Fractional integrals and their commutators on martingale Orlicz spaces

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1 Introduction

This is an announcement of [2].

It is well known as the Hardy-Littlewood-Sobolev theorem that the fractional integral operators I_α on the Euclidean space $\mathbb{R}^n$ is bounded from L_p to L_q for $1 < p < q < \infty$, $0 < \alpha < n$ and $-n/p + \alpha = -n/q$. For any BMO function b, Chanillo [4] proved the same boundedness of the commutator $[b, I_\alpha]$. Paluszyński [19] proved that, for any β-Lipschitz function b, $0 < \beta < 1$, the commutator $[b, I_\alpha]$ is bounded from L_p to L_q for $-n/p+\alpha+\beta = -n/q$ and from L_p to the Triebel-Lizorkin space $\dot{F}^{\beta}_{p,\infty}$.

In martingale theory, based on the result by Watari [23, Theorem 1.1], Chao and Ombe [5] proved the boundedness of the fractional integrals for H_p, L_p, BMO and Lipschitz spaces of the dyadic martingales. These fractional integrals were defined for more general martingales in [14, 20] and studied in [6, 15, 16]. In this paper we investigate the fractional integrals on martingale Orlicz spaces.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_n\}_{n\ge 0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. We suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms, where $B \in \mathcal{F}_n$ is called an atom (more precisely a $(\mathcal{F}_n, P)$-atom), if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $P(A) = P(B)$ or $P(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n, respectively.

We say a sequence $(f_n)_{n\geq 0}$ in L_1 is a martingale relative to $\{\mathcal{F}_n\}_{n\geq 0}$ if it is adapted to $\{\mathcal{F}_n\}_{n\geq 0}$ and satisfies $E_n[f_m] = f_n$ for every $n \leq m$. It is known as the Doob theorem that, if $p \in (1, \infty)$, then any L_p-bounded martingale converges in L_p. Moreover, if $p \in [1, \infty)$, then, for any $f \in L_p$, its corresponding martingale $(f_n)_{n\geq 0}$ with $f_n = E_n f$ is an L_p-bounded martingale and converges to f in L_p (see for example [17]). For this reason a function $f \in L_1$ and the corresponding martingale $(f_n)_{n\geq 0}$ will be denoted by the same symbol f.

We first recall the definition of generalized fractional integrals of martingales.

Definition 1.1 ([16])**.** Let $(\gamma_n)_{n\geq 0}$ be a non-increasing sequence of non-negative bounded functions adapted to $\{\mathcal{F}_n\}_{n\geq 0}$. For a martingale $(f_n)_{n\geq 0}$, its generalized fractional integral $I_\gamma f = ((I_\gamma f)_n)_{n\geq 0}$ is defined as a martingale by

$$(I_\gamma f)_n = \sum_{k=0}^{n} \gamma_{k-1}(f_k - f_{k-1})$$

with convention $\gamma_{-1} = \gamma_0$ and $f_{-1} = 0$.

Our definition of I_γ is based on the notion of martingale transform in the sense of Burkholder [3]. For quasi-normed spaces M_1 and M_2 of functions, we denote by $B(M_1,\, M_2)$ the set of all bounded martingale transforms from M_1 to M_2, that is, $I_\gamma \in B(M_1,\, M_2)$ means that

$$\sup_{n\geq 0} \|(I_\gamma f)_n\|_{M_2} \leq C \sup_{n\geq 0} \|f_n\|_{M_1},$$

for all M_1-bounded martingales $f = (f_n)_{n\geq 0}$.

Let

$$\beta_n = \sum_{B\in A(\mathcal{F}_n)} P(B)\chi_B, \quad n = 0, 1, 2, \cdots . \tag{1.1}$$

For $\alpha > 0$, let $\gamma_n = \beta_n^\alpha$, $n \geq 0$. Then $I_\gamma f$ is the fractional integral of f introduced in [14].

