdash \varepsilon \leq \dot{x}(\alpha)$ which is a contradiction.

For $\alpha < \kappa$ set $z \in \kappa^\kappa$ such that $\Vdash_{\mathbb{P}} \dot{x}(\alpha) < z(\alpha)$. This z work. □

Now we are ready to prove the Main Theorem.

Proof of the Main Theorem. In V, we start with $\mathbb{P}_0 := \mathrm{Fn}_{<\kappa}(\nu \times \kappa, \kappa)$. Note that $\mathbb{P}_0$ is κ^+-cc and $<\kappa$-closed. Then $\Vdash_{\mathbb{P}_0} \mathfrak{d}_\kappa = 2^\kappa = \nu$ by Lemma 2.3.

In $V^{\mathbb{P}_0}$, let $\mathbb{P}_1$ be the FS iteration of amoeba forcing of length $\lambda\kappa$. Then, $\Vdash_{\mathbb{P}_1} \mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \kappa$ and $\mathfrak{c} = \lambda$. In particular, $\mathrm{add}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) = \kappa$ and minadd $=$ supcof $= \kappa$. On the other hand, $\mathrm{cov}(\mathcal{SN}) = \kappa$ because the length of the FS iteration has cofinality κ (see e.g. [BJ95, Lemma 8.2.6]). Therefore, $\Vdash_{\mathbb{P}_1} \mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) = \kappa$ and $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa = \nu$ by Theorem 2.5 and Lemma 2.6. □

3. Open problems

Very quite recently, the author with Mejía and Rivera-Madrid [CMRM] constructed a poset forcing $\mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$. This is first result where 3 cardianl invariants associated with $\mathcal{SN}$ are pairwise different, but its still unknown for 4, so we ask.

Question 3.1. Is it consistent with ZFC that $\mathrm{add}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

In a work in progress, the author with Mejía and Yorioka have improved methods and results known from [Yor02] to prove the consistency of $\mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$. However its still unknown the following problem.

Question 3.2. Is it consistent with ZFC that $\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$?

[1]A poset $\mathbb{P}$ is κ^κ-*bounding* if for any $p \in \mathbb{P}$ and any $\mathbb{P}$-name $\dot{x}$ of a member for κ^κ, there are a function $z \in \kappa^\kappa$ and some $q \leq p$ that forces $\dot{x}(\alpha) \leq z(\alpha)$ for any $\alpha < \kappa$.

The method of *κ-uf-extendable matrix iterations*, introduced recently by the author with Brendle and Mejía [BCM], could be useful to answer the question above. For example they constructed a ccc poset forcing

$$\text{add}(\mathcal{N}) = \text{add}(\mathcal{M}) < \text{cov}(\mathcal{N}) = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) < \text{cof}(\mathcal{M}) = \text{cof}(\mathcal{N}).$$

In the same model, $\text{cov}(\mathcal{SN}) = \text{cov}(\mathcal{N}) < \text{non}(\mathcal{SN}) = \text{non}(\mathcal{N})$ by Theorem A and because this model is obtained by a FS iteration of length with cofinality ν (where ν is the desired value for $\text{non}(\mathcal{M})$), and it is well known that such cofinality becomes an upper bound of $\text{cov}(\mathcal{SN})$ (see e.g. [BJ95, Lemma 8.2.6]). But it is unknown how to deal with $\text{add}(\mathcal{SN})$ and $\text{cof}(\mathcal{SN})$ in this context.

References

[BCM] Jörg Brendle, Miguel A. Cardona, and Diego A. Mejía. Filter-linkedness and its effect on the preservation of cardinal characteristics. `arXiv:1809.05004`.

[BJ95] Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line*. A K Peters, Wellesley, Massachusetts, 1995.

[Bor19] Émile Borel. Sur la classification des ensembles de mesure nulle. *Bulletin de la Société Mathématique de France*, 47:97–125, 1919.

[Car93] Timothy J. Carlson. Strong measure zero and strongly meager sets. *Proc. Amer. Math. Soc.*, 118(2):577–586, 1993.

[CM19] Miguel A. Cardona and Diego A. Mejía. On cardinal characteristics of Yorioka ideals. *MLQ*, 2019. In press. `arXiv:1703.08634`.

[CMRM] Miguel A. Cardona, Diego A. Mejía, and Ismael E. Rivera-Madrid. The covering number of the strong measure zero ideal can be above almost everything else. `arXiv:1902.01508v1`.

[GJS93] Martin Goldstern, Haim Judah, and Saharon Shelah. Strong measure zero sets without Cohen reals. *J. Symbolic Logic*, 58(4):1323–1341, 1993.

[Lav76] Richard Laver. On the consistency of Borel's conjecture. *Acta Math.*, 137(3-4):151–169, 1976.

[Mil81] Arnold W. Miller. Some properties of measure and category. *Trans. Amer. Math. Soc.*, 266(1):93–114, 1981.

[Osu08] Noboru Osuga. The cardinal invariants of certain ideals related to the strong measure zero ideal. *Kyōto Daigaku Sūrikaiseki Kenkyūsho Kōkyūroku*, 1619:83–90, 2008.

[Paw90] Janusz Pawlikowski. Finite support iteration and strong measure zero sets. *J. Symbolic Logic*, 55(2):674–677, 1990.

[Yor02] Teruyuki Yorioka. The cofinality of the strong measure zero ideal. *J. Symbolic Logic*, 67(4):1373–1384, 2002.

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