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<td><strong>Citation</strong></td>
<td>数理解析研究所講究録 = RIMS Kokyuroku (2019), 2141: 49-55</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2019-12</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/2433/254939">http://hdl.handle.net/2433/254939</a></td>
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<tr>
<td><strong>Type</strong></td>
<td>Departmental Bulletin Paper</td>
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<tr>
<td><strong>Textversion</strong></td>
<td>publisher</td>
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TODORČEVIĆ’S FRAGMENTS OF MARTIN’S AXIOM AND VARIATIONS OF UNIFORMIZATIONS OF LADDER SYSTEM COLORINGS

TERUYUKI YORIOKA

INTRODUCTION

In this article, we introduce parametrized versions of Devlin-Shelah’s assertion about uniformizations of ladder system colorings. For a subset $S$ of the power set of $\omega_1 \cap \text{Lim}$, $U(S)$ is the assertion that, for any coloring $\langle f_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$ of the ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$, there exist $S \in S$ and a function from $\omega_1$ into $\omega$ which uniformizes the restricted coloring $\langle f_\alpha : \alpha \in S \rangle$. Devlin-Shelah’s original assertion is the assertion $U(\{\omega_1 \cap \text{Lim}\})$. This follows from $\text{MA}_{\omega_1}$, and is equivalent to the existence of non-free Whitehead group. The axiom $\mathcal{K}_2$, which is one of Todorcevic’s fragments of Martin’s Axiom, implies the assertion $U(\{\omega_1 \cap \text{Lim}\}^{\omega_1})$. Todorcevic-Velickovic pointed out that $\mathcal{K}_4$ implies $U(\{\omega_1 \cap \text{Lim}\})$. We show that the axiom $\mathcal{K}_3$ implies the assertion $U(\text{stat})$, and similarly, $\mathcal{K}_2'$ implies $U(\text{club})$. By Larson-Todorcevic’s result, it is shown that it is consistent that $U(\{\omega_1 \cap \text{Lim}\}^{\omega_1})$ holds and $U(\text{club})$ fails.

1. BACKGROUND AND PRELIMINARIES

1.1. Todorcevic’s fragments of Martin’s Axiom. In the 1980s, Todorcevic investigated Martin’s Axiom from the view point of Ramsey theory, and introduced the following fragments of Martin’s Axiom: $\mathcal{K}_{< \omega}$ denotes the assertion that every ccc forcing notion has precaliber $\aleph_1$; $\mathcal{K}_\alpha$ denotes the assertion that every ccc forcing notion has the property $K_\alpha$; $\mathcal{K}_{< \omega}'$ denotes the assertion that every ccc partition $K_0 \cup K_1 = [\omega_1]^{< \omega}$ has an uncountable $K_0$-homogeneous set; $\mathcal{K}_\alpha'$ denotes the assertion that every ccc partition $K_0 \cup K_1 = [\omega_1]^n$ has an uncountable $K_0$-homogeneous set.* The following diagram is a summary of implications of these fragments of $\text{MA}_{\omega_1}$. The triangle on the left side of the diagram is Todorcevic-Velickovic theorem [11, COROLLARY 2.7].

\[
\begin{array}{cccccccc}
\mathcal{K}_{< \omega} & \rightarrow & \cdots & \rightarrow & \mathcal{K}_{n} & \rightarrow & \cdots & \rightarrow & \mathcal{K}_3 & \rightarrow & \mathcal{K}_2 \\
\mathcal{K}_{< \omega}' & \rightarrow & \cdots & \rightarrow & \mathcal{K}_n' & \rightarrow & \cdots & \rightarrow & \mathcal{K}_3' & \rightarrow & \mathcal{K}_2' \\
\mathcal{K}_\alpha & \rightarrow & \mathcal{K}_{\alpha+1} & \rightarrow & \mathcal{K}_\alpha & \rightarrow & \mathcal{K}_{\alpha+1} & \rightarrow & \mathcal{K}_\alpha & \rightarrow
\end{array}
\]

It is not known whether any other implications in this diagram hold under ZFC.

*They are defined by Todorcevic in several papers. In [5, Definition 4.9] and [11, §2], $\mathcal{K}_\alpha$’s are defined as assertions for ccc forcing notions, however in [6, §4] and [8, §7], $\mathcal{K}_\alpha$’s are defined as assertions for ccc partitions. To separate them, we use the notations as above. These notations are same to ones in [12].

