Singular integral operators bounded on Orlicz spaces and Orlicz type Hardy spaces

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Abstract

We study the singular integral operators which have standard kernels and which are bounded on Orlicz spaces. We generalize the T1 theorem on Orlicz spaces and investigate Their boundedness from the Orlicz type Hardy space H^1_Φ to L^1 and from L^∞ to the Orlicz type BMO (namely BMO_Φ). Furthermore, we prove that these operators can be extended to bounded operators on some homogenous Sovolev spaces.

1 Introduction

The Calderón–Zygmund singular integral operator T (CZ operator for a short) is a bounded linear operator on $L^2(\mathbb{R}^n)$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \qquad (x \notin \mathrm{supp}(f))$$

for all $f \in L^2(\mathbb{R}^n)$ with compact support, where $K(x,y)$ is a standard kernel, that is, a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ and satisfying following conditions:

$$(1)|K(x,y)| \le \frac{A_1}{|x-y|^n}, \tag{1.1}$$

$$(2)|K(x,y) - K(x,y_0)| \le \frac{A_2|y-y_0|^\delta}{|x-y|^{n+\delta}} \qquad \left(|y-y_0| < \frac{|x-y|}{2}\right), \tag{1.2}$$

$$(3)|K(x,y) - K(x_0,y)| \le \frac{A_3|x-x_0|^\delta}{|x-y|^{n+\delta}} \qquad \left(|x-x_0| < \frac{|x-y|}{2}\right). \tag{1.3}$$

Here $\delta \in (0,1]$ is a positive constant. In this paper, we denote $T \in \mathrm{CZO}(\delta)$ if T satisfies above conditions (1.1)−(1.3). It is well known that any CZ operator extends to a bounded linear operator on L^p for $1 < p < \infty$. CZ operators play an important role in harmonic analysis (see [2, 3]).

Orlicz spaces are introduced by Orlicz in [4, 5]. They generalize Lebesgue spaces, and are useful tools on harmonic analysis. As for the CZ operators on Orlicz spaces, Cianchi [1] gave a nesseary and sufficient conditions on Φ and Ψ for the boundedness from the Orlicz space L^Φ to another Orlicz space L^Ψ.

Definition 1.1. An $L^\Phi(\mathbb{R}^n)$ bounded linear integral operator T is said to be Φ-Calderón-Zygmund operator (Φ-CZ operator for a short) if $T \in \mathrm{CZO}(\delta)$ for some $0 < \delta \le 1$.

If $\Phi(r) = r^2$, then Φ-CZ operators are classical ones. In this sense, Φ-CZ operators generalize CZ operators.

In this paper, in Section 2, we recall some definitions and results of classical harmonic analysis, and in Section 3, we give a definition of Orlicz type BMO and Orlicz type Hardy space. Furthermore, we give some results of those spaces and Φ-CZ operators. Finally, in Section 4, we prove the following main theorem:

Theorem 1.2. *For a Young function* Φ *and a linear operator* $T : \mathcal{S} \to \mathcal{S}'$*, we have the following*

(i) *If* T *is a* Φ*-CZ operator with* $\varepsilon = \varepsilon(\Phi) < \delta/4$ *and* $T1, T^*1 \in \mathrm{BMO}$*, then* T *can be extended to a bounded operator from* $\dot{W}^{2,\varepsilon} \cap \dot{W}^{2,-\varepsilon}$ *to* $\dot{W}^{2,\varepsilon} + \dot{W}^{2,-\varepsilon}$.

(ii) *In particular, if* $\Phi \in \Delta_2 \cap \nabla_2$ *and* T *is a* Φ*-CZ operator, then* T *can be extended to a bounded operator on* L^2.

2 Preliminaries

A function $\Phi : [0, \infty] \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r \to +0} \Phi(r) = 0$ and $\lim_{r \to \infty} \Phi(r) = \infty$. There are many equivalent norms for Orlicz spaces. The following norm is called Luxenburg-Nakano norm.

Definition 2.1. For a Young function Φ and a measurable function f, we set

$$L^\Phi(\mathbb{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \ \text{ for some } k > 0 \right\}$$

$$||f||_\Phi := \inf \left\{ \lambda > 0 \mid \int_{\mathbb{R}^n} \Phi(\frac{|f(x)|}{\lambda})dx \leq 1 \right\}$$

The Orlicz space L^Φ is a Banach space with its norm.

