

Boundedness of the maximal operator for double phase functionals with variable exponents

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Abstract

Our aim in this note is to establish the boundedness of the maximal operator for double phase functionals $\phi(x, t) = t^{p(x)} + \{b(x)t\}^{q(x)}$, where $p(\cdot), q(\cdot)$ are log-Hölder continuous exponents and b is a nonnegative, bounded and Hölder continuous function of order $\theta \in (0, 1]$.

1 Introduction

Regarding regularity theory of differential equations, Baroni, Colombo and Mingione [2, 3, 4, 5] studied a double phase functional $\Phi(x, t) = t^p + a(x)t^q$, $x \in \mathbf{R}^N$, $t \geq 0$, where $1 < p < q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0, 1]$. In [4], the minimization problem of the double phase functional was discussed under the assumption $q < (1 + \theta/N)p$. Hästö [9, Theorem 4.7] showed the boundedness of the maximal operator on $L^\Phi(G)$ when $\Phi(x, t) = t^p + a(x)t^q$, $1 < p < q$, $G \subset \mathbf{R}^N$ is bounded, $a \in C^\theta(\bar{G})$ is non-negative and $q \leq (1 + \theta/N)p$.

In this note, let us consider the double phase functional

$$\phi(x, t) = t^{p(x)} + \{b(x)t\}^{q(x)}, \quad (1.1)$$

where $p(\cdot), q(\cdot)$ are log-Hölder continuous and b is nonnegative, bounded and Hölder continuous of order $\theta \in (0, 1]$. For an open set $G \subset \mathbf{R}^N$, recall that the Lebesgue

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space $L^{p(\cdot)}(G)$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(G)} = \inf \left\{ \lambda > 0 : \int_G \left(\frac{|f(y)|}{\lambda} \right)^{p(y)} dy \leq 1 \right\} < \infty$$

(see e.g. [6], [7], [8], [13]). Further let us consider the Musielak-Orlicz space $\phi(G)$ of all functions f such that

$$\|f\|_{\phi(G)} = \inf \left\{ \lambda > 0 : \int_G \phi \left(y, \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\} < \infty$$

(see e.g. [9], [11]).

Our main aim in this note is to establish the boundedness of the maximal operator in $\phi(\mathbf{R}^N)$ (see Theorem 2.1 in Section 2). We also extend Theorem 2.1 to the Herz case (Theorem 5.1).

2 Variable exponents

Let $p(\cdot)$ be a real valued measurable function on \mathbf{R}^N such that

$$(P1) \quad 1 < p^- := \operatorname{ess\,inf}_{x \in \mathbf{R}^N} p(x) \leq \operatorname{ess\,sup}_{x \in \mathbf{R}^N} p(x) =: p^+ < \infty;$$

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_p}{\log(e + 1/|x - y|)} \quad \text{for all } x, y \in \mathbf{R}^N$$

with a constant $C_p \geq 0$.

Moreover we suppose that

(P3) $p(\cdot)$ is log-Hölder continuous at ∞ , namely, there exists a constant $p(\infty) > 1$ such that

$$|p(x) - p(\infty)| \leq \frac{C_{p,\infty}}{\log(e + |x|)} \quad \text{for all } x \in \mathbf{R}^N$$

with a constant $C_{p,\infty} \geq 0$.

In this case, we write $p(\cdot) \in \mathcal{P}_0$.

Our aim in this note is to give the boundedness of the maximal operator $f \rightarrow Mf$ in the Musielak-Orlicz space $\phi(\mathbf{R}^N)$.

THEOREM 2.1. Let $p(\cdot)$ and $q(\cdot)$ be variable exponents in \mathcal{P}_0 . Suppose

$$0 \leq 1/p(x) - 1/q(x) \leq \min\{1/p(\infty), \theta/N\}$$

for $x \in \mathbf{R}^N$. Then there is a constant $C > 0$ such that

$$\|Mf\|_{\phi(\mathbf{R}^N)} \leq C\|f\|_{\phi(\mathbf{R}^N)}. \quad (2.1)$$

In what follows, we always assume that $p(\cdot), q(\cdot) \in \mathcal{P}_0$.

3 Maximal functions

Consider the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $B(x,r)$ denotes the open ball with center at x and radius r .

For the boundedness of the maximal operator in $L^p(\mathbf{R}^N)$ of constant exponent, we refer to [1], [12] and [15].