Supported by Grant-in-Aid for Scientific Research (C) 18K03393, Japan Society for the Promotion of Science.
Larson-Todorcević introduced a property of ccc partitions on $[\omega_1]^2$, called the rectangle refining property, and introduced the assertion $\mathcal{K}_2^2(\text{rec})$ that every partition on $[\omega_1]^2$ with the rectangle refining property has an uncountable homogeneous set. Larson-Todorcević proved that it is consistent that a Suslin tree can force $\mathcal{K}_2^2(\text{rec})$ [6]. More precisely, they introduced the assertion $\text{MA}_{\omega_1}(S)$ which asserts that there exists a coherent Suslin tree $S$ such that the forcing axiom for all ccc forcing notions which preserves $S$ to be Suslin holds, and showed that, under $\text{MA}_{\omega_1}(S)$, $S$ forces $\mathcal{K}_2^2(\text{rec})$. In [13], the author developed their result to $\mathcal{K}_{<\omega}(\text{rec})$ in some sense, that is, under $\text{MA}_{\omega_1}(S)$, $S$ forces $\mathcal{K}_{<\omega}(\text{rec})$ in some sense.

1.2. Uniformizations of ladder system colorings. The notion of uniformization of a ladder system coloring was introduced by Devlin-Shelah, in order to study the non-free Whitehead groups [2]. The following (4) is a parametrized version of their assertion introduced in [2, 5.2 THEOREM].

**Definition 1.1.**

1. A ladder system on $\omega_1$ is a sequence $(C_\alpha : \alpha \in \omega_1 \cap \text{Lim})$ such that, for each $\alpha \in \omega_1 \cap \text{Lim}$, $C_\alpha$ is an unbounded subset of $\alpha$ and the order type of $C_\alpha$ is $\omega$.

2. A coloring of a ladder system $(C_\alpha : \alpha \in \omega_1 \cap \text{Lim})$ is a sequence $(f_\alpha : \alpha \in \omega_1 \cap \text{Lim})$ such that, for each $\alpha \in \omega_1 \cap \text{Lim}$, $f_\alpha$ is a function from $C_\alpha$ into $\omega$.

3. For each coloring $(f_\alpha : \alpha \in \omega_1 \cap \text{Lim})$ of a ladder system $(C_\alpha : \alpha \in \omega_1 \cap \text{Lim})$ and a subset $S$ of $\omega_1$, a function $\varphi$ from $\omega_1$ into $\omega$ uniformizes the restricted coloring $(f_\alpha : \alpha \in S)$ if for every $\alpha \in S$, $f_\alpha$ and $\varphi \upharpoonright C_\alpha$ are almost equal, that is, the set $$\{ \xi \in C_\alpha : f_\alpha(\xi) \neq \varphi(\xi) \}$$ is finite.

4. For a subset $S$ of the power set of $\omega_1 \cap \text{Lim}$, $U(S)$ is the assertion that, for any coloring $(f_\alpha : \alpha \in \omega_1 \cap \text{Lim})$ of a ladder system $(C_\alpha : \alpha \in \omega_1 \cap \text{Lim})$, there exist $S \in S$ and a function from $\omega_1$ into $\omega$ which uniformizes the restricted coloring $(f_\alpha : \alpha \in S)$.

Devlin-Shelah introduced the assertion $U(\{\omega_1 \cap \text{Lim}\})$ in [2, 5.2 THEOREM]. They pointed out that $U(\{\omega_1 \cap \text{Lim}\})$ is a sufficient condition of the existence of a non-free Whitehead group [2, §6]. Moreover, Eklof-Shelah showed that $U(\{\omega_1 \cap \text{Lim}\})$ is equivalent to the existence of a non-free Whitehead group [4], [3, Ch. XIII].

For any nonstationary subset $N$ of $\omega_1 \cap \text{Lim}$, one can prove the assertion $U(\{N\})$ from ZFC [3, Ch. II Exercise 20 (a)]. Devlin-Shelah showed that $\text{MA}_{\omega_1}$ implies $U(\{\omega_1 \cap \text{Lim}\})$ [2, 5.2 THEOREM]. It follows from their proof that $\mathcal{K}_2^2$ implies $U([\omega_1 \cap \text{Lim}]^{<\omega})$. In [13], the author proved that $\mathcal{K}_2^2(\text{rec})$ implies the assertion $U([\omega_1 \cap \text{Lim}]^{<\omega})$. Todorcević-Veličkovič pointed out that $U(\{\omega_1 \cap \text{Lim}\})$ is followed from $\mathcal{K}_4^{1,1}$ [11, §2].

