Polarized Partition on Successors of Singular Cardinals with the GCH

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Abstract

In this paper, we prove that $\forall\mu < \kappa \binom{\kappa^{++}}{\kappa^{+}} \to \binom{\mu}{\kappa^{+}}_{\kappa}$ is consistent with $2^{\kappa^{+}} = \kappa^{++}$ for singular κ.

1 Introduction

Definition 1.1 (Erdős–Hajnal–Rado [2])**.** *For any cardinals $\kappa_0, \kappa_1, \lambda_0, \lambda_1, \theta$,*

$$\binom{\kappa_0}{\kappa_1} \to \binom{\lambda_0}{\lambda_1}_{\theta}$$

means for any $c : \kappa_0 \times \kappa_1 \to \theta$, there are $H_0 \in [\kappa_0]^{\lambda_0}$ and $H_1 \in [\kappa_1]^{\lambda_1}$ such that $c \restriction H_0 \times H_1$ is a constant function.

We say $\binom{\kappa_0}{\kappa_1} \to \binom{\lambda_0}{\lambda_1}_{<\theta}$ iff $\forall\theta' < \theta \binom{\kappa_0}{\kappa_1} \to \binom{\lambda_0}{\lambda_1}_{\theta'}$. For this partition relation, we are interested in the case of $\kappa_0 = \lambda_0$ and $\kappa_1 = \lambda_1$. But if $\kappa_0 > 2^{\kappa_1}$ then $\binom{\kappa_0}{\kappa_1} \to \binom{\kappa_0}{\kappa_1}_{<cf(\kappa_1)}$ is obviously satisfied. In addition, $\binom{\kappa}{\kappa} \to \binom{\kappa}{\kappa}_2$ fails for every κ. Therefore we consider a coloring on $\kappa_0 \times \kappa_1$ where κ_0 is in between κ_1^+ and 2^{κ_1}.

For such coloring, the following theorem is known.

Theorem 1.2 (Sierpiński [9] for $\kappa = \omega$; Erdős–Hajnal–Rado [2])**.** *For any infinite cardinal κ, if $2^\kappa = \kappa^+$,*

$$\binom{\kappa^+}{\kappa} \not\to \binom{\kappa^+}{\kappa}_2.$$

Let us consider the following question.

Question 1.3. *How about polarized partition on $\kappa^+ \times \kappa$ under the assumption $2^\kappa = \kappa^+$?*

Note that $\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_{<cf(\kappa)}$ is the maximal form under the $2^\kappa = \kappa^+$. For large λ, we ask whether $\binom{\kappa^+}{\kappa} \to \binom{\lambda}{\kappa}_{<cf(\kappa)}$ is consistent with $2^\kappa = \kappa^+$ or not.

In the case of κ is a limit cardinal, the following are known.

Theorem 1.4 (Baumgartner–Hajnal [1])**.** *If κ is weakly compact, then*

$$\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_{<\kappa}.$$

Theorem 1.5 (Erdös–Hajnal–Rado [2])**.** *If κ is singular of cofinality ω, then*

$$\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_{<\omega}.$$

Theorem 1.6 (Shelah [8])**.** *If κ is singular limit of measurable cardinals, then*

$$\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_{<cf(\kappa)}.$$

On the other hand, for successor cardinals, the following consistency result is known.

Theorem 1.7 (Jensen)**.** *In L, for every infinite κ,*

$$\binom{\kappa^{++}}{\kappa^+} \not\to \binom{2}{\kappa^+}_{\kappa}.$$

In fact, Weak Kurepa Hypothesis over κ^+ gives a such coloring. For successors of regular cardinals, a positive polarized partition is also known:

Theorem 1.8 (Laver [5])**.** *If κ is regular below some HUGE cardinals then there is a κ-directed closed poset $\mathbb{P}$ which forces that*

$$\binom{\kappa^{++}}{\kappa^+} \to \binom{\kappa^+}{\kappa^+}_{\kappa}.$$

We will prove that we can force $2^{\kappa^+} = \kappa^{++}$ without destroying $\binom{\kappa^{++}}{\kappa^+} \to \binom{\kappa^+}{\kappa^+}_{\kappa}$ as lemma 3.5. So it is consistent that $\binom{\kappa^{++}}{\kappa^+} \to \binom{\kappa^+}{\kappa^+}_{\kappa}$ with $2^{\kappa^+} = \kappa^{++}$. Therefore a case of successors of regular cardinals is solved. However, the following is still open.