LEMMA 3.1 (cf. [14, Lemma 3.5]). *There is a constant $C > 0$ such that*

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \leq C \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^{p(y)} dy \right)^{1/p(x)} + C(1+|x|)^{-N}$$

for all $x \in \mathbf{R}^N$, $r > 0$ and measurable functions f on \mathbf{R}^N such that $\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} < \infty$.

COROLLARY 3.2. *There is a constant $C > 0$ such that*

$$\|Mf\|_{L^{p(\cdot)}(\mathbf{R}^N)} \leq C\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} \quad \text{for all } f \in L^{p(\cdot)}(\mathbf{R}^N).$$

In fact, take $1 < p_0 < p^-$ and apply Lemma 3.1 with $p(\cdot)$ replaced by $p_0(\cdot) = p(\cdot)/p_0$. If $\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} \leq 1$, then we find

$$Mf(x) \leq C (M[f_0](x))^{1/p_0(x)} + C(1+|x|)^{-N},$$

where $f_0(y) = |f(y)|^{p_0(y)}$. Hence

$$\begin{aligned} \int (Mf(x))^{p(x)} dx &\leq C \int (M[f_0](x))^{p_0} dx + C \int (1+|x|)^{-Np(x)} dx \\ &\leq C \int (f_0(y))^{p_0} dy + C \int (1+|x|)^{-Np^-} dx \\ &\leq C \int |f(y)|^{p_0(y)} dy + C \leq C. \end{aligned}$$

4 Fractional maximal functions

Consider the fractional maximal function

$$M_{\tau(\cdot)}f(x) = \sup_{r>0} \frac{r^{\tau(x)}}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $\tau(\cdot)$ is a measurable function on \mathbf{R}^N satisfying

$$(\tau) \quad 0 \leq \tau^- = \inf_{x \in \mathbf{R}^N} \tau(x) \leq \tau^+ = \sup_{x \in \mathbf{R}^N} \tau(x) \leq N.$$

THEOREM 4.1 (Sobolev type inequality). *Suppose*

$$0 \leq 1/p(x) - 1/q(x) = \tau(x)/N \leq 1/p(\infty)$$

for all $x \in \mathbf{R}^N$. Then there is a constant $C > 0$ such that

$$\|M_{\tau(\cdot)}f\|_{L^{q(\cdot)}(\mathbf{R}^N)} \leq C\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)}$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^N)$.

Proof. Let f be a measurable function on \mathbf{R}^N with $\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} \leq 1$. Let $x \in \mathbf{R}^N$ and $r > 0$. First note that for $0 < r \leq \delta$ we have

$$\begin{aligned} \frac{r^{\tau(x)}}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy &\leq \delta^{\tau(x)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \\ &\leq \delta^{\tau(x)} Mf(x) \end{aligned}$$

and for $0 < \delta < r < (1 + |x|)^{p(x)}$ we have by Lemma 3.1

$$\begin{aligned} &\frac{r^{\tau(x)}}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \\ &\leq Cr^{\tau(x)} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^{p(y)} dy \right)^{1/p(x)} + Cr^{\tau(x)}(1 + |x|)^{-N} \\ &\leq Cr^{\tau(x)-N/p(x)} \\ &\leq C\delta^{\tau(x)-N/p(x)} \end{aligned}$$

since $1/q(x) = 1/p(x) - \tau(x)/N > 0$. If $r \geq (1 + |x|)^{p(x)}$, then, for $z = rx/|x|$, we

obtain by Lemma 3.1

$$\begin{aligned}
& \frac{r^{\tau(x)}}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy \\
\leq & C \frac{r^{\tau(x)}}{|B(z, 3r)|} \int_{B(z, 3r)} |f(y)| dy \\
\leq & Cr^{\tau(x)} \left(\frac{1}{|B(z, 3r)|} \int_{B(z, 3r)} |f(y)|^{p(y)} dy \right)^{1/p(z)} + Cr^{\tau(x)}(1 + |z|)^{-N} \\
\leq & Cr^{\tau(x)-N/p(z)} + Cr^{\tau(x)}(1 + |z|)^{-N} \\
\leq & Cr^{\tau(x)-N/p(z)} \\
\leq & Cr^{\tau(x)-N/p(\infty)} \\
\leq & C(1 + |x|)^{(\tau(x)-N/p(\infty))p(x)} \\
\leq & C(1 + |x|)^{(\tau(x)-N/p(x))p(x)}.
\end{aligned}$$

Hence

$$M_{\tau(x)}f(x) \leq \delta^{\tau(x)}Mf(x) + C\delta^{\tau(x)-N/p(x)} + C(1 + |x|)^{(\tau(x)-N/p(x))p(x)}.$$