On the other hand, Larson-Todorcević essentially proved that a Suslin tree forces the negation of the assertion $U(\text{club})$, where club stands for the set of all club subsets of $\omega_1 \cap \text{Lim}$ [5, THEOREM 6.2]. Therefore, it is proved that under $\text{MA}_{\omega_1}(S)$, $S$ forces that $U([\omega_1 \cap \text{Lim}]^{<\omega})$ holds and $U(\text{club})$ fails. The author does not know whether $U(\text{club})$ implies $U(\{\omega_1 \cap \text{Lim}\})$.
In the next section, it is proved that $K'_3$ implies $U(\text{stat})$, where stat stands for the set of all stationary subsets of $\omega_1 \cap \text{Lim}$. By a similar argument, it is proved that $K'_4$ implies $U(\text{stat})$.

2. $K'_3$ IMPLIES $U(\text{stat})$

In this section, we prove the title of the section. Here, for ordinals $\alpha$, $\beta$ and $\gamma$, we write $\{\alpha, \beta\} < \gamma$, or $\{\alpha, \beta, \gamma\} < \gamma$, when $\alpha < \beta$, or $\alpha < \beta < \gamma$.

Let $\langle e_\alpha : \alpha \in \omega_1 \rangle$ be a sequence such that

- each $e_\alpha$ is an injective function from $\alpha$ into $\omega$, and
- $\langle e_\alpha : \alpha \in \omega_1 \rangle$ is a coherent sequence, that is, for each $\alpha, \beta \in \omega_1$ with $\alpha < \beta$, the set
  $$\{ \xi \in \omega : e_\beta(\xi) \neq e_\alpha(\xi) \}$$

is finite [7, 9, 10].

Let $\langle r_\alpha : \alpha \in \omega_1 \rangle$ be an injective sequence of members of the set $\omega^2$. For each $\alpha, \beta \in \omega_1$ with $\alpha < \beta$, and each $n \in \omega$, define

$$\sigma(\alpha, \beta) := \min \{ n \in \omega : r_\alpha(n) \neq r_\beta(n) \},$$

$$F_n(\beta) := \{ \xi \in \beta : e_\beta(\xi) \leq n \} \cup \{ \beta \},$$

and

$$b(\alpha, \beta) := \min \{ F_{\sigma(\alpha, \beta)}(\beta) \setminus \alpha \}. $$

(See e.g. [7, §6].) Then, similar to [11, THEOREM 2.1], the following is proved.

**Lemma 2.1.** Let $X$ be an uncountable subset of $\omega_1$ and $M$ a countable elementary submodel of $H_{\aleph_2}$ such that the set

$$\{ \langle e_\alpha : \alpha \in \omega_1 \rangle, \langle r_\alpha : \alpha \in \omega_1 \rangle, X \}$$

belongs to the model $M$. Then, for any $\beta \in X \setminus M$, there exists $\alpha \in X \cap M$ such that $b(\alpha, \beta) = \omega_1 \cap M$ and, for any $\xi \in \beta \setminus \alpha$, $e_\beta(\xi) = e_{\omega_1 \cap M}(\xi)$.

**Proof.** Let $\beta \in X \cap M$. Since the sequence $\langle r_\alpha : \alpha \in \omega_1 \rangle$ is injective and $X$ is uncountable, we can find $\alpha' \in X \setminus \{ \beta \cup M \}$ such that

$$r_{\alpha'} \upharpoonright e_\beta(\omega_1 \cap M) = r_\beta \upharpoonright e_\beta(\omega_1 \cap M).$$

(Here, we do not mind whether $\alpha'$ is less than $\beta$ or not.) We should notice that there exist uncountably many such $\alpha'$. Define $n := \sigma(\beta, \alpha')$ (or $\sigma(\alpha', \beta)$). Then we notice that $e_{\beta}(\omega_1 \cap M) \leq n$. Since the function $e_\beta$ is injective and the sequence $\langle e_\alpha : \alpha \in \omega_1 \rangle$ is coherent, we can find $\gamma \in \omega_1 \cap M$ such that for any $\xi \in \beta \setminus \gamma$, $e_{\omega_1 \cap M}(\xi) = e_\beta(\xi) > n$.