For Young functions Φ and Ψ, we write $\Phi \simeq \Psi$ if there exists a constant C such that $\Phi(C^{-1}r) \leq \Psi(r) \leq \Phi(Cr)$. Note that, if $\Phi \simeq \Psi$, then $||f||_{L^\Phi} \sim ||f||_{L^\Psi}$.

For a Young function Φ, the complementary function $\tilde{\Phi}$ is defined by

$$\tilde{\Phi}(r) = \begin{cases} \sup \{ \, rs - \Phi(s) \mid s \in [0, \infty) \, \} & r \in [0, \infty), \\ \infty & r = \infty. \end{cases}$$

To consider Orlicz spaces (that is to say, to consider Young functions), there are two important conditions of Young functions. Those come up from speciality of spaces L^1 and L^∞.

Definition 2.2. For a Young function Φ,

(1) $\Phi \in \Delta_2$ if any $a > 1$, there exists a constant $C_a > 0$ such that $\Phi(ar) \leq C_a \Phi(r)$ for all $r \geq 0$,

(2) $\Phi \in \nabla_2$ if there exists a constant $k > 1$ such that $\Phi \leq \dfrac{1}{2k}\Phi(kr)$ for all $r \geq 0$.

Note that $\Phi \in \Delta_2 \Leftrightarrow \tilde{\Phi} \in \nabla_2$. It is well known that for a Young function Φ, if $\Phi \in \Delta_2$, we have $(L^\Phi)^* = L^{\tilde{\Phi}}$, that is to say, the dual space of L^Φ is equal to $L^{\tilde{\Phi}}$(see [15] for these results). From this, we obtain the following remark.

Remark 2.3. Let $\Phi \in \Delta_2$ and T be a Φ-CZ operator. Then, T^* is a $\tilde{\Phi}$-CZ operator.
Where T^* denotes the adjoint operator of T.

To make matters clearer, we give the following numbers depending on Young functions:

Definition 2.4. Let Φ be a Young function. Then, we define

$$p_+ = p_+(\Phi) := \inf \{ \, 1 \leq p \leq \infty \mid \Phi(\lambda r) \leq \lambda^p \Phi(r) \ \text{ for } r \geq 0, \ \lambda > 1 \, \},$$
$$p_- = p_-(\Phi) := \sup \{ \, 1 \leq p \leq \infty \mid \Phi(\lambda r) \leq \lambda^p \Phi(r) \ \text{ for } r \geq 0, \ 0 < \lambda \leq 1 \, \}.$$

Remark that, if $\Phi(r) \simeq r^p$, then $p_+ = p_- = p$.

Those numbers have an important relationship with ∇_2 and Δ_2, in the sense of the following lemma.

Lemma 2.5 ([14]). *For a Young function Φ, we have*

$$(1)\ p_+(\Phi) < \infty \ \Leftrightarrow\ \Phi \in \Delta_2$$
$$(2)\ p_-(\Phi) > 1 \ \Leftrightarrow\ \Phi \in \nabla_2$$

From this, if $\Phi \notin \nabla_2$, then L^Φ is possible to have a property close to L^1. Conversely, if $\Phi \notin \Delta_2$, then L^Φ is possible to have a property close to L^∞.

By an easy calculation, we have the following:

Lemma 2.6. *For a Young function Φ,*

$$\tilde{p}_\pm := p_\pm(\tilde{\Phi}) = p'_\mp,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ for $0 \le p \le \infty$.

We note some results on Orlicz spaces below. Refer to [15,16] for these results.

Lemma 2.7. *Let Φ be a Young function. Then, following conditions are equivalent.*

(1) $\Phi \in \nabla_2$.

(2) *The maximal operator M is bounded on L^Φ.*

Lemma 2.8. *Let Φ be a Young function and $\Psi_p^{-1}(s) := s^{1/p}\Psi^{-1}(s)$. Then, following conditions are equivalent.*

(1) *There exists $a > 0$ such that $\Psi_{\alpha/n}(r) \le \Phi(ar)$ for any $r > 0$, and*

$$\int_0^\varepsilon \frac{\Psi(t)}{t^{1+n/(n-\alpha)}} dt < \infty,$$

for enough small $\varepsilon > 0$.

(2) *The fractional maximal operator M_α is bounded from L^Φ to L^Ψ.*

Next, we recall definitions of Hardy spaces, BMO and their properties. At first, we define q-aroms, those are also called Hardy atoms.