In this paper we prove $I_\gamma \in B(L_\Phi, L_\Psi)$ for the Orlicz spaces L_Φ and L_Ψ under suitable conditions. Moreover, we consider the commutator $[b, I_\gamma]$ generated by a function b. For $f \in L_\infty$, which is regarded as an L_∞-bounded martingale $f = (f_n)_{n\geq 0}$ with $f_n = E_n f$, $((I_\gamma f)_n)_{n\geq 0}$ is also an L_∞-bounded martingale. We denote by $I_\gamma f$ the limit function, that is, $I_\gamma f = ((I_\gamma f)_n)_{n\geq 0}$. In this case the commutator $[b, I_\gamma]f = bI_\gamma f - I_\gamma(bf)$ is well-defined for all $b \in L_\infty$. In this paper we prove that, for functions b in Campanato spaces and $f \in L_\Phi$, $[b, I_\gamma]f$ is well-defined and bounded from L_Φ to L_Ψ under suitable conditions.

The definition of the Campanato space is the following:

Definition 1.2. For $p \in [1,\infty)$ and $\psi : (0,1] \to (0,\infty)$, let

$$\mathcal{L}^-_{p,\psi} = \{f \in L_p : \|f\|_{\mathcal{L}^-_{p,\psi}} < \infty\},$$

where

$$\|f\|_{\mathcal{L}^-_{p,\psi}} = \sup_{n\geq 0} \sup_{B\in A(\mathcal{F}_n)} \frac{1}{\psi(P(B))} \left(\frac{1}{P(B)} \int_B |f - E_{n-1}f|^p \, dP \right)^{1/p}.$$

We say that a function $\theta : (0,1] \to (0,\infty)$ satisfies the doubling condition if there exists a positive constant C such that, for all $r, s \in (0,1]$,

$$\frac{1}{C} \leq \frac{\theta(r)}{\theta(s)} \leq C, \quad \text{if } \frac{1}{2} \leq \frac{r}{s} \leq 2. \tag{1.2}$$

We say that θ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $r, s \in (0,1]$,

$$\theta(r) \leq C\theta(s) \quad (\text{resp. } \theta(s) \leq C\theta(r)), \quad \text{if } r < s. \tag{1.3}$$

The stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is said to be regular, if there exists a constant $R \geq 2$ such that

$$f_n \leq R f_{n-1} \tag{1.4}$$

holds for all $n \geq 1$ and all nonnegative martingales $(f_n)_{n\geq 0}$.

It is known by [12, Theorem 2.9] that, if $\{\mathcal{F}_n\}_{n\geq 0}$ is regular and ψ is almost increasing, then

$$\|f\|_{\mathcal{L}^-_{1,\psi}} \leq \|f\|_{\mathcal{L}^-_{p,\psi}} \leq C_p \|f\|_{\mathcal{L}^-_{1,\psi}}. \tag{1.5}$$

2 Orlicz spaces

First we define a set $\bar{\Phi}$ of increasing functions $\Phi : [0,\infty] \to [0,\infty]$ and give some properties of functions in $\bar{\Phi}$.

For an increasing function $\Phi : [0,\infty] \to [0,\infty]$, let

$$a(\Phi) = \sup\{t \geq 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \geq 0 : \Phi(t) = \infty\},$$

with convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. Then $0 \leq a(\Phi) \leq b(\Phi) \leq \infty$. Let $\bar{\Phi}$ be the set of all increasing functions $\Phi : [0,\infty] \to [0,\infty]$ such that

$$0 \leq a(\Phi) < \infty, \quad 0 < b(\Phi) \leq \infty, \tag{2.1}$$

$$\lim_{t\to+0} \Phi(t) = \Phi(0) = 0, \tag{2.2}$$

$$\Phi \text{ is left continuous on } [0, b(\Phi)), \tag{2.3}$$

$$\text{if } b(\Phi) = \infty, \text{ then } \lim_{t\to\infty} \Phi(t) = \Phi(\infty) = \infty, \tag{2.4}$$

$$\text{if } b(\Phi) < \infty, \text{ then } \lim_{t\to b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) \ (\leq \infty). \tag{2.5}$$

In what follows, if an increasing and left continuous function $\Phi : [0,\infty) \to [0,\infty)$ satisfies (2.2) and $\lim_{t\to\infty} \Phi(t) = \infty$, then we always regard that $\Phi(\infty) = \infty$ and that $\Phi \in \bar{\mathit{\Phi}}$.