By elementarity of $M$, we can find $\alpha \in (X \cap M) \setminus \gamma$ that is a copy of $\alpha'$, which means here that $\alpha \geq \gamma$ and $r_\alpha \upharpoonright (n + 1) = r_{\alpha'} \upharpoonright (n + 1)$. Then

$$\sigma(\alpha, \beta) = \sigma(\beta, \alpha') = n.$$ 

Therefore $b(\alpha, \beta) = \omega_1 \cap M$. 

The following is the main preliminary lemma of the proof.
Lemma 2.2. Let \( \langle C_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle \) be a ladder system, \( \{ \eta_n^\alpha : n \in \omega \} \) the increasing enumeration of \( C_\alpha \) for each \( \alpha \in \omega_1 \cap \text{Lim} \), \( I \) an uncountable subset of the set \( [\omega_1]^\kappa \), \( \kappa \) an enough large regular cardinal, \( M \) a countable elementary submodel of \( H_\kappa \) that contains the set

\[
\{ \langle e_\alpha : \alpha \in \omega_1 \rangle, \langle r_\alpha : \alpha \in \omega_1 \rangle, \langle C_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle, I, H_{\aleph_2} \},
\]

and \( \tau \in I \setminus M \). Then there exists \( J \in [I]^{\aleph_1} \cap M \) such that, for every \( \nu \in J \cap M \),

1. \( \nu \) is an end-extension of \( \tau \cap M \), that is, \( \tau \cap M \subseteq \nu \) and \( \min(\nu \setminus (\tau \cap M)) > \max(\tau \cap M) \),

2. for any \( \{ \alpha, \beta \} < \) and \( \{ \gamma, \delta \} < \) in the set \([\nu \cup \tau]^2\), if \( \{ \alpha, \beta, \gamma, \delta \} \not\subseteq \nu \) and \( \{ \alpha, \beta, \gamma, \delta \} \not\subseteq \tau \) and \( \{ b(\alpha, \beta), b(\gamma, \delta) \} \subseteq \text{Lim} \), then

\[
\max \left( \left\{ \eta_n^{b(\alpha, \beta)} : n \geq e_\beta(\alpha) \right\} \cap \left\{ \eta_n^{b(\gamma, \delta)} : n \geq e_\delta(\gamma) \right\} \right) < \max \left( \bigcup \left\{ C_{b(\alpha', \beta')} \cap M : \{ \alpha', \beta' \} < \in [\tau]^2, b(\alpha', \beta') \geq \omega_1 \cap M \right\} \right).
\]

Proof. By simplifying the argument, for each \( \gamma \in \omega_1 \setminus \text{Lim} \), we define \( C_\gamma := \{ \gamma - 1 \} \) and \( \eta_n^\gamma := \gamma - 1 \) for every \( n \in \omega \). Define

\[
L_0 := \{ b(\alpha, \beta) : \{ \alpha, \beta \} \in [\tau \setminus M]^2 \}
\]

and

\[
L_1 := \left\{ \min \left( F_{\sigma(\alpha, \beta)}(\beta) \cap [(\omega_1 \cap M) + 1, \beta] \right) : \beta \in \tau \setminus M, \alpha \in \tau \cap \beta \right\},
\]

where

\[
[(\omega_1 \cap M) + 1, \beta] := \{ \xi \in \beta + 1 : (\omega_1 \cap M) + 1 \leq \xi \}.
\]

We notice that

\[
(L_0 \cup L_1) \cap M = \emptyset.
\]

Take a number \( \overline{m} \in \omega \) such that

- for any \( \delta \in L_0 \cup L_1 \), \( \{ \eta_n^\delta : n \geq \overline{m} \} \cap M = \emptyset \),
- the set \( \{ \eta_n^\delta : n \geq \overline{m} \} : \delta \in \{ \omega_1 \cap M \} \cup L_0 \cup L_1 \) is pairwise disjoint,
- for any \( \delta \in \{ \omega_1 \cap M \} \cup L_0 \cup L_1 \) and any \( \{ \alpha, \beta \} \in [\tau]^2 \), if \( b(\alpha, \beta) \in \text{Lim} \setminus \{ \delta \} \), then

\[
\{ \eta_n^{b(\alpha, \beta)} : n \geq e_\beta(\alpha) \} \cap \{ \eta_n^\delta : n \geq \overline{m} \} = \emptyset,
\]

- \( \overline{m} > \max \{ e_\beta(\alpha), e_\beta(\omega_1 \cap M) : \beta \in \tau \setminus M, \alpha \in \tau \cap \beta \} \), and
- \( \overline{m} > \max \{ \sigma(\alpha, \beta) : \{ \alpha, \beta \} \in [\tau]^2 \} \).