Question 1.9. *Is it consistent that $\binom{\kappa^{++}}{\kappa^+} \to \binom{\kappa^+}{\kappa^+}_{\kappa}$ for singular κ?*

As a partial answer for this question, we will show the following theorem.

Theorem 1.10. *If κ is supercompact below HUGE and $2^\kappa = \kappa^+$, there is a poset which forces that*

1. $\forall \mu < \kappa \binom{\kappa^{++}}{\kappa^+} \to \binom{\mu}{\kappa^+}_{\kappa}$,
2. *κ is strong limit singular,*
3. $2^{\kappa^+} = \kappa^{++}$.

2 Polarized Partition and Saturated Ideal

In this paper, if we say that I is an ideal over κ^+, I denotes κ^+-complete non-principal ideal over κ^+.

Definition 2.1. *An ideal I is (λ, μ, κ)-saturated if and only if for every $X \in [I^+]^\lambda$, there is a $Y \in [X]^\mu$ such that $\forall Z \in [Y]^\kappa \bigcap Z \in I^+$.*

Note that κ-saturation is $(\kappa, 2, 2)$-saturation. So this is an extended concept of saturation property.

Theorem 2.2 (Laver [5])**.** *If λ is HUGE and $\kappa < \lambda$ is regular, then there is a κ-directed closed $\mathbb{P}$ which forces that there is a $(\kappa^{++}, \kappa^{++}, \kappa)$-saturated ideal over κ^+.*

Laver's polarized partition theorem is shown by using $(\kappa^{++}, \kappa^{++}, \kappa)$-saturation properties. We give a direct proof of Laver's theorem.

Theorem 2.3 (Laver [5])**.** *If $2^\kappa = \kappa^+$ and there is a $(\kappa^{++}, \kappa^{++}, \kappa)$-saturated ideal over κ^+, then*

$$\binom{\kappa^{++}}{\kappa^+} \to \binom{\kappa^+}{\kappa^+}_\kappa .$$

Lemma 2.4. *Suppose that $2^\kappa = \kappa^+$ and $\langle X_\alpha \mid \alpha < \kappa^{++} \rangle \in {}^{\kappa^{++}}([\kappa^+]^{\kappa^+})$ satisfies $\forall \mathcal{X} \in [\kappa^{++}]^\kappa (|\bigcap_{\alpha \in \mathcal{X}} X_\alpha| = \kappa^+)$. Then there is a $\mathcal{Y} \in [\kappa^{++}]^{\kappa^+}$ such that $|\bigcap_{\alpha \in \mathcal{Y}} X_\alpha| = \kappa^+$.*

Proof. Let Ψ be sufficiently large regular and let $M \prec \mathcal{H}_\Psi$ be an elementary substructure such that:

- $|M| = \kappa^+$.
- $\kappa^+ + 1 \subseteq M$.
- $\overline{X_\alpha}, \kappa^{++} \in M$.
- M is closed under the taking κ-sequence. i.e. ${}^\kappa M \subseteq M$.
- $\delta = M \cap \kappa^{++}$ is an ordinal.

This M can be taken because we have $2^\kappa = \kappa^+$. Note that δ has cofinality κ^+ since M closed under the taking κ-sequence. We construct a sequence $\langle \beta_\xi \mid \xi < \kappa^+ \rangle$ and $\langle \alpha_\xi \mid \xi < \kappa^+ \rangle$ by the following way:

- $\alpha_0 = \min X_\delta$.
- $\beta_0 =$ the least $\gamma \in \delta$ such that $\alpha_0 \in X_\gamma$.
- $\alpha_\xi = \min \bigcap_{\eta<\xi} X_{\beta_\eta} \cap X_\delta \setminus \{\alpha_\eta \mid \eta < \xi\}$.
- $\beta_\xi =$ the least $\gamma \in \delta \setminus \sup\{\beta_\eta \mid \eta < \xi\}$ such that $\{\alpha_\eta \mid \eta \leq \xi\} \subseteq X_\gamma$.

