A DOUBLY NONLINEAR DEGENERATE SINGULAR PARABOLIC EQUATION AND A NONLINEAR EIGENVALUE PROBLEM

(二重非線形退化特異放物型方程式と非線形固有値問題)

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary $\partial\Omega$. We consider the following doubly nonlinear degenerate and singular parabolic equation

$$(1.1)\quad \begin{cases} \partial_t(|u|^{q-1}u) - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda(t)|u|^{q-1}u & \text{in } \Omega_\infty := (0,\infty)\times\Omega \\ u = u_0 & \text{on } \partial_p\Omega_\infty \\ \|u(t)\|_{q+1} = 1, \quad t \geq 0 \end{cases}$$

where $2 \leq p < n$, $q = \frac{np}{n-p} - 1$, and $u = u(t,x)$ is a real-valued function defined for $(t,x) \in \Omega_\infty$, $\nabla_\alpha = \partial/\partial x_\alpha$, $\alpha = 1,\dots,n$, $\nabla u = (\nabla_\alpha u)$ is the spatial gradient of a function u, $|\nabla u|^2 = \sum_{\alpha=1}^n (\nabla_\alpha u)^2$ and $\partial_t u$ is the derivative on time t. The initial and boundary data $u_0 = u_0(x)$ is in the Sobolev space $\mathrm{W}_0^{1,p}(\Omega)$ and satisfies $u_0 > 0$ in Ω and $\|u_0\|_{q+1} = 1$. By multiplying the equation by u and integration by parts on space,

$$\frac{d}{dt}\frac{q}{q+1}\|u(t)\|_{q+1}^{q+1} + \|\nabla u(t)\|_p^p = \lambda(t)\|u(t)\|_{q+1}^{q+1} \Longrightarrow \lambda(t) = \|\nabla u(t)\|_p^p,$$

where $\|f\|_p$ is the $p-$th powered integral norm on Ω of a measurable function f, $E(u) := \|\nabla u\|_p^p/p$ is the $p-$energy of a function u. The corresponding stationary elliptic type equation is concerned with a nonlinear eigenvalue problem and but, has only trivial zero solution, by the so-called Pohožaev identity and Hopf's maximum principle, provided that the domain Ω is star-shaped with the origin, and thus, a solution of the evolution equation may have any concentration point of volume, local $(q+1)-$th powered integral, at infinite time, by the volume conservation $\|u(t)\|_{q+1} = 1$. Our main purpose is to study such asymptotic behavior of a solution to the evolution equation above.

We report the following main theorem in this paper. The definition of a weak solution of the p-Sobolev flow (1.1) is given in Definition 1 in Section 2.3.

Theorem 1 *Let Ω be a bounded domain with smooth boundary. Suppose that the initial data u_0 is positive in Ω, belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and satisfies the volume constraint $\|u_0\|_{L^{q+1}(\Omega)} = 1$. Let u be a weak solution of (1.1) in $\Omega_\infty \equiv \Omega \times (0,\infty)$ with the initial and boundary data u_0. Then, u is positive and bounded in Ω_∞ and, together with its spatial gradient, are locally Hölder continuous in Ω_∞. Moreover, for $t \in [0,\infty)$,*

$$(1.2)\qquad \lambda(t) = \|\nabla u(t)\|_{L^p(\Omega)}^p; \qquad \lambda(t) \leq \lambda(0).$$

The global existence of the p-Sobolev flow and its asymptotic behavior, that is the volume concentration at infinite time, will be treated in our forthcoming paper, based on the a-priori regularity estimates for the p-Sobolev flow, obtained in the main theorem above.

We show the boundedness and non-negativity of a solution by a comparison type argument, and derive an expansion of positivity of a solution by some local energy estimates.

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2 Preliminaries

We prepare some notations and technical analysis tools, which are used later.

2.1 Notation

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 3$) with smooth boundary $\partial\Omega$ and for a positive $T \leq \infty$ let $\Omega_T := \Omega \times (0, T)$ be the cylindrical domain. Let us define the parabolic boundary of Ω_T by

$$\partial_p \Omega_T := (\partial\Omega \times [0, T)) \cup (\Omega \times \{t = 0\}).$$