Definition 2.9. Let $1 \le q \le \infty$. We say a function A is a q-atom if there exists a ball $B \subset \mathbb{R}^n$ and A satisfies following conditions:

$$(1)\ \operatorname{supp} A \subset B, \qquad (2)\ ||A||_q \le |B|^{1-\frac{1}{q}},$$
$$(3)\ \int A = 0.$$

Hereafter, we denote A_q by the set of all q-atoms.

Definition 2.10. Let $1 < q \le \infty$. For $f \in \mathcal{S}'$, $f \in H^1_{q,\text{atom}}$ if there exist $\{\lambda_j\} \in \ell^1$ and $\{a_j\} \subset A_q$ such that$f = \sum_{j=1}^\infty \lambda_j a_j$. Furthermore we define

$$||f||_{H^1_{q,\text{atom}}}(\mathbb{R}^n) = \inf \left\{ ||\lambda_j||_{\ell^1} \mid f = \sum_j \lambda_j a_j,\ a_j \in A_q \right\}.$$

Where infimum token over all of decompositions of f.

It is well known that $H^1_{q,\text{atom}}$ is independent from choice of $1 < q \le \infty$ (see [2] foe example). From this, we define $H^1_{q,\text{atom}}$ by H^1 again.

Definition 2.11. For a measurable function f, we define

$$(1) \qquad ||f||_{\mathrm{BMO}} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x)-f_B|dx.$$
$$(2) \qquad \mathrm{BMO}(\mathbb{R}^n) = \left\{ f\in L^1_{\mathrm{loc}} \mid ||f||_{\mathrm{BMO}} < \infty \right\}/\mathcal{P},$$

where $\mathcal{P}$ denotes the set of all polynomial functions.

The following lemma means the equivalence $\dot{F}^0_{\infty,2} = \mathrm{BMO}$. See [2] for example.

Lemma 2.12. *Let $\phi\in C^\infty_c(\mathbb{R}^n)$ be a function satisfies $\int\phi = 0$ and supported on unit ball on $\mathbb{R}^n$. Also, let $\phi_t(x) := \frac{1}{t^n}\phi(\frac{x}{t})$. Then, the norm*

$$||f||_{\dot{F}^0_{\infty,2}} = \sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_0^{r_B}\int_B |\phi_t * f(y)|^p dy\frac{dt}{t}\right)^{\frac{1}{p}}$$

is independent from choice of $1\le p<\infty$ and ϕ satisfying above conditions. Furthermore, we have

$$||f||_{\mathrm{BMO}} \sim ||f||_{\dot{F}^0_{\infty,2}}.$$

3 Φ-CZ singular operator,Orlicz type BMO

In this section, we give some results of Φ-CZ operators, Orlicz type Hardy spaces and Orlicz type BMO.

Remark 3.1. Let $\Phi\in\Delta_2\cap\nabla_2$ be a Young function and T be a CZ operator. Then T is also a Φ-CZ operator.

Remark 3.1 follows directly from the following theorem.

Theorem 3.2 ([6]). *Let $\Phi\in\Delta_2\cap\nabla_2$ be a Young function and T be a CZ operator. Then T is bounded on L^Φ.*

To consider about singular integral operators, "Weak Bounded Properties" have an important role. To define that, we prepare functions called bump functions.

Definition 3.3. A $C^{2[n/2]+2}$-function ϕ is said to be a bump function if it satisfies following conditions:

$$(1)\,\mathrm{supp}\,\phi\subset B(0,10) \qquad ,(2)|\partial^\alpha_x\phi(x)|\le 1 \quad (|\alpha|\le 2[n/2]+2).$$

Next, we give the definitions of the classical and generalized weakly bounded properties. Here and below, Let $p_+ = p_+(\Phi), p_- = p_-(\Phi)$ and $\tau_{x_0}(f)(x) := f(x-x_0)$, $f_R(x) := R^{-n}f(x/R)$ for $x\in\mathbb{R}^n$.

Definition 3.4. For a linear operator $T:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$, T is said to be Φ-weakly bounded (Φ-WB) if there exists a constant C such that

$$|\langle T\tau_{x_0}(f_R), \tau_{y_0}(g_R)\rangle| \le \begin{cases} CR^{-n+n(\frac{1}{p_-}-\frac{1}{p_+})} & (1<R) \\ CR^{-n+n(\frac{1}{p_+}-\frac{1}{p_-})} & (0<R\le 1), \end{cases} \tag{3.1}$$

for all bump functions f, g and $x_0, y_0\in\mathbb{R}^n$.