This construction will be succeed. In every stage, α_ξ can be defined by $|\bigcap_{\eta<\xi} X_{\beta_\eta} \cap X_\delta| = \kappa^+$. β_ξ can also be defined in every $\xi < \kappa^+$. By construction, we have $\{\alpha_\eta \mid \eta \leq \xi\} \subseteq X_\delta$. By assumption for M, we have $\beta := \sup_{\eta<\xi} \beta_\eta < \delta$ and $\{\alpha_\eta \mid \eta \leq \xi\}$ is in M. By elementarity of M,

$$M \models \exists \delta' > \beta \, \{\alpha_\eta \mid \eta \leq \xi\} \subseteq X_{\delta'}.$$

So we can take β_ξ as δ'.

As conclusion, let $\mathcal{Y} := \{\beta_\xi \mid \xi < \kappa^+\}$, then

$$\{\alpha_\xi \mid \xi < \kappa^+\} \subseteq \bigcap_{\xi<\kappa^+} X_{\beta_\xi} = \bigcap_{\alpha\in\mathcal{Y}} X_\alpha.$$

So we have $|\bigcap_{\alpha\in\mathcal{Y}} X_\alpha| = \kappa^+$. □

Proof of Theorem 2.3. Let I be $(\kappa^{++}, \kappa^{++}, \kappa)$-saturated ideal over κ^+. Let $c : \kappa^{++} \times \kappa^+ \to \kappa$ be an arbitrary coloring. For every $\alpha < \kappa^{++}$, by κ-completeness of I, there is an η_α such that $X_\alpha = \{\xi < \kappa^+ \mid c(\alpha, \xi) = \eta_\alpha\} \in I^+$.

For simplicity, we may assume that there is an η such that $\eta_\alpha = \eta$ for all $\alpha < \kappa^{++}$. By $(\kappa^{++}, \kappa^{++}, \kappa)$-saturated, we may assume that $\forall \mathcal{X} \in [\kappa^{++}]^\kappa (\bigcap_{\alpha\in\mathcal{X}} X_\alpha \in I^+)$. In particular, $\bigcap_{\alpha\in\mathcal{X}} X_\alpha$ is of size κ^+ for every $\mathcal{X} \in [\kappa^{++}]^\kappa$.

By lemma 2.4, we can pick $H_0 \in [\kappa^{++}]^{\kappa^+}$ such that $H_1 := \bigcap_{\alpha\in H_0} X_\alpha$ is of size κ^+. We have $c \restriction H_0 \times H_1$ is monochromatic with color η. □

Remark 2.5. *We can proof more strong form of theorem 2.3. In fact, under the same assumption, we can take $H_0 \in [\kappa^{++}]^\alpha$ and $H_1 \in [\kappa^+]^{\kappa^+}$ such that $c \restriction H_0 \times H_1$ is monochromatic for every $c : \kappa^{++} \times \kappa^+ \to \kappa$ and ordinal $\alpha < \kappa^{++}$.*

Definition 2.6 (Garti [3])**.** *For any family $\mathcal{A} \subseteq \mathcal{P}(\lambda)$,*

1. *Coloring $c : \kappa \times \lambda \to \theta$ is $\mathcal{A}$-amenable if and only if*

$$\forall \alpha < \kappa \exists \eta < \theta \exists X \in \mathcal{A} (\forall \xi \in X \; c(\alpha, \xi) = \eta).$$

2. $\binom{\kappa_0}{\kappa_1} \to_{\mathcal{A}} \binom{\lambda_0}{\lambda_1}_\theta$ *says that for every $\mathcal{A}$-amenable coloring $c : \kappa_0 \times \kappa_1 \to \theta$, there are $H_0 \in [\kappa_0]^{\lambda_0}, H_1 \in [\kappa_1]^{\lambda_1}$ such that $c \restriction H_0 \times H_1$ is monochromatic.*

The above lemma gives a polarized partition for amenable colorings:

Corollary 2.7. *If $2^\kappa = \kappa^+$ and $\mathcal{A} \subseteq [\kappa^+]^{\kappa^+}$ has κ^+-completeness(or every intersection of κ-many sets in $\mathcal{A}$ have size κ^+), $\binom{\kappa^{++}}{\kappa^+} \to_{\mathcal{A}} \binom{\kappa^+}{\kappa^+}_\kappa$ holds.*

In particular, $\binom{\kappa^{++}}{\kappa^+} \to_{Club(\kappa^+)} \binom{\kappa^+}{\kappa^+}_\kappa$ holds.