Next, take an ordinal \( \overline{\xi} \in \omega_1 \cap M \) such that

- \( \tau \cap M \subseteq \overline{\xi} \),
- for any \( \{ \alpha, \beta \} \in [\tau]^2 \), if \( b(\alpha, \beta) \in \omega_1 \cap M \), then \( b(\alpha, \beta) < \overline{\xi} \), and
- if \( b(\alpha, \beta) \in \text{Lim} \setminus M \), then

\[
C_{b(\alpha, \beta)} \cap M = C_{b(\alpha, \beta)} \cap \overline{\xi},
\]

and

- for any \( \beta \in \tau \setminus M \) and any \( \zeta \in (\omega_1 \cap M) \setminus \overline{\xi} \), \( e_\beta(\zeta) > \overline{m} \).

Then we notice that

\[
\max \left( \bigcup \{ C_{b(\alpha, \beta)} \cap M : \{ \alpha, \beta \} \in [\tau]^2, b(\alpha, \beta) \geq \omega_1 \cap M \} \right) < \overline{\xi}.
\]

Let \( \beta_i^\tau : i \in n \) be the increasing enumeration of the set \( \tau \setminus M \). Define
\[ J := \{ \nu \in I : \quad \bullet \, \nu \cap \xi = \tau \cap M, \\
\bullet \, |\nu \setminus \xi| = n, \text{ and let } \{ \beta_i^\nu : i \in n \} \text{ be the increasing enumeration of the set } \nu \setminus \xi, \\
\bullet \, \text{for each } \alpha \in \tau \cap M \text{ and } i \in n, \\
C_b(\alpha, \beta_i^\nu) \cap \xi = C_b(\alpha, \beta_i^\nu) \cap \xi, \]
\[ \bullet \, \text{for each } \{i, j\} \in [n]^2, \quad C_b(\beta_i^\nu, \beta_j^\nu) \cap \xi = C_b(\beta_i^\nu, \beta_j^\nu) \cap \xi, \quad \text{and} \\
\bullet \, \text{for each } i \in n, \quad r_{\beta_i^\nu} \setminus \overline{m} = r_{\beta_i^\nu} \setminus \overline{m}. \]

Since \( \tau \in J \in M \) and \( \tau \not\in M \), \( J \) is uncountable. Moreover, we notice that, for any \( \nu \in J \),
\[ \overline{m} > \max \{ \sigma(\alpha, \beta) : \{\alpha, \beta\} \in [\nu \cup \tau]^2 \}. \]

Let \( \nu \in J \). Show that \( \nu \) satisfies the condition (2) of the lemma.

Let \( \alpha \in \tau \cap M \) and \( i \in n \). Then
\[ C_b(\alpha, \beta_i^\nu) \cap \xi = C_b(\alpha, \beta_i^\nu) \cap \xi. \]
Moreover,
\[ \text{either both } b(\alpha, \beta_i^\nu) < \omega_1 \cap M \text{ and } b(\alpha, \beta_i^\nu) < \xi \text{ hold,} \]
\[ \text{or both } b(\alpha, \beta_i^\nu) \in \text{Lim} \setminus M \text{ and } C_b(\alpha, \beta_i^\nu) \cap M = C_b(\alpha, \beta_i^\nu) \cap \xi \text{ hold.} \]

Therefore, for any \( \alpha, \alpha' \in \tau \cap M \) and any \( i, i' \in n \) with \( \langle \alpha, i \rangle \neq \langle \alpha', i' \rangle \), the pair of the sets \( \{\beta_i^\nu, \beta_i^{\nu'}\} \) and \( \{\alpha', \beta_j^\nu\} \) satisfies the condition (2).

Let \( \{i, j\} \in [n]^2 \). Then
\[ C_b(\beta_i^\nu, \beta_j^\nu) \cap \xi = C_b(\beta_i^\nu, \beta_j^\nu) \cap \xi. \]
Therefore, by a similar observation in the previous paragraph, for any \( \{i, j\} \in [n]^2 \), the pair of the sets \( \{\beta_i^\nu, \omega_1 \cap M\} \) and \( \{\beta_j^\nu, \beta_j^\nu\} \) satisfies the condition (2).