Remark that, if $\Phi(r)\simeq r^p$ for some $1<p<\infty$, then $T\in$ Φ-WB is equivalent to $T\in WB$, the classical Weak Boundedness.

Lemma 3.5. *Let T be a Φ-CZ operator. Then, $T\in$ Φ-WB.*

Proof. Let f, g be bump functions. Then, we have

$$
\begin{aligned}
||f_R||_\Phi &= R^{-n}\inf\left\{\lambda>0 \mid \int_{\mathbb{R}^n}\Phi(\frac{|f(\frac{x}{R})|}{\lambda})\le 1\right\}\\
&= R^{-n}\inf\left\{\lambda>0 \mid \int_{\mathbb{R}^n}R^n\Phi(\frac{|f(x)|}{\lambda})\le 1\right\}\\
&\le \begin{cases} R^{-n}\inf\left\{\lambda \mid \int_{\mathbb{R}^n}\Phi(R^{\frac{n}{p_-}}\frac{|f(x)|}{\lambda>0})\le 1\right\} & (1<R),\\ R^{-n}\inf\left\{\lambda \mid \int_{\mathbb{R}^n}\Phi(R^{\frac{n}{p_+}}\frac{|f(x)|}{\lambda>0})\le 1\right\} & (0<R\le 1),\end{cases}\\
&\le \begin{cases} R^{-n(1-\frac{1}{p_-})}||f||_\Phi & (1<R),\\ R^{-n(1-\frac{1}{p_+})}||f||_\Phi & (0<R\le 1).\end{cases}
\end{aligned}
$$

Keeping in ind that $||\tau(f)||_\Phi = ||f||_\Phi$, applying Lemma 2.6 by Hölder's inequality, we have

$$
\begin{aligned}
|\langle T\tau_{x_0}(f_R), \tau_{y_0}(g_R)\rangle| &\lesssim ||T\tau_{x_0}(f_R)||_\Phi ||\tau_{y_0}(g_R)||_{\tilde{\Phi}}\\
&\lesssim ||f_R||_\Phi ||g_R||_{\tilde{\Phi}}\\
&\le \begin{cases} R^{-2n+n(\frac{1}{p_-}+\frac{1}{p_-})}||f||_\Phi||g||_{\tilde{\Phi}} & (1<R),\\ R^{-2n+n(\frac{1}{p_+}+\frac{1}{p_+})}||f||_\Phi||g||_{\tilde{\Phi}} & (0<R\le 1)\end{cases}\\
&= \begin{cases} CR^{-n+n(\frac{1}{p_-}-\frac{1}{p_+})} & (1<R)\\ CR^{-n+n(\frac{1}{p_+}-\frac{1}{p_-})} & (0<R\le 1),\end{cases}
\end{aligned}
$$

As a result, we have the consulsion. □

Next, we define Orlicz type Hardy spaces and BMO.

Definition 3.6. A compactly supported integrable function A is said to be a Φ-*atom* if there exists a ball $B \subset \mathbb{R}^n$ and satisfying following conditions.
(1) $\operatorname{supp} A \subset B$,
(2) $||A||_\Phi \le ||\chi_B||_{\tilde{\Phi}}^{-1}$,
(3) $\int A = 0$.

Here and below, we denote A_Φ by the set of all Φ-*atoms*.

Definition 3.7. Let Φ be a Young function. We define

$$
H^1_\Phi(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \mid f = \sum_{j=1}^{\infty}\lambda_j a_j,\ \{\lambda_j\}\in l^1,\ \{a_j\}\subset A_\Phi \right\}
$$

$$
||f||_{H^1_\Phi} := \inf\left\{ ||\{\lambda_j\}||_{l^1} \mid f = \sum_{j=1}^{\infty}\lambda_j a_j,\ \{a_j\}\subset A_\Phi \right\}
$$

Note that for any Φ-atom, its L^1 norm is bounded. So that, we have $H^1_\Phi \subset L^1$.