Proof. By the similar proof of Theorem 2.3. □

Theorem 2.8 (Garti [3])**.** *If $2^{\aleph_1} = \aleph_2$ holds, then*

$$\binom{\aleph_2}{\aleph_1} \not\to_{Club(\aleph_1)} \binom{\aleph_2}{\aleph_1}_2.$$

By these observations, we have

$$\text{GCH implies } \binom{\aleph_2}{\aleph_1} \not\rightarrow_{Club(\aleph_1)} \binom{\aleph_2}{\aleph_1}_2 \text{ but } \binom{\aleph_2}{\aleph_1} \rightarrow_{Club(\aleph_1)} \binom{\aleph_1}{\aleph_1}_{\aleph_0}.$$

Question 2.9. *Is it consistent that there is a $(\kappa^{++}, \kappa^{++}, \kappa)$-saturated ideal over κ^+ for singular κ?*

3 Proof of Main Theorem

Definition 3.1. *We say that $\mathbb{P}$ satisfies $(*)_\kappa$ if and only if there is a $\langle \mathbb{F}_\eta \mid \eta < \kappa \rangle$ such that $\mathbb{F}_\eta$ is κ-directed and $\mathbb{P} = \cup_{\eta<\kappa}\mathbb{F}_\eta$.*

Here, X is κ-directed iff every subset of X of size $< \kappa$ have a common extension. Cleary, $(*)_\kappa$ implies κ-c.c. Many Prikry-type forcing satisfies $(*)_\kappa$. $(*)_\kappa$ poset preserves saturation property for ideal over κ^+ as follows.

Lemma 3.2. *For regular cardinals $\mu < \kappa$, suppose that*

1. *There is a (κ^{++}, μ, μ)-saturated ideal I over κ^+.*
2. *$\mathbb{P}$ is a poset which satisfies $(*)_\kappa$ and preserves $\mu < \kappa$ and κ.*

Then

$$\mathbb{P} \Vdash \check{I} \text{ generates } (\check{\kappa}^{++}, \check{\mu}, \check{\mu})\text{-saturated ideal}$$

Proof. Let $\langle \mathbb{F}_\eta \mid \eta < \kappa \rangle$ witnesses to $(*)_\kappa$. Let $\dot{I}$ be a $\mathbb{P}$-name which denotes an ideal generated by $\check{I}$.

Since $\mathbb{P}$ has the κ^+-c.c., $\mathbb{P}$ forces $\dot{I}$ is $\check{\kappa}^+$-complete.

Consider a p and $\langle \dot{X}_\alpha \mid \alpha < \kappa^{++} \rangle$ such that

$$p \Vdash \dot{X}_\alpha \in \dot{I}^+.$$

It is enough to show that there is a $q \leq p$ such that $q \Vdash \exists H \in [\check{\kappa}^{++}]^{\check{\mu}} \bigcap_{\alpha\in H} \dot{X}_\alpha \in \dot{I}^+$.

For each $\alpha < \kappa^{++}, \eta < \kappa$, let $A_\alpha^\eta := \{\xi < \kappa^+ \mid \exists q \in \mathbb{F}_\eta(q \leq p \wedge q \Vdash \check{\xi} \in \dot{X}_\alpha)\}$. Since $\bigcup_{\eta<\kappa} A_\alpha^\eta \in I^+$ and I is κ^+-complete, there is an η_α such that $A_\alpha^{\eta_\alpha} \in I^+$. And $A_\alpha := A_\alpha^{\eta_\alpha}$. Then there are $Z \in [\kappa^{++}]^{\kappa^{++}}$ and $\eta < \kappa$ such that $\eta = \eta_\alpha$ for all $\alpha \in Z$. By the (κ^{++}, μ, μ)-saturation for I, we can pick $H \in [Z]^\mu$ such that $\bigcap_{\alpha\in H} A_\alpha \in I^+$.