Definition 2.1. A function $\Phi \in \bar{\mathit{\Phi}}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is convex on $[0, b(\Phi))$.

By the convexity, any Young function Φ is continuous on $[0, b(\Phi))$ and strictly increasing on $[a(\Phi), b(\Phi)]$. Hence Φ is bijective from $[a(\Phi), b(\Phi)]$ to $[0, \Phi(b(\Phi))]$. Moreover, Φ is absolutely continuous on any closed subinterval in $[0, b(\Phi))$. That is, its derivative Φ' exists a.e. and

$$\Phi(t) = \int_0^t \Phi'(s)\, ds, \quad t \in [0, b(\Phi)). \tag{2.6}$$

For $\Phi, \Psi \in \bar{\mathit{\Phi}}$, we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \le \Psi(t) \le \Phi(Ct) \quad \text{for all } t \in [0,\infty].$$

Definition 2.2. (i) Let $\mathit{\Phi}_Y$ be the set of all Young functions.

(ii) Let $\bar{\mathit{\Phi}}_Y$ be the set of all $\Phi \in \bar{\mathit{\Phi}}$ such that $\Phi \approx \Psi$ for some $\Psi \in \mathit{\Phi}_Y$.

(iii) Let $\mathcal{Y}$ be the set of all $\Phi \in \mathit{\Phi}_Y$ such that $a(\Phi) = 0$ and $b(\Phi) = \infty$.

For $\Phi \in \bar{\mathit{\Phi}}$, we recall the generalized inverse of Φ in the sense of O'Neil [18, Definition 1.2].

Definition 2.3. For $\Phi \in \bar{\mathit{\Phi}}$ and $u \in [0,\infty]$, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \ge 0 : \Phi(t) > u\}, & u \in [0,\infty), \\ \infty, & u = \infty. \end{cases} \tag{2.7}$$

Let $\Phi \in \bar{\mathit{\Phi}}$. Then Φ^{-1} is finite, increasing and right continuous on $[0,\infty)$ and positive on $(0,\infty)$. If Φ is bijective from $[0,\infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ. Moreover, we have the following proposition, which is a generalization of Property 1.3 in [18].

Proposition 2.1 ([22]). *Let $\Phi \in \bar{\mathit{\Phi}}$. Then*

$$\Phi(\Phi^{-1}(u)) \le u \le \Phi^{-1}(\Phi(u)) \quad \text{for all } u \in [0,\infty]. \tag{2.8}$$

For functions $P, Q : [0,\infty] \to [0,\infty]$, we write $P \sim Q$ if there exists a positive constant C such that

$$C^{-1}P(t) \le Q(t) \le CP(t) \quad \text{for all } t \in [0,\infty].$$

Then, for $\Phi, \Psi \in \bar{\mathit{\Phi}}$,

$$\Phi \approx \Psi \quad \Leftrightarrow \quad \Phi^{-1} \sim \Psi^{-1}. \tag{2.9}$$

For a Young function Φ, its complementary function is defined by

$$\widetilde{\Phi}(t) = \begin{cases} \sup\{tu - \Phi(u) : u \in [0,\infty)\}, & t \in [0,\infty), \\ \infty, & t = \infty. \end{cases}$$

Then $\widetilde{\Phi}$ is also a Young function, and $(\Phi, \widetilde{\Phi})$ is called a complementary pair. For example, $\Phi(t) = t$, then

$$\widetilde{\Phi}(t) = \begin{cases} 0, & t \in [0,1], \\ \infty, & t \in (1,\infty]. \end{cases}$$