Let \( i \in n \). Then, for any \( \zeta \in [\beta_i^\nu, \omega_1 \cap M] \),
\[ e_{\beta_i^\nu}(\zeta) > \overline{m} \geq \sigma(\beta_i^\nu, \beta_j^\nu). \]
Moreover, in this case,
\[ \sigma(\beta_i^\nu, \beta_j^\nu) \geq e_{\beta_j^\nu}(\omega_1 \cap M). \]
Hence then, \( b(\beta_i^\nu, \beta_j^\nu) = \omega_1 \cap M \) and
\[ \left\{ \eta_n^{(\beta_i^\nu, \beta_j^\nu)} : n \geq e_{\beta_j^\nu}(\beta_j^\nu) \right\} \subseteq \left\{ \eta_n^{\omega_1 \cap M} : n \geq \overline{m} \right\}. \]

Therefore, by the third condition of the number \( \overline{m} \), for any \( \{\alpha, \beta\} \in [\tau]^2 \) and any \( i \in n \), the pair of the sets \( \{\beta_i^\nu, \beta_j^\nu\} \) and \( \{\alpha', \beta_j^\nu\} \) satisfies the condition (2).

Let \( \{i, j\} \in [n]^2 \). Then, by the previous observation,
\[ b(\beta_i^\nu, \beta_j^\nu) \left\{ \begin{array}{ll}
= \omega_1 \cap M & \text{if } \sigma(\beta_i^\nu, \beta_j^\nu) \geq e_{\beta_j^\nu}(\omega_1 \cap M), \\
\in \text{Lim} & \text{otherwise.}
\end{array} \right. \]
Therefore, for any \( \{\alpha, \beta\} \in [\tau]^2 \) and any \( \{i, j\} \in [n]^2 \), the pair of the sets \( \{\alpha, \beta\} \) and \( \{\beta_i^\nu, \beta_j^\nu\} \) satisfies the condition (2).

In the following proof, for each \( \tau \in [\omega_1]^{<\omega_0} \), define
\[ L(\tau) := \{ b(\alpha, \beta) : \{\alpha, \beta\} < \in [\tau]^2 \} \cap \text{Lim}. \]

\( L(\tau) \) is a finite set of ordinals. For each \( \tau \in [\omega_1]^{<\omega_0} \), \( m_\tau \) denotes the size of \( \tau \), and let \( \{\beta_i^\nu : i \in m_\tau \} \) be the increasing enumeration of \( \tau \).
Theorem 2.3. $\mathcal{K}^3_3$ implies $U(\text{stat})$.

Proof. Let $(C_\alpha : \alpha \in \omega_1 \cap \text{Lim})$ be a ladder system, and $(f_\alpha : \alpha \in \omega_1 \cap \text{Lim})$ a coloring of the ladder system $(C_\alpha : \alpha \in \omega_1 \cap \text{Lim})$. Define the set $K_0$ that consists of all sets $\{\alpha, \beta, \gamma\} \in [\omega_1]^3$ with the property that the set

$$\left( b(\alpha, \beta) \uparrow \left\{ \eta_n^{b(\alpha, \beta)} : n \geq e_\beta(\alpha) \right\} \cup \left( b(\alpha, \gamma) \uparrow \left\{ \eta_n^{b(\alpha, \gamma)} : n \geq e_\gamma(\alpha) \right\} \right)$$

forms a function. We show that $K_0$ is a ccc partition.

Let $I \subseteq [\omega_1]^{< \aleph_0} \cap \omega_1$ be an uncountable set of finite $K_0$-homogeneous sets, and $M$ a countable elementary submodel of $H_{\omega_1}$ that contains the set

$$\{ (e_\alpha : \alpha \in \omega_1), (r_\alpha : \alpha \in \omega_1), (C_\alpha : \alpha \in \omega_1 \cap \text{Lim}), (f_\alpha : \alpha \in \omega_1 \cap \text{Lim}), I, H_{\omega_1} \}.$$