We recall some function spaces, defined on space-time region. For $1 \leq p, q \leq \infty$, $L^q(t_1, t_2\,;\, L^p(\Omega))$ is a function space of measurable real-valued functions on a space-time region $\Omega \times (t_1, t_2)$ with a finite norm

$$\|v\|_{L^q(t_1,t_2\,;\,L^p(\Omega))} := \begin{cases} \left(\displaystyle\int_{t_1}^{t_2} \|v(t)\|^q_{L^p(\Omega)}\, dt \right)^{1/q} & (1 \leq q < \infty) \\ \displaystyle\sup_{t_1 \leq t \leq t_2} \|v(t)\|_{L^p(\Omega)} & (q = \infty) \end{cases}$$

where

$$\|v(t)\|_{L^p(\Omega)} := \begin{cases} \left(\displaystyle\int_{\Omega} |v(x,t)|^p\, dx \right)^{1/p} & (1 \leq p < \infty) \\ \displaystyle\sup_{x \in \Omega} |v(x,t)| & (p = \infty) \end{cases}$$

When $p = q$, we write $L^p(\Omega \times (t_1, t_2)) = L^p(t_1, t_2\,;\, L^p(\Omega))$ for brevity. For $1 \leq p < \infty$ the Sobolev space $W^{1,p}(\Omega)$ is consists of measurable real-valued functions that are weakly differentiable and their weak derivatives are p-th integrable on Ω, with the norm

$$\|v\|_{W^{1,p}(\Omega)} := \left(\int_{\Omega} |v|^p + |\nabla v|^p\, dx \right)^{1/p},$$

where $\nabla v = (v_{x_1}, \ldots, v_{x_n})$ denotes the gradient of v in a distribution sense, and let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with resptect to the norm $\|\cdot\|_{W^{1,p}}$. Also let $L^q(t_1, t_2\,;\, W_0^{1,p}(\Omega))$ denote a function space of measurable real-valued functions on space-time region with a finite norm

$$\|v\|_{L^q(t_1,t_2\,;\,W_0^{1,p}(\Omega))} := \left(\int_{t_1}^{t_2} \|v(t)\|^q_{W^{1,p}(\Omega)}\, dt \right)^{1/q}.$$

Let $B = B_\rho(x_0) := \{x \in \mathbb{R}^n \,:\, |x - x_0| < \rho\}$ denote the open ball with radius $\rho > 0$ centered at some point $x_0 \in \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a bounded domain. For a real number k and a function v in $L^1(E)$, we define the *truncation* of v by

$$(v - k)_+ := \max\{(v - k), 0\}; \quad (k - v)_+ := \max\{(k - v), 0\}. \tag{2.1}$$

For a measurable function v in $L^1(E)$ and a pair of real numbers $k < l$, we set

$$\begin{aligned} &E \cap \{v > l\} := \{x \in E \,:\, v(x) > l\}, \quad E \cap \{v < k\} := \{x \in E \,:\, v(x) < k\}, \\ &E \cap \{k < v < l\} := \{x \in E \,:\, k < v(x) < l\}. \end{aligned} \tag{2.2}$$

Let $z = (x, t) \in \mathbb{R}^n \times \mathbb{R}$ be a space-time variable and $dz = dxdt$ be the space-time volume element.

2.2 Technical tools

The following is called De Giorgi's inequality (see [5]).

Proposition 2 (De Giorgi's inequality) *Let $v \in W^{1,1}(B)$ for a ball $B \subset \mathbb{R}^n$ and let $k < l$ be real numbers. Then there exists a positive constant C depending only on p, n such that*

$$(l-k)|B \cap \{v > l\}| \le C \frac{\rho^{n+1}}{|B \cap \{v < k\}|} \int_{B \cap \{k < v < l\}} |\nabla v|\, dx. \tag{2.3}$$

Since $v \in W^{1,1}(B)$, of course, $-v \in W^{1,1}(B)$. Let $k > l$ be any pair of real numbers. Applying (2.3) above for $-v$ and $-l > -k$, we have

$$(k-l)|B \cap \{v < l\}| \le C \frac{\rho^{n+1}}{|B \cap \{v > k\}|} \int_{B \cap \{l < v < k\}} |\nabla v|\, dx.$$

Let $q = np/(n-p) - 1$ as before. Following [5], we define the auxiliary function

$$A^+(k,u) := \int_{k^q}^{u^q} \left(\xi^{1/q} - k\right)_+ d\xi; \quad A^-(k,u) := \int_{u^q}^{k^q} \left(k - \xi^{1/q}\right)_+ d\xi \tag{2.4}$$

for $u \ge 0$ and $k \ge 0$. Changing a variable $\eta = \xi^{1/q}$, we have

$$\begin{aligned} A^+(k,u) &= q \int_k^u (\eta - k)_+ \eta^{q-1}\, d\eta = q \int_0^{(u-k)_+} (\eta - k)^{q-1} \eta\, d\eta; \\ A^-(k,u) &= q \int_u^k (k - \eta)_+ \eta^{q-1}\, d\eta = q \int_0^{(k-u)_+} (k - \eta)^{q-1} \eta\, d\eta. \end{aligned}$$

Then we formally get

$$\frac{\partial}{\partial t} A^+(k,u) = \frac{\partial u^q}{\partial t}(u-k)_+, \quad \frac{\partial}{\partial t} A^-(k,u) = -\frac{\partial u^q}{\partial t}(k-u)_+. \tag{2.5}$$

If $k = 0$, we abbreviate as

$$A^+(u) = A^+(0,u), \quad A^-(u) = A^-(0,u).$$

Let $0 < t_1 < t_2 \le T$ and let K be any domain in Ω. We denote a parabolic cylinder by $K_{t_1,t_2} := K \times (t_1, t_2)$. We recall the Sobolev embedding of parabolic type.