Definition 2.4. For a function $\Phi \in \bar{\mathit{\Phi}}_Y$, let

$$\begin{aligned} L_\Phi &= \left\{ f \in L^0 : E[\Phi(\epsilon|f|)] < \infty \text{ for some } \epsilon > 0 \right\}, \\ \|f\|_{L_\Phi} &= \inf\left\{ \lambda > 0 : E[\Phi(|f|/\lambda)] \le 1 \right\}, \\ \mathrm{w}L_\Phi &= \left\{ f \in L^0 : \sup_{t \in (0,\infty)} \Phi(t) P(\epsilon f, t) < \infty \text{ for some } \epsilon > 0 \right\}, \\ \|f\|_{\mathrm{w}L_\Phi} &= \inf\left\{ \lambda > 0 : \sup_{t \in (0,\infty)} \Phi(t) P(f/\lambda, t) \le 1 \right\}, \\ &\text{where} \quad P(f,t) = P(\{\omega \in \Omega : |f(\omega)| > t\}). \end{aligned}$$

Remark 2.1. It is known that

$$\sup_{t \in (0,\infty)} \Phi(t) P(f,t) = \sup_{t \in (0,\infty)} t\, P(\Phi(|f|), t), \tag{2.10}$$

see [7, Proposition 4.2] for example.

Let $(\Phi, \widetilde{\Phi})$ be a complementary pair of functions in $\mathit{\Phi}_Y$. Then it is known that

$$t \le \Phi^{-1}(t) \widetilde{\Phi}^{-1}(t) \le 2t, \quad t \in [0,\infty]. \tag{2.11}$$

It is also known that

$$E[|fg|] \le 2\|f\|_{L_\Phi} \|g\|_{L_{\widetilde{\Phi}}}. \tag{2.12}$$

Lemma 2.2. *Let $\Phi \in \mathit{\Phi}_Y$. Then, for all $A \in \mathcal{F}$, its characteristic function χ_A is in $\mathrm{w}L_\Phi$ and*

$$\|\chi_A\|_{L_\Phi} = \|\chi_A\|_{\mathrm{w}L_\Phi} = \frac{1}{\Phi^{-1}(1/P(A))}. \tag{2.13}$$

Definition 2.5. (i) A function $\Phi \in \bar{\mathit{\Phi}}$ is said to satisfy the Δ_2-condition, denote $\Phi \in \bar{\Delta}_2$, if there exists a constant $C > 0$ such that

$$\Phi(2t) \le C\Phi(t) \quad \text{for all } t > 0. \tag{2.14}$$

(ii) A function $\Phi \in \bar{\varPhi}$ is said to satisfy the ∇_2-condition, denote $\Phi \in \bar{\nabla}_2$, if there exists a constant $k > 1$ such that

$$\Phi(t) \le \frac{1}{2k}\Phi(kt) \quad \text{for all } t > 0. \tag{2.15}$$

(iii) Let $\Delta_2 = \varPhi_Y \cap \bar{\Delta}_2$ and $\nabla_2 = \varPhi_Y \cap \bar{\nabla}_2$.

Remark 2.2. (i) $\Delta_2 \subset \mathcal{Y}$ and $\bar{\nabla}_2 \subset \bar{\varPhi}_Y$ ([10, Lemma 1.2.3]).

(ii) Let $\Phi \in \bar{\varPhi}_Y$. Then $\Phi \in \bar{\Delta}_2$ if and only if $\Phi \approx \Psi$ for some $\Psi \in \Delta_2$, and, $\Phi \in \bar{\nabla}_2$ if and only if $\Phi \approx \Psi$ for some $\Psi \in \nabla_2$.

(iii) Let $\Phi \in \varPhi_Y$. Then $\Phi \in \Delta_2$ if and only if the set of simple functions is dense in L_Φ.