Claim 3.3. *For every $\xi \in \bigcap_{\alpha\in H} A_\alpha$, there is a $q \leq p$ such that $q \Vdash \check{\xi} \in \bigcap_{\alpha\in\check{H}} \dot{X}_\alpha$.*

Proof of Claim. By the definition of A_α, for each $\alpha \in H$, we can pick $q_\alpha \in \mathbb{F}_\eta$ such that $q_\alpha \Vdash \check{\xi} \in \dot{X}_\alpha$. Since each q_α are in $\mathbb{F}_\eta$, there is a $q \leq q_\alpha$ for every α. q is an extension of p and forces that $\check{\xi} \in \bigcap_{\alpha\in\check{H}} \dot{X}_\alpha$. □

We show that there is a $q \leq p$ which forces that $\bigcap_{\alpha\in\check{H}} \dot{X}_\alpha \in \dot{I}$. Consider a set $A = \{q \leq p \mid \exists Z_q \in I(q \Vdash \bigcap_{\alpha\in\check{H}} \dot{X}_\alpha \subseteq \check{Z}_q)\}$. Note that A is the set of all $q \leq p$ which forces $\bigcap_{\alpha\in\check{H}} \dot{X}_\alpha$ is $\dot{I}$-measure zero. Let $\mathcal{A} \subseteq A$ be a maximal antichain below p. By the κ^+-c.c., $\mathcal{A}$ has size at most κ. So $Z := \bigcup_{q\in\mathcal{A}} Z_q$ is also I-measure zero.

Therefore $(\bigcap_{\alpha\in H} A_\alpha) \setminus Z \neq \emptyset$. Pick $\xi \in (\bigcap_{\alpha\in H} A_\alpha) \setminus Z$, by claim, there is a $q \leq p$ which forces that $\check{\xi} \in \bigcap_{\alpha\in\check{H}} \dot{X}_\alpha$. Cleary, q is incompatible with any element in $\mathcal{A}$. By the maximality of $\mathcal{A}$, $q \Vdash \bigcap_{\alpha\in\check{H}} \dot{X}_\alpha \in \dot{I}^+$. □

Corollary 3.4. *In $V^{\mathbb{P}}$, $\binom{\kappa^{++}}{\kappa^+} \to \binom{\mu}{\kappa^+}_\kappa$ holds.*

Proof. Let G be an arbitrary $(V, \mathbb{P})$-generic. We discuss in $V[G]$. Let $\hat{I}$ be an ideal generated by I. By lemma, $\hat{I}$ satisfies (κ^{++}, μ, μ)-saturated.

Take any coloring $c : \kappa^{++} \times \kappa^+ \to \kappa$. For each $\alpha < \kappa^{++}$, there is a η_α such that $X_\alpha = \{\xi < \kappa^+ \mid c(\alpha, \xi) = \eta_\alpha\}$ in $\hat{I}^+$. We may assume that there is an η such that $\eta_\alpha = \eta$ for every $\alpha < \kappa^{++}$.

By the saturation property of $\hat{I}$, there is an $H_0 \in [\kappa^{++}]^\mu$ such that $H_1 := \bigcap_{\alpha \in H_0} X_\alpha \in I^+$. Trivially, $c \restriction H_0 \times H_1$ is monochromatic with color η.

□

Lemma 3.5 (Folklore)**.** *Suppose that $\binom{\kappa^{++}}{\kappa^+} \to \binom{\mu}{\kappa^+}_\kappa$ holds for some $\mu < \kappa^{++}$ and $2^{\kappa^+} > \kappa^{++}$. Then there is a poset $\mathbb{P}$ such that*

1. *$\mathbb{P}$ forces $2^{\kappa^+} = \kappa^{++}$ and $\binom{\kappa^{++}}{\kappa^+} \to \binom{\mu}{\kappa^+}_\kappa$,*
2. *$\mathbb{P}$ preserves all cardinals below κ^{++}.*