By elementarity of $M$, we can take $\tau \in I \setminus M$ such that $\omega_1 \cap M \not\subseteq L(\tau)$, and can take $\overline{\eta} \in \omega_1 \cap M$ such that

$$\bigcup_{\delta \in L(\tau)} (C_\delta \cap M) \subseteq \overline{\eta}.$$

Define the subset $I'$ of the set $I$ that consists of all sets $\nu$ in $I$ such that

- $m_\nu = m_\tau$,
- for any $\{i, j\} \in [m_\tau]^2$,
  $$b(\beta_i^\tau, \beta_j^\tau) \in \text{Lim} \iff b(\beta_i^\tau, \beta_j^\tau) \in \text{Lim},$$
  and
- for any $\{i, j\} \in [m_\tau]^2$, whenever $b(\beta_i^\tau, \beta_j^\tau) \in \text{Lim} \setminus M$,
  $$f_b(\beta_i^\tau, \beta_j^\tau) \uparrow \overline{\eta} = f_b(\beta_i^\tau, \beta_j^\tau) \uparrow \overline{\eta}.$$

Then, since $\tau \in I' \subseteq M$ and $\tau \not\in M$, $I'$ is uncountable. By applying $I'$ and $\tau$ to Lemma 2.2, we obtain $J \subseteq [I']^{\aleph_1} \cap M$ that satisfies the condition in the lemma. Then we can conclude that, for each $\nu \in J$, $\nu \cup \tau$ is $K_0$-homogeneous.

By $\mathcal{K}^3_3$, there exists an uncountable $K_0$-homogeneous subset $X$ of $\omega_1$. Take a continuous $\in$-chain $\langle M_\xi : \xi \in \omega_1 \rangle$ of countable elementary submodels of $H_{\omega_1}$ such that $M_0$ contains the set

$$\{ (e_\alpha : \alpha \in \omega_1), (r_\alpha : \alpha \in \omega_1), (C_\alpha : \alpha \in \omega_1 \cap \text{Lim}), X \}.$$

Then the set $D := \{ \omega_1 \cap M_\xi : \xi \in \omega_1 \}$ is club in $\omega_1$. For each $\xi \in \omega_1$, by Lemma 2.1, there are $\beta_\xi \in X \setminus M_\xi$ and $\alpha_\xi \in X \cap M_\xi$ such that $b(\alpha_\xi, \beta_\xi) = \omega_1 \cap M_\xi$. Then there are $\overline{\alpha} \in \omega_1$ and a subset $\Gamma$ of $\omega_1$ such that

- for every $\xi \in \Gamma$, $\alpha_\xi = \overline{\alpha}$, and
- the set $S := \{ b(\overline{\alpha}, b_\xi) : \xi \in \Gamma \}$ is a stationary subset of $D$.

Then, since $X$ is $K_0$-homogeneous, the set

$$\bigcup_{\xi \in \Gamma} \left( f_b(\overline{\alpha}, b_\xi) \cup \left\{ \eta_n^{b(\overline{\alpha}, b_\xi)} : n \geq e_{b_\xi}(\overline{\alpha}) \right\} \right)$$

forms a function, and uniformizes the restricted coloring $(f_\delta : \delta \in S)$. \hfill \Box

Remark 2.4. It follows from the proof of the previous theorem that the forcing notion of finite $K_0$-homogeneous sets in the proof satisfies the property $R_{1, \omega_1}$ \cite{12, 14}, and so satisfies Chodounsky-Zapletal's $Y$-cc \cite{1}.
Remark 2.5. For a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$ and a coloring $\langle f_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$ of the ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$, define the set $K_0$ that consists of all sets $\tau$ in the set $[\omega_1]^4$ with the property that, for any $\{\{\alpha, \beta\}, \{\gamma, \delta\} \in [\tau]^2$, the set

$$\left( f_{b(\alpha, \beta)} \upharpoonright \left\{ n \geq e_\beta(\alpha) \right\} \right) \cup \left( f_{b(\gamma, \delta)} \upharpoonright \left\{ n \geq e_\delta(\gamma) \right\} \right)$$

forms a function. As in the previous proof, it also follows from Lemma 2.2 that $K_0$ is a ccc partition. By the previous proof, we notice that an uncountable $K_0$-homogeneous set produces a club subset $D$ of $\omega_1$ as above and a function that uniformizes the restricted coloring $\langle f_\delta : \delta \in D \rangle$. It concludes that $K_0'$ implies $U(\text{club})$.

Acknowledgement. I would like to thank Deigo A. Mejía for giving a useful idea of this article.

References


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