Definition 3.8. Let Φ be a Young function.

$$
\begin{aligned}
&(1)\quad ||f||_{\mathrm{BMO}_\Phi} := \sup_B \frac{1}{||\chi_B||_\Phi}||f-f_B||_\Phi,\\
&(2)\quad \mathrm{BMO}_\Phi(\mathbb{R}^n) := \left\{ f\in L^1_{\mathrm{loc}}(\mathbb{R}^n) \mid ||f||_{\mathrm{BMO}_\Phi}<\infty \right\}/\mathcal{P}.
\end{aligned}
$$

If $\Phi(r) = r^p$ $(1 < p \le \infty)$, then $H^1_\Phi = H^1$. Furthermore, if $\Phi(r) = r^p$ $(1 \le p < \infty)$, then $\mathrm{BMO}_\Phi = \mathrm{BMO}$ by the John-Nirenberg inequality. Using Hölder's inequality, we can obtain the following easily.

Remark 3.9. For any Young function Φ, we have

$$\mathrm{BMO}_\Phi \subset \mathrm{BMO}.$$

The following lemma was proved by Guilyev [18, 2014]. However, it is covered by general equivalence proved by Ho [19, 2012].

Lemma 3.10. *If* $\Phi \in \Delta_2$*, then* $\mathrm{BMO}_\Phi = \mathrm{BMO}$*, that is*

$$||f||_{\mathrm{BMO}_\Phi} \sim ||f||_{\mathrm{BMO}}.$$

Theorem 3.11. *Let* $\Phi \in \Delta_2$*. If* T *is a* Φ*-CZ operator, then* T^* *is bounded from* L^∞ *to* $\mathrm{BMO}_{\tilde{\Phi}}$*.*

To prove Theorem 3.11, we need following two lemmas.

Lemma 3.12. *Let* Φ *be a Young function and* T *be a* Φ*-CZ operator. Then,* T *is bounded from* H^1_Φ *to* L^1*.*

Proof. From $f \in H^1_\Phi$ can be decomposed into the sum of Φ-*atom*s, considering the boundedness of Φ-*atom*. Let $a(x)$ be a Φ-*atom* bounded on B and $B^* := 2B$. Then,

$$\begin{aligned}\int_{\mathbb{R}^n} |Ta(x)|dx &= \int_{B^*} |Ta(x)|dx + \int_{(B^*)^c} |Ta(x)|dx \\ &= I + \mathit{II}.\end{aligned}$$

Thanks to Hölder's inequality, the size condition of $a(x)$ and $||\chi_{B^*}|| = (\Phi^{-1}(\frac{1}{|B^*|}))^{-1} \le 2^n(\Phi^{-1}(\frac{1}{|B|}))^{-1}$, we have

$$\begin{aligned}I &\lesssim ||\chi^*_B||_{\tilde{\Phi}} ||a||_{\Phi(B)} \\ &\le \tilde{\Phi}^{-1}(\frac{1}{|B|})/\Phi^{-1}(\frac{1}{|B^*|}) \\ &\lesssim 1.\end{aligned}$$

Next, we estimate II. By (1.2),we obtain

$$\begin{aligned}\mathit{II} &= \int_{(B^*)^c} \left| \int_B K(x,y)a(y)dy \right| dx \\ &= \int_{(B^*)^c} \left| \int_B K(x,y) - K(x,x_0)a(y)dy \right| dx \\ &\le \int_{(B^*)^c} (\int_B \frac{A_2|y-x_0|^\delta}{|x-y|^{n+\delta}} |a(y)|dy)dx \\ &\le A_2 r^\delta ||a||_{L^1(B)} \int_{|x-x_0|\ge 2r} \frac{dx}{|x-x_0|^{n+\delta}} \\ &\lesssim 1.\end{aligned}$$

□

Let L^Φ_{comp} be the set of all L^Φ-functions with compact support.

Lemma 3.13. *Let* $\Phi \in \Delta_2$*. Then the dual of* H^1_Φ *is* $\mathrm{BMO}_{\tilde{\Phi}}$*. More precisely, we have the following assertions:*

(1) *If* $b \in \mathrm{BMO}_{\tilde{\Phi}}$*, then the mapping*

$$\ell = \ell_b : f \in L^\Phi_{\mathrm{comp}} \mapsto \int f(x)b(x)dx \in \mathbb{C}$$

can be extended to a bounded linear functional on H^1_Φ*. Also, we have*

$$||\ell|| \lesssim ||b||_{\mathrm{BMO}_{\tilde{\Phi}}}.$$

(2) *Conversely, if* l *is a continuous liner functional on* $H^1_{\tilde{\Phi}}$*, then there exists* $b \in \mathrm{BMO}_{\tilde{\Phi}}$ *such that* $\ell(f) = \int f(x)b(x)dx$ *for all* L^{Φ}_{comp} *and that*

$$||b||_{\mathrm{BMO}_{\tilde{\Phi}}} \lesssim ||\ell||.$$

Proof. (1) Let A be any Φ-*atom* supported on B. From the moment condition of A,

$$\ell_b(A) = \int bA = \int_B (b - b_B)A.$$

From Hölder's inequity on Orlicz spaces,

$$|\ell_b(A)| \lesssim ||b - b_B||_{\tilde{\Phi}} ||A||_{\Phi} \leq ||b||_{\mathrm{BMO}_{\tilde{\Phi}}}.$$

So, we have $\ell_b \in (H^1_{\Phi})^*$.