Proof. Let $\mathbb{P} := \mathrm{coll}(\kappa^{++}, 2^{\kappa^+})$. Note that $\mathbb{P}$ is κ^{++}-closed. Cleary $\mathbb{P}$ forces that $2^{\kappa^+} = \kappa^{++}$. It is enough to show that $\mathbb{P}$ preserves polarized partition relation. Let $p \in \mathbb{P}$ and $\dot{c} \in V^{\mathbb{P}}$ be such that:

$$p \Vdash \dot{c} : \check{\kappa}^{++} \times \check{\kappa}^+ \to \check{\kappa}.$$

By induction on $\alpha < \kappa^{++}$, we define a sequence $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$ and coloring $f : \kappa^{++} \times \kappa^+ \to \kappa$ such that:

- $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$ is decreasing sequence in $\mathbb{P}$ and $p_0 \leq p$.
- $p_\alpha \Vdash \dot{c}(\check{\alpha}, \check{\xi}) = \check{f}(\check{\alpha}, \check{\xi})$ for every $\xi < \kappa^+$.

By κ^{++}-closedness, we can construct this sequence. By $\binom{\kappa^{++}}{\kappa^+} \to \binom{\mu}{\kappa^+}_\kappa$, there are $H_0 \in [\kappa^{++}]^\mu$ and $H_1 \in [\kappa^+]^{\kappa^+}$ such that $f \restriction H_0 \times H_1$ is monochromatic.

Let $\alpha := (\sup H_0) + 1 < \kappa^{++}$. Then

$$p_\alpha \Vdash \dot{c} \restriction \check{H}_0 \times \check{H}_1 = \check{f} \restriction \check{H}_0 \times \check{H}_1 \text{ is monochromatic.}$$

□

We recall about Prikry forcing in [7]. For a normal ultrafilter U over κ, Prikry forcing $\mathbb{P}_U$ is $[\kappa]^{<\omega} \times U$ ordered by $\langle a, A\rangle \leq \langle b, B\rangle$ iff a is an end-extension of b (i.e. $a \cap \max(b) = b$), $A \subseteq B$ and $a \setminus b \subseteq B$. Prikry forcing preserves all cardinals but forces that $cf(\kappa) = \omega$.

Lemma 3.6. *$\mathbb{P}_U$ satisfies $(*)_\kappa$.*

Proof. For each $a \in [\kappa]^{<\omega}$, let $\mathbb{F}_a := \{\langle a, A\rangle \mid A \in U\}$, which is κ-directed. Cleary, $\mathbb{P} = \bigcup_{a \in [\kappa]^{<\omega}} \mathbb{F}_a$. □

Proof of Theorem 1.10. First, by Laver's theorem and indestructibility for supercompactness, there is a poset $\mathbb{P}$ which forces that there is $(\kappa^{++},\kappa^{++},\kappa)$-saturated ideal over κ^{+} and κ remains supercompact.

In $V^{\mathbb{P}}$, Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name which denotes Prikry forcing over κ. By $\dot{\mathbb{Q}}$ satisfies that $(*)_\kappa$ in $V^{\mathbb{P}}$ and lemma 3.2, $V^{\mathbb{P}*\dot{\mathbb{Q}}} \models \forall\mu<\kappa \binom{\kappa^{++}}{\kappa^{+}} \to \binom{\mu}{\kappa^{+}}_{\kappa}$ holds and $cf(\kappa)=\omega$. Further, by lemma 3.5, we can forces that $2^{\kappa^{+}} = \kappa^{++}$ without destroying polarized partition relation. So this proof is done. □

Lemma 3.7. *If κ is supercompact below HUGE, then there is a poset which forces that*

1. $\binom{\kappa^{++}}{\kappa^{+}} \to \binom{\lambda}{\kappa^{+}}_{\kappa}$ *for every* $\lambda<\kappa$,
2. $2^{\kappa^{+}} = \kappa^{++}$,
3. *κ is strong limit singular with uncountable cofinality.*

Proof. Note that Magidor's forcing in [6] satisfies $(*)_\kappa$. We can do the same proof for theorem 1.10. □

Lemma 3.8. *Suppose that there is a $(\kappa^{++},\kappa^{++},\kappa)$-saturated ideal over κ^{+} and κ is supercompact, $2^{\kappa}=\kappa^{+}$. Then there is a poset which forces that*

1. $\binom{\aleph_{\omega+2}}{\aleph_{\omega+1}} \to \binom{\aleph_n}{\aleph_{\omega+1}}_{\aleph_\omega}$ *for every* $n<\omega$,
2. $\kappa=\aleph_\omega$,
3. $2^{\aleph_{\omega+1}} = \aleph_{\omega+2}$.