(iv) Let $\Phi \in \varPhi_Y$. Then Φ^{-1} satisfies the doubling condition by its concavity, that is,

$$\Phi^{-1}(u) \le \Phi^{-1}(2u) \le 2\Phi^{-1}(u) \quad \text{for all } u \in [0, \infty].$$

(v) If $\Phi \in \bar{\nabla}_2$, then there exists $\theta \in (0,1)$ such that $\Phi((\cdot)^\theta) \in \bar{\nabla}_2$ ([22, Lemma 4.5]).

3 Main results

We denote by $\mathcal{M}_{L_\Phi}$ the set of all L_Φ bounded martingales $f = (f_n)_{n\ge 0}$.

Theorem 3.1. *Let $\Phi, \Psi \in \bar{\varPhi}_Y$. Assume that $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing and that there exists a positive constant C such that, for all $n \ge 0$,*

$$\sum_{k=0}^{n} (\gamma_{k-1} - \gamma_k)\, \Phi^{-1}\left(\frac{1}{\beta_k}\right) + \gamma_n\, \Phi^{-1}\left(\frac{1}{\beta_n}\right) \le C\Psi^{-1}\left(\frac{1}{\beta_n}\right). \tag{3.1}$$

Then, for any positive constant C_Φ, there exists a positive constant C'_Φ such that, for all $f \in \mathcal{M}_{L_\Phi}$ with $f \not\equiv 0$,

$$\Psi\left(\frac{M(I_\gamma f)}{C'_\Phi \sup_{n\ge 0} \|f_n\|_{L_\Phi}}\right) \le \Phi\left(\frac{Mf}{C_\Phi \sup_{n\ge 0} \|f_n\|_{L_\Phi}}\right). \tag{3.2}$$

Consequently, $I_\gamma \in B(L_\Phi, \mathrm{w}L_\Psi)$. Moreover, if $\Phi \in \nabla_2$, then $I_\gamma \in B(L_\Phi, L_\Psi)$.

Next, for a function $\rho : (0,1] \to (0,\infty)$ such that

$$\int_0^1 \frac{\rho(t)}{t}\, dt < \infty, \tag{3.3}$$

let

$$\gamma_n = \int_0^{\beta_n} \frac{\rho(t)}{t}\,dt, \quad \beta_n = \sum_{B\in A(\mathcal{F}_n)} P(B)\chi_B, \quad n = 0,1,2,\cdots. \tag{3.4}$$

In this case we denote I_γ by I_ρ, namely, for a martingale $f = (f_n)_{n\geq 0}$,

$$I_\rho f = ((I_\rho f)_n)_{n\geq 0}, \quad (I_\rho f)_n = \sum_{k=0}^{n} \left(\int_0^{\beta_{k-1}} \frac{\rho(t)}{t}\,dt\right)(f_k - f_{k-1}). \tag{3.5}$$

If $\rho(t) = \alpha t^\alpha$ and $\alpha > 0$, then $\int_0^{\beta_{k-1}} \frac{\rho(t)}{t}\,dt = (\beta_{k-1})^\alpha$ and I_ρ is the fractional integrals introduced by [14] as a generalization of I_α on dyadic martingales investigated in [5].

If $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, that is, there exists $R \geq 2$ such that

$$E_n f \leq R E_{n-1} f \tag{3.6}$$

for all non-negative integrable function f, then the inequality $\beta_n \leq \beta_{n-1} \leq R\beta_n$ holds, see [14, Lemma 3.1]. Hence,

$$\begin{aligned}
\sum_{k=0}^{n} (\gamma_{k-1} - \gamma_k)\Phi^{-1}(1/\beta_k) &= \sum_{k=1}^{n} \Phi^{-1}(1/\beta_k) \int_{\beta_k}^{\beta_{k-1}} \frac{\rho(t)}{t}\,dt \\
&\sim \sum_{k=1}^{n} \int_{\beta_k}^{\beta_{k-1}} \frac{\Phi^{-1}(1/t)\rho(t)}{t}\,dt \\
&= \int_{\beta_n}^{\beta_0} \frac{\Phi^{-1}(1/t)\rho(t)}{t}\,dt.
\end{aligned}$$