(2) Let $\ell \in (H^1_{\Phi})^*$, $B_j := B(0, 2^j)$, and define

$$L^{\Phi}_0(B_j) := \left\{ f \in L^{\Phi}(B_j) \mid \int f = 0 \right\}.$$

Then $\ell : L^{\Phi}(B_j)_0 \to \mathbb{C}$ is a bounded linear operator. In fact, if $\operatorname{supp} f \in B_j$,

$$|\ell(f)| \leq ||\ell||_{(H^1_{\Phi})^*} ||f||_{H^1_{\Phi}} \lesssim ||\ell||_{(H^1_{\Phi})^*} ||\chi_{B_j}||_{\tilde{\Phi}} ||f||_{L^{\Phi}(B_j)}$$

From $\Phi \in \Delta_2$, the dual of $L^{\Phi}_0(B_j)$ is $L^{\tilde{\Phi}}_0(B_j)$. So, there exists a unique $g_j \in L^{\tilde{\Phi}}_0(B_j)$ which satisfies the following condition for all $f \in L^{\Phi}_0(B_j)$

$$\ell(f) = \int g_j f \ , \qquad ||g_j||_{L^{\tilde{\Phi}}} \lesssim ||\ell||_{(H^1_{\Phi})^*} ||\chi_{B_j}||_{\Phi}$$

For $j < k$ and $f \in L^{\Phi}_0(B_j)$, we have

$$\int (g_k - m_{B_j}(g_k)) f = \int g_k f.$$

Thanks to the uniqueness of g_j, we have

$$g_j = (g_k - m_{B_j}(g_k))\chi_{B_j}.$$

Let $h_j := g_j - m_{Q(1)(g_j)}$. By (2), for $1 \leq j < k$ and $x \in B_j$ we have

$$h_k - h_j = g_k - g_j - m_{B_1}(g_k) + m_{B_1}(g_j) = 0.$$

Let

$$g = \begin{cases} \lim_{j\to\infty} h_j & \text{(if the limit exists)}, \\ 0 & \text{(otherwise)}. \end{cases}$$

Next, we prove $g \in \mathrm{BMO}_{\tilde{\Phi}}$. For any ball $B \subset \mathbb{R}^n$, choose j satisfying $B \subset B_j$. Then, we have

$$||g_j - m(g_j)||_{L^{\tilde{\Phi}}} \leq ||g_j||_{\tilde{\Phi}} \lesssim ||\ell||_{(H^1_{\Phi})^*} ||\chi_{B_j}||_{\Phi}$$

From this, we have $g \in \mathrm{BMO}_{\tilde{\Phi}}$. □

Proof of Theorem 3.11. Immediately from Lemma 3.12 and Lemma 3.13. □

4 Proof of Main teorems

To prove Theorem 1.2 (i), we prepare the following proposition:

Proposition 4.1. *For a singular integral operator* $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, *If* $\varepsilon = n(\frac{1}{p_+} - \frac{1}{p_-}) < \frac{\delta}{4}$ *and* $T \in \Phi\text{-}WB$ *and* $T1, T^*1 \in \mathrm{BMO}$, *then* T *extends to a bounded operator from* $\dot{W}^{2,\varepsilon} \cap \dot{W}^{2,-\varepsilon}$ *to* $\dot{W}^{2,\varepsilon} + \dot{W}^{2,-\varepsilon}$.

We leave proof by the convenience of pages.

Proof of Theorem 1.2 (i). Immediately from Proposition 4.1 and Lemma 3.5. □

Next, we prove Theorem 1.2 (ii). At first, we prove the interpolation theorem on Orlicz spaces.

Theorem 4.2. *Let* $\Phi, \Psi \in \Delta_2 \cap \nabla_2$ *be Young functions such that* $p_+(\Psi) < p_-(\Phi)$ *and* T *be a* Φ*-CZ operator. Then, for* $f \in L^\Phi \cap L^\Psi$, *we have* $||Tf||_\Psi \lesssim ||f||_\Psi$.