Proof. Let U be a normal ultrafilter over κ and $j: V \to M$ be an elementary embedding induced by U. By $2^{\kappa}=\kappa^{+}$, there is an $(M, \mathrm{Coll}((\kappa^{++})^{M}, < j(\kappa)))$-generic G in V. Then G induces a Gitik's forcing in [4] which forces $\kappa=\aleph_\omega$. Note that Gitik's forcing satisfies $(*)_\kappa$. We can do the same proof for theorem 1.10 again. □

Note that we don't need a HUGE cardinal to give such a polarized partition over $\aleph_1 \times \aleph_2$.

Lemma 3.9 (Zhang [10])**.** *Suppose that there is a presaturated ideal I over κ^{+}. For every $n<\omega$,*

$$\binom{\kappa^{++}}{\kappa^{+}} \to \binom{n}{\kappa^{+}}_{\kappa}.$$

In particular, if there is a presaturated ideal over $\aleph_1$ then $\binom{\aleph_2}{\aleph_1} \to \binom{n}{\aleph_1}_{\aleph_0}$ for every $n<\aleph_0$.

Proof. Let G be an arbitrary $(V,\mathbb{P})$-generic and $j: V \to M \subseteq V[G]$ be generic ultrapower induced by G.

Since I is presaturated, we have

- $crit(j) = \kappa^+$.
- $j(\kappa^+) = (\kappa^{++})^V$.

Let $c : \kappa^{++} \times \kappa^+ \to \kappa$ be an arbitrary coloring.

Let $A := j''\kappa^{++}$. $|A| = (\kappa^{++})^V$. Since $cf((\kappa^{++})^V) > \kappa$, we can take unbounded subset $B \subseteq A$ and $\eta < \kappa$ such that

$$\forall \xi \in B(j(c)(\xi, (\kappa^+)^V) = \eta).$$

Pick any finite sequence $j(\beta_0), ..., j(\beta_{n-1}) \in B$. Note that $\langle \beta_0, ..., \beta_{n-1} \rangle \in V$.

We construct a sequence $\langle \alpha_\xi \mid \xi < \kappa^+ \rangle$ below $(\kappa^+)^V$ by induction. Suppose that $\langle \alpha_\xi \mid \xi < \eta \rangle$ has been defined. Let $\alpha := \sup_{\xi<\eta} \alpha_\xi < \kappa^+$. In M, the following holds:

$$(\kappa^+)^V > \alpha \wedge \forall i < n(j(c)(j(\beta_i), (\kappa^+)^V) = j(\eta)).$$

So $M \models \exists \zeta > \alpha \forall i < n(j(c)(j(\beta_i), \zeta) = j(\eta))$. By elementarity,

$$\exists \zeta > \alpha \forall i < n(c(\beta_i, \zeta) = \eta).$$

Let α_η be defined as such ζ.

Cleary, $c \restriction \{\beta_i \mid i < n\} \times \{\alpha_\xi \mid \xi < \kappa^+\}$ is monochromatic. $\square$

The same proof shows the following lemma.

Lemma 3.10. *Suppose that there are presaturated ideal I over κ^+ and regular $\mu < \kappa$ such that $\langle I^+, \subseteq \rangle$ is μ^+-Baire. Then*

$$\binom{\kappa^{++}}{\kappa^+} \to \binom{\mu}{\kappa^+}_\kappa.$$

Remark 3.11. *In [10], Zhang proved it is consistent that $\binom{\aleph_2}{\aleph_1} \to \binom{\aleph_0}{\aleph_1}_{\aleph_0}$ fails but there is a presaturated ideal over $\aleph_1$.*

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