Now, letting $\delta = \{Mf(x)\}^{-p(x)/N}$, we find

$$\begin{aligned}
M_{\tau(x)}f(x) & \leq C\{Mf(x)\}^{1-\tau(x)p(x)/N} + C(1 + |x|)^{(\tau(x)-N/p(x))p(x)} \\
& \leq C\{Mf(x)\}^{p(x)/q(x)} + C(1 + |x|)^{(\tau(x)-N/p(x))p(x)},
\end{aligned}$$

so that

$$\begin{aligned}
\int \{M_{\tau(x)}f(x)\}^{q(x)} dx & \leq C \int \{Mf(x)\}^{p(x)} dx + C \int (1 + |x|)^{-Np(x)} dx \\
& \leq C,
\end{aligned}$$

as required. □

Proof of Theorem 2.1. Let f be a measurable function on \mathbf{R}^N with

$$\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} + \|bf\|_{L^{q(\cdot)}(\mathbf{R}^N)} \leq 1.$$

Let $x \in \mathbf{R}^N$ and $r > 0$. Set $\tau(x) = N/p(x) - N/q(x)$. First note that

$$\begin{aligned}
& b(x) \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy \\
= & \frac{1}{|B(x, r)|} \int_{B(x, r)} \{b(x) - b(y)\} |f(y)| dy + \frac{1}{|B(x, r)|} \int_{B(x, r)} b(y) |f(y)| dy \\
\leq & \frac{1}{|B(x, r)|} \int_{B(x, r)} C|x - y|^{\tau(x)} |f(y)| dy + \frac{1}{|B(x, r)|} \int_{B(x, r)} b(y) |f(y)| dy,
\end{aligned}$$

so that

$$b(x)Mf(x) \leq CM_{\tau(x)}f(x) + M[bf](x).$$

Since $1/q(x) = 1/p(x) - \tau(x)/N$, Theorem 4.1 gives

$$\begin{aligned} \|bMf\|_{L^{q(\cdot)}(\mathbf{R}^N)} &\leq C\|M_{\tau(\cdot)}f\|_{L^{q(\cdot)}(\mathbf{R}^N)} + \|M[bf]\|_{L^{q(\cdot)}(\mathbf{R}^N)} \\ &\leq C\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)} + C\|bf\|_{L^{q(\cdot)}(\mathbf{R}^N)}, \end{aligned}$$

which proves the result. \square

EXAMPLE 4.2. Let $1/q(0) < 1/p(0) - \theta/N$. For $0 < \theta \leq 1$ and $0 < \beta < N$, consider

$$b(x) = (\min\{\max\{0, x_N\}, 1\})^\theta \quad \text{and} \quad f(y) = |y|^{-\beta} \chi_{B_-}(y),$$

where $B_- = \{x = (x_1, \dots, x_N) \in B(0, 1) : x_N < 0\}$. Then note that

- (1) $Mf(x) \geq C|x|^{-\beta}$ on $B(0, 1)$;
- (2) $\int_{B(0,1)} \phi(y, f(y)) dy < \infty$ when $-\beta p(0) + N > 0$;
- (3) $\int_{B(0,1)} \phi(x, Mf(x)) dx = \infty$ when $(\theta - \beta)q(0) + N \leq 0$.

Now take θ and β such that $0 < \theta \leq 1$, $\theta < \beta < N$ and

$$\frac{1}{q(0)} = \frac{\beta - \theta}{N} \quad \text{and} \quad \frac{1}{q(0)} + \frac{\theta}{N} = \frac{\beta}{N} < \frac{1}{p(0)}.$$

Then (2) and (3) hold.

5 Herz spaces

Let $A(r) = B(0, 2r) \setminus B(0, r)$ for $r > 0$. For a real number ν and $0 < q < \infty$, consider the Herz space

$$\|f\|_{\phi, q, \nu} = \|f\|_{\phi(B(2))} + \left(\int_1^\infty (r^\nu \|f\|_{\phi(A(r))})^q \frac{dr}{r} \right)^{1/q} < \infty.$$

Theorem 2.1 is extended in the Herz settings.

THEOREM 5.1. Suppose $-N/q(\infty) < \nu < N - N/p(\infty)$ and

$$0 \leq 1/p(x) - 1/q(x) = \tau(x)/N \leq \min\{1/p(\infty), \theta/N\}$$

for all $x \in \mathbf{R}^N$. Then there is a constant $C > 0$ such that

$$\|Mf\|_{\phi, q, \nu} \leq C \|f\|_{\phi, q, \nu} \quad (5.1)$$

when $\|f\|_{\phi, q, \nu} < \infty$.