To prove this theorem, we need some lemmas and a theorem.

Definition 4.3. Let Φ be a Young function and f be a measurable function. Then, we define

$$M_\Phi(f)(x) = \sup_{Q \ni x} \frac{1}{||\chi_Q||_\Phi} ||f||_{L^\Phi(Q)},$$

where Q moves over all cubes containing x.

Here and below, let M be a Hardy–Littlewood maximal operator. Remark that if a Young function Φ satisfies $\Phi \in \nabla_2$, then M is bounded on L^Φ[15,16].

Lemma 4.4. *For a Young function* $\Phi \in \Delta_2$ *and any non-negative functions, we have*

$$(M(f^{p_-})(x))^{1/p_-} \lesssim M_\Phi(f)(x) \lesssim (M(f^{p_+}))^{1/p_+},$$

for all $x \in \mathbb{R}^n$.

We leave the proof again.

Lemma 4.5. *For Young functions* $\Phi \in \Delta_2$ *and* $\Psi \in \nabla_2$, *satisfying* $p_+(\Phi) < p_-(\Psi)$, M_Φ *is bounded on* L^Ψ.

Proof. From Lemma 3.15, we have

$$\begin{aligned} ||M_\Phi(f)||_\Psi &\leq ||(M(f^{p_+(\Phi)}))^{\frac{1}{p_+(\Phi)}}||_\Psi \\ &= (||M(f^{p_+(\Phi)})||_{\Psi^{\frac{1}{p_+(\Phi)}}})^{\frac{1}{p_+(\Phi)}}, \end{aligned}$$

where $\Psi^q(r) := \Psi(r^q)$. Then, by an easy calculation, we obtain

$$p_-(\Psi^{\frac{1}{p_+(\Phi)}}) = \frac{p_-(\Psi)}{p_+(\Phi)} > 1.$$

Consequently $\Psi^{\frac{1}{p_+(\Phi)}} \in \nabla_2$, that is M is bounded on $L^{\Psi^{\frac{1}{p_+(\Phi)}}}$. As result,

$$\begin{aligned} (||M(f^{p_+(\Phi)})||_{\Psi^{\frac{1}{p_+(\Phi)}}})^{\frac{1}{p_+(\Phi)}} &\lesssim (||f^{p_+(\Phi)}||_{\Psi^{\frac{1}{p_+(\Phi)}}})^{\frac{1}{p_+(\Phi)}} \\ &= ||f||_\Psi. \end{aligned}$$

□

Finally, we invoke the following theorem:

Theorem 4.6 ([17]). *Let $d \in \mathbb{N} \cup \{0\}$. Assume that $\Phi \in \Delta_2 \cap \nabla_2$. Then, the following are equivalent:*

(i) $F \in L^{\Phi}(\mathbb{R}^n)$.

(ii) *There exist a sequence of functions $\{a_j\}_{j=1}^{\infty}$, a sequence of non-negative numbers $\{\lambda\}_{j=1}^{\infty}$, a sequence of cubes $\{Q_j\}_{j=1}^{\infty}$ with the following properties:*

(a) $|| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} ||_{\Phi} < \infty$,

(b) $\mathrm{supp}(a_j) \subset Q_j$,

(c) $||a_j||_{\infty} \leq 1$,

(d) $\int_{Q_j} a(x) x^{\alpha} dx = 0$ *for all multi-index α satisfying $|\alpha| < d$,*

(e) $f = \sum_{j=1}^{\infty} \lambda_j a_j$ *in $L^{\Phi}(\mathbb{R}^n)$.*

Proof of Theorem 4.2. Decompose $f \in L^{\Phi} \cap L^{\Psi}$ as in Theorem 4.6. Remark that we archive $|| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} ||_{\Phi} \lesssim ||f||_{\Phi}$. Then, for Φ-CZ operator T, we get

$$\begin{aligned} Tf(x) &= \sum_{j=1}^{\infty} \lambda_j T a_j(x) \\ &= \sum_{j=1}^{\infty} \lambda_j T a_j(x) \chi_{2Q_j}(x) + \sum_{j=1}^{\infty} \lambda_j T a_j(x) \chi_{\mathbb{R}^n \setminus 2Q_j}(x) \\ &:= I(x) + I\!I(x), \end{aligned}$$