That is, (3.1) is equivalent to

$$\int_0^{\beta_n} \frac{\rho(t)}{t}\,dt\,\Phi^{-1}(1/\beta_n) + \int_{\beta_n}^{b_0} \frac{\rho(t)\Phi^{-1}(1/t)}{t}\,dt \leq C\Psi^{-1}(1/\beta_n) \quad \text{for all} \quad n \geq 0. \tag{3.7}$$

Corollary 3.2. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be regular, and let $\Phi, \Psi \in \bar{\Phi}_Y$. Assume that $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing and that there exists a positive constant A such that, for all $r \in (0,1]$,*

$$\int_0^{r} \frac{\rho(t)}{t}\,dt\;\Phi^{-1}(1/r) + \int_r^1 \frac{\rho(t)\,\Phi^{-1}(1/t)}{t}\,dt \leq A\Psi^{-1}(1/r). \tag{3.8}$$

Then, for any positive constant C_Φ, there exists a positive constant C_1 such that, for all $f \in \mathcal{M}_{L_\Phi}$ with $f \not\equiv 0$,

$$\Psi\left(\frac{M(I_\rho f)}{C'_\Phi \sup_{n\geq 0}\|f_n\|_{L_\Phi}}\right) \leq \Phi\left(\frac{Mf}{C_\Phi \sup_{n\geq 0}\|f_n\|_{L_\Phi}}\right). \tag{3.9}$$

Consequently, $I_\rho \in B(L_\Phi, \mathrm{w}L_\Psi)$. Moreover, if $\Phi \in \nabla_2$, then $I_\rho \in B(L_\Phi, L_\Psi)$.

For a sequence $\gamma = (\gamma_n)_{n\ge 0}$ of positive measurable functions, let

$$M_\gamma f = \sup_{n\ge 0} \gamma_n |E_n f|, \quad f \in L_1. \tag{3.10}$$

Theorem 3.3. *Let $\Phi, \Psi \in \bar{\varPhi}_Y$. Assume that $u \mapsto \Psi^{-1}(u)/\Phi^{-1}(u)$ is almost decreasing and that there exists a positive constant A such that, for all $n \ge 0$,*

$$\gamma_n \, \Phi^{-1}(1/\beta_n) \le A\Psi^{-1}(1/\beta_n). \tag{3.11}$$

Then, for any positive constant C_Φ, there exists a positive constant C'_Φ such that, for all $f \in L_\Phi$ with $f \not\equiv 0$,

$$\Psi\left(\frac{M_\gamma f}{C'_\Phi \|f\|_{L_\Phi}}\right) \le \Phi\left(\frac{Mf}{C_\Phi \|f\|_{L_\Phi}}\right). \tag{3.12}$$

Consequently, M_γ is bounded from L_Φ to $\mathrm{w}L_\Psi$. Moreover, if $\Phi \in \bar{\nabla}_2$, then M_γ is bounded from L_Φ to L_Ψ.

For the commutator $[b, I_\rho]f = bI_\rho f - I_\rho(bf)$, we have the following theorem.

Theorem 3.4. *Let $\psi : (0,1] \to (0,\infty)$, and let $\Phi, \Psi \in \bar{\varPhi}_Y$.*

(i) *Assume that ψ is almost increasing and that there exists a positive constant A and a function $\Theta \in \bar{\nabla}_2$ such that, for all $n \ge 0$,*

$$\sum_{k=0}^{n} (\gamma_{k-1} - \gamma_k)\Phi^{-1}\left(\frac{1}{\beta_k}\right) + \gamma_n \Phi^{-1}\left(\frac{1}{\beta_n}\right) \le A\Theta^{-1}\left(\frac{1}{\beta_n}\right), \tag{3.13}$$

$$\psi(\beta_n)\Theta^{-1}\left(\frac{1}{\beta_n}\right) \le A\Psi^{-1}\left(\frac{1}{\beta_n}\right), \tag{3.14}$$

$$\psi(\beta_n)\gamma_{n-1}\Phi^{-1}\left(\frac{1}{\beta_n}\right) \le A\Psi^{-1}\left(\frac{1}{\beta_n}\right). \tag{3.15}$$