For a proof of Theorem 5.1, we use Theorem 2.1 and the following lemmas.

LEMMA 5.2. There are constants $C_1, C_2 > 0$ such that

$$C_1 r^{N/p(\infty)} \leq \|\chi_{B(x, r)}\|_{L^{p(\cdot)}} \leq C_2 r^{N/p(\infty)}$$

for all $x \in \mathbf{R}^N$ and $r > 1$.

LEMMA 5.3 ([10, Lemma 2.5]). There is a constant $C > 0$ such that

$$\frac{1}{|A(r)|} \int_{A(r)} |f(y)| dy \leq C r^{-N/p(\infty)} \|f\|_{L^{p(\cdot)}(A(r))}$$

for all $x \in \mathbf{R}^N$, $r > 1$ and measurable functions f on \mathbf{R}^N such that $\|f\|_{L^{p(\cdot)}(A(r))} < \infty$.

LEMMA 5.4. If $\varepsilon + \nu - N + N/p(\infty) < 0$ and $\varepsilon > 0$, then

$$\int_{B(0, r) \setminus B(0, 1)} |f(y)| dy \leq C r^{-\varepsilon + N - N/p(\infty) - \nu} \left(\int_{1/2}^r (t^{\varepsilon + \nu} \|f\|_{L^{p(\cdot)}(A(t))})^q \frac{dt}{t} \right)^{1/q}$$

for all $r \geq 1$ and $f \in L^1_{\text{loc}}(\mathbf{R}^N)$.

Proof. We may assume that $f(x) = 0$ for $x \in B(0, 1)$. Let j_0 be the smallest integer such that $2^{j_0} \geq r$. By Lemma 5.3, we have

$$\begin{aligned} \int_{B(0, r) \setminus B(0, 1)} |f(y)| dy &\leq C \sum_{j=1}^{j_0} (2^{-j} r)^N \frac{1}{|A(2^{-j} r)|} \int_{A(2^{-j} r)} |f(y)| dy \\ &\leq C \sum_{j=1}^{j_0} (2^{-j} r)^{N - N/p(\infty)} \|f\|_{L^{p(\cdot)}(A(2^{-j} r))}. \end{aligned}$$

In case $q > 1$, by Hölder's inequality, we have

$$\begin{aligned}
& \sum_{j=1}^{j_0} (2^{-j}r)^{N-N/p(\infty)} \|f\|_{L^{p(\cdot)}(A(2^{-j}r))} \\
& \leq \left(\sum_{j=1}^{j_0} ((2^{-j}r)^{-\varepsilon+N-N/p(\infty)-\nu})^{q'} \right)^{1/q'} \left(\sum_{j=1}^{j_0} ((2^{-j}r)^{\varepsilon+\nu} \|f\|_{L^{p(\cdot)}(A(2^{-j}r))})^q \right)^{1/q} \\
& \leq Cr^{-\varepsilon+N-N/p(\infty)-\nu} \left(\int_{1/2}^r (t^{\varepsilon+\nu} \|f\|_{L^{p(\cdot)}(A(t))})^q \frac{dt}{t} \right)^{1/q}.
\end{aligned}$$

Therefore, we obtain the required result in this case.

For the case $0 < q \leq 1$, by the fact that $(a+b)^q \leq a^q + b^q$ for all $a, b \geq 0$ instead of Hölder's inequality, we also obtain the required result. \square

In the same manner we have the following result.

LEMMA 5.5. *Let $\beta \in \mathbf{R}$. If $\varepsilon + \beta - N/p(\infty) - \nu < 0$ and $\varepsilon > 0$, then*

$$\int_{\mathbf{R}^N \setminus B(0,r)} |y|^{\beta-N} |f(y)| dy \leq Cr^{\varepsilon+\beta-N/p(\infty)-\nu} \left(\int_{\tau/2}^{\infty} (t^{-\varepsilon+\nu} \|f\|_{L^{p(\cdot)}(A(t))})^q \frac{dt}{t} \right)^{1/q}$$

for all $r \geq 1$ and $f \in L_{\text{loc}}^1(\mathbf{R}^N)$.

Proof of Theorem 5.1. Suppose $\|f\|_{\phi,q,\nu} \leq 1$. Set $\tau(x) = N/p(x) - N/q(x)$ and $\tau(\infty) = N/p(\infty) - N/q(\infty)$. We show that

$$\int_1^{\infty} (r^\nu \|bMf\|_{L^{q(\cdot)}(A(r))})^q \frac{dr}{r} \leq C$$

when $1 < q < \infty$.