in $L^{\Phi} \cap L^{\Psi}$. At first, we estimate $I(x)$. Let g be a function such that $g \in L^{\widetilde{\Psi}}$ with $||g||_{\widetilde{\Psi}} = 1$. Then, from Hölder's inequality and the boundedness of an Orlicz type maximal operator $M_{\tilde{\Phi}}$ on $L^{\widetilde{\Psi}}$ (assumption and Lemma 4.5), we have

$$\begin{aligned} \int_{\mathbb{R}^n} |I(x) \cdot g(x)| dx &\leq \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} |\lambda_j \chi_{2Q_j}(x)| |T a_j(x) \cdot g(x)| dx \\ &\lesssim \sum_{j=1}^{\infty} |\lambda_j| ||T a_j \chi_{2Q_j}||_{\Phi} ||g \chi_{2Q_j}||_{\tilde{\Phi}} \\ &\lesssim \sum_{j=1}^{\infty} |\lambda_j| ||a_j||_{\infty} \frac{1}{||\chi_{2Q_j}||_{\tilde{\Phi}}} ||g \chi_{2Q_j}||_{\tilde{\Phi}} |2Q_j| \\ &\lesssim \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} |\lambda_j| \chi_{2Q_j} M_{\tilde{\Phi}}(g)(x) dx \\ &\lesssim || \sum_{j=1}^{\infty} |\lambda_j| \chi_{2Q_j} ||_{\Psi} ||M_{\tilde{\Phi}}(g)||_{\widetilde{\Psi}} \\ &\lesssim ||g||_{\widetilde{\Psi}} = 1 \end{aligned}$$

Next, we estimate $I\!I(x)$. Since $\int a_j = 0$, we get

$$T a_j(x) = \int_{Q_j} K(x,y) a_j(y) dy = \int_{Q_j} (K(x,y) - K(x,x_j)) a_j(y) dy,$$

where x_j is the center of Q_j. Keeping in mind that $||\chi_{2^kB}||_\Phi = 1/\Phi(\frac{1}{|2^kB|}) \leq 2^{\frac{k}{p_+}}||\chi_B||$ for $k \geq 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus 2Q_j} |I\!I(x)\cdot g(x)|dx &= \sum_{k=1}^{\infty}\int_{2^{k+1}Q_j\setminus 2^kQ_j} |I\!I(x)\cdot g(x)|dx \\
&\leq \sum_{k=1}^{\infty}\int_{2^{k+1}\setminus 2^kQ_j}\left(\sum_{j=1}^{\infty}|\lambda_j|\int_{Q_j}|K(x,y)-K(x,x_j)||a_j(y)|dy\right)|g(x)|dx \\
&\lesssim \sum_{k=1}^{\infty}\int_{2^{k+1}Q_j\setminus 2^kQ_j}\left(\sum_{j=1}^{\infty}|\lambda_j|\int_{Q_j}\frac{|y-x_0|^\delta}{|x-x_j|^{n+\delta}}|a_j(y)|dy\right)|g(x)|dx \\
&\lesssim \sum_{k=1}^{\infty}\frac{1}{2^{kn}}\sum_{j=1}^{\infty}\int_{2^{k+1}Q_j}|\lambda_j||g(x)|dx \\
&\lesssim \sum_{k=1}^{\infty}\frac{1}{2^{kn(1-1/p_+(\Psi))}}||\sum_{j=1}^{\infty}\lambda_j\chi_{Q_j}||_\Psi||g(x)||_{\widetilde{\Psi}} \\
&\lesssim 1.
\end{aligned}$$

Thus, we have $Tf \in L^\Psi$, together with the desired estimate. □

Theorem 4.7 ([13]). *For a singular integral operator* $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$*, If* $T \in WB$ *and* $T1, T^*1 \in \mathrm{BMO}$*, then* T *extends to a bounded operator on* L^2.

Proof of Theorem 1.2 (ii). Let $\Phi \in \Delta_2 \cap \nabla_2$ and T be a Φ-CZ operator. From Theorem 4.2 and $p_- > 1$, there exists $p > 1$ such that T is bounded on L^p. Thus, we have $T \in \mathrm{WB}$. Furthermore, from Theorem 3.11 and Lemma 3.10, we have $T1 \in \mathrm{BMO}_\Phi = \mathrm{BMO}$ and $T^*1 \in \mathrm{BMO}_{\tilde{\Phi}} = \mathrm{BMO}$. Then, applying theorem 4.7, we have the desired result. □

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