If $\Phi, \Psi \in \bar{\Delta}_2 \cap \bar{\nabla}_2$, then there exist constants $\nu \in (1,\infty)$ and $C \in (0,\infty)$ such that, for all $b \in \mathcal{L}^-_{\nu,\psi}$ and all $f \in L_\Phi$,

$$\|[b, I_\gamma]f\|_{L_\Psi} \le C\|b\|_{\mathcal{L}^-_{\nu,\psi}} \|f\|_{L_\Phi}. \tag{3.16}$$

Moreover, if $\{\mathcal{F}_n\}_{n\ge 0}$ be regular, then, for all $b \in \mathcal{L}^-_{1,\psi}$ and all $f \in L_\Phi$,

$$\|[b, I_\gamma]f\|_{L_\Psi} \le C\|b\|_{\mathcal{L}^-_{1,\psi}} \|f\|_{L_\Phi}, \tag{3.17}$$

without the assumption (3.15).

(ii) *Conversely, let* $\{\mathcal{F}_n\}_{n\ge0}$ *be regular and* $\alpha > 0$. *Assume that* ψ *satisfies the doubling condition and that there exists a positive constant* A *such that, for all* $n \ge 0$,

$$\Psi^{-1}\left(\frac{1}{\beta_n}\right) \le A{\beta_n}^{\alpha}\psi(\beta_n)\Phi^{-1}\left(\frac{1}{\beta_n}\right). \tag{3.18}$$

Assume also that

$$\|b\|_{\mathcal{L}^-_{1,\psi}(\mathcal{F}_0)} = \sup_{B\in A(\mathcal{F}_0)} \frac{1}{\psi(B)P(B)}\int_B |b|\,dP < \infty. \tag{3.19}$$

If $[b, I_\alpha]$ *is bounded from* L_Φ *to* L_Ψ *with operator norm* $\|[b, I_\alpha]\|_{L_\Phi\to L_\Psi}$, *then* b *is in* $\mathcal{L}^-_{1,\psi}$ *and there exists a positive constant* C, *independently* b, *such that*

$$\|b\|_{\mathcal{L}^-_{1.\psi}} \le C\left(\|[b, I_\alpha]\|_{L_\Phi\to L_\Psi} + \|b\|_{\mathcal{L}^-_{1,\psi}(\mathcal{F}_0)}\right).$$

For an almost increasing function $\psi : (0, 1] \to (0,\infty)$, we define the sharp maximal function $M^\sharp_\psi$ by

$$M^\sharp_\psi f = \sup_{n\ge0} \psi(\beta_n)^{-1}E_n|f - E_{n-1}f|, \quad f \in L_1, \tag{3.20}$$

with the convention $E_{-1}f = 0$. If $\psi \equiv 1$ we denote $M^\sharp_\psi$ by $M^\sharp$, that is,

$$M^\sharp f = \sup_{n\ge0} E_n|f - E_{n-1}f|. \tag{3.21}$$

Then we define the Triebel-Lizorkin-Orlicz space as follows.

Definition 3.1. For $\Phi \in \bar{\varPhi}$ and $\psi : (0, 1] \to (0,\infty)$, let

$$F^\psi_{L_\Phi} = \{f \in L_1 : \|f\|_{F^\psi_{L_\Phi}} < \infty\},$$

where

$$\|f\|_{F^\psi_{L_\Phi}} = \|M^\sharp_\psi f\|_{L_\Phi}.$$

We can extend Theorem 3.4 to Triebel-Lizorkin-Orlicz spaces

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