For $r > 1$, write

$$\begin{aligned}
f &= f\chi_{B(0,r/2)} + f\chi_{B(0,4r) \setminus B(0,r/2)} + f\chi_{\mathbf{R}^N \setminus B(0,4r)} \\
&= f_{1,r} + f_{2,r} + f_{3,r}.
\end{aligned}$$

By Theorem 2.1 we have

$$\int_1^{\infty} (r^\nu \|bMf_{2,r}\|_{L^{q(\cdot)}(A(r))})^q \frac{dr}{r} \leq C.$$

Note that

$$\begin{aligned}
& b(x) \frac{1}{|B(x,t)|} \int_{B(x,t)} |f_{1,r}(y)| dy \\
& \leq \frac{1}{|B(x,t)|} \int_{B(x,t)} C|x-y|^{\tau(x)} |f_{1,r}(y)| dy + \frac{1}{|B(x,t)|} \int_{B(x,t)} b(y) |f_{1,r}(y)| dy \\
& \leq Cr^{\tau(\infty)-N} \int_{B(0,r)} |f(y)| dy + Cr^{-N} \int_{B(0,r)} b(y) |f(y)| dy
\end{aligned}$$

and

$$\begin{aligned}
& b(x) \frac{1}{|B(x,t)|} \int_{B(x,t)} |f_{3,r}(y)| dy \\
& \leq \frac{1}{|B(x,t)|} \int_{B(x,t)} C|x-y|^{\tau(y)} |f_{3,r}(y)| dy + \frac{1}{|B(x,t)|} \int_{B(x,t)} b(y) |f_{3,r}(y)| dy \\
& \leq C \int_{\mathbf{R}^N \setminus B(0,2r)} |y|^{\tau(\infty)-N} |f(y)| dy + C \int_{\mathbf{R}^N \setminus B(0,2r)} |y|^{-N} b(y) |f(y)| dy
\end{aligned}$$

for all $x \in A(r)$ and $t > 0$. By Lemmas 5.2 and 5.4 we have for $0 < \varepsilon_1 < N - N/p(\infty) - \nu$ and $0 < \varepsilon_2 < N - N/q(\infty) - \nu$

$$\begin{aligned}
& \int_1^\infty (r^\nu \|bMf_{1,r}\|_{L^{q(\cdot)}(A(r))})^q \frac{dr}{r} \\
& \leq C \int_1^\infty \left(r^{\nu+N/q(\infty)+\tau(\infty)} \left(r^{-\varepsilon_1-N/p(\infty)-\nu} \left(\int_{1/2}^r (t^{\varepsilon_1+\nu} \|f\|_{L^{p(\cdot)}(A(t))})^q \frac{dt}{t} \right)^{1/q} \right) \right)^q \frac{dr}{r} \\
& \quad + C \int_1^\infty \left(r^{\nu+N/q(\infty)} \left(r^{-\varepsilon_2-N/q(\infty)-\nu} \left(\int_{1/2}^r (t^{\varepsilon_2+\nu} \|bf\|_{L^{q(\cdot)}(A(t))})^q \frac{dt}{t} \right)^{1/q} \right) \right)^q \frac{dr}{r} \\
& \leq C \int_{1/2}^\infty (t^{\varepsilon_1+\nu} \|f\|_{L^{p(\cdot)}(A(t))})^q \left(\int_t^\infty r^{-\varepsilon_1 q} \frac{dr}{r} \right) \frac{dt}{t} \\
& \quad + C \int_{1/2}^\infty (t^{\varepsilon_2+\nu} \|bf\|_{L^{q(\cdot)}(A(t))})^q \left(\int_t^\infty r^{-\varepsilon_2 q} \frac{dr}{r} \right) \frac{dt}{t} \\
& \leq C \int_{1/2}^\infty (t^\nu \|f\|_{L^{p(\cdot)}(A(t))})^q \frac{dt}{t} + C \int_{1/2}^\infty (t^\nu \|bf\|_{L^{q(\cdot)}(A(t))})^q \frac{dt}{t} \leq C.
\end{aligned}$$

In the same way, by Lemmas 5.2 and 5.5, we obtain

$$\int_1^\infty (r^\nu \|bMf_{3,r}\|_{L^{q(\cdot)}(A(r))})^q \frac{dr}{r} \leq C,$$

which completes the proof. \square

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