Proposition 3 ([5]) *There exists a constant C depending only on n, p, r such that for every $v \in L^\infty(t_1, t_2; L^r(K)) \cap L^p(t_1, t_2; W_0^{1,p}(K))$*

$$\int_{K_{t_1,t_2}} |v|^{p\frac{n+r}{n}}\, dz \le C \left(\int_{K_{t_1,t_2}} |\nabla v|^p\, dz \right) \left(\sup_{t_1 < t < t_2} \int_\Omega |v|^r\, dx \right)^{\frac{p}{n}} \tag{2.6}$$

The so-called *fast geometric convergence* is crucially used later. See [5] for details.

Lemma 4 (Fast geometric convergence, [5]) *Let $\{Y_m\}_{m=0}^\infty$ be a sequence of positive numbers, satisfying the recursive inequlities*

$$Y_{m+1} \le C b^m Y_m^{1+\alpha}, \quad n = 0, 1, \ldots, \tag{2.7}$$

where $C, b > 1$ and $\alpha > 0$ are given constants independent of m. If the initial value Y_0 satisfies

$$Y_0 \le C^{-1/\alpha} b^{-1/\alpha^2}, \tag{2.8}$$

then $\lim_{m \to \infty} Y_m = 0$.

The following fundamental algebraic inequality, associated with the p -Laplace operator is well-known(see [4]).

Lemma 5 *For every $p \in (1, \infty)$ there exist positive constants C_1 and C_2 depending only on p and n such that for any ξ, $\eta \in R^{nk}$*

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq C_1(|\xi| + |\eta|)^{p-2}|\xi - \eta|,$$
$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq C_2|\xi - \eta|^p, \tag{2.9}$$

where dot $\cdot$ denotes the inner product in $\mathbb{R}^n$.

2.3 A weak solution

We state the definition of weak solutions of the p-Sobolev flow (1.1) here. We will study a weak solution of the p-Sobolev flow (1.1).

Definition 1 *A measurable function u defined on Ω_∞ is called a weak solution to (1.1) if the following conditions (i)-(iv) are satisfied.*

(D1) $u \in L^\infty(0, \infty\,; W_0^{1,p}(\Omega))$; $\quad \partial_t(|u|^{q-1}u) \in L^2(\Omega_\infty)$;

(D2) *There exists a function $\lambda(t) \in L^1(0, \infty)$ such that, for every $\varphi \in C_0^\infty(\Omega_\infty)$,*

$$-\int_{\Omega_\infty} |u|^{q-1}u\varphi_t \, dz + \int_{\Omega_\infty} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dz = \int_{\Omega_\infty} \lambda(t)|u|^{q-1}u\varphi dz,$$

where $dz = dxdt$;

(D3) $\|u(t)\|_{L^{q+1}(\Omega)} = 1$ *for all* $t \geq 0$;

(D4) $\|u(t) - u_0\|_{L^{q+1}(\Omega)} \to 0$ *as* $t \to 0$.

3 Fundamental properties of the p-Sobolev flow

In what follows, we consider some fundamental properties of the p-Sobolev flow (1.1).

3.1 Nonnegativity and boundedness

Firstly, we claim that any weak solution to (1.1) must be nonnegative under nonnegativity of the initial value u_0.

Lemma 6 (Nonnegativity) *Suppose $u_0 \geq 0$ in Ω. Then, a weak solution u of (1.1) satisfies*

$$u \geq 0 \quad in \quad \Omega_\infty. \tag{3.1}$$

Proof. Let $0 < t_1 < t \leq \infty$ be arbitrarily taken and fixed. Put $\Omega_{t_1,t} = \Omega \times (t_1, t)$. Let δ be any positive number such that $\delta \leq (t - t_1)/3$. We define a Lipschitz cut-off function on time, $\sigma_{t_1,t}$ such that

$$0 \leq \sigma_{t_1,t} \leq 1, \quad \sigma_{t_1,t} = 1 \quad \text{in} \quad (t_1 + \delta, t - \delta) \quad \text{and} \quad (\sigma_{t_1,t}) \subset (t_1, t).$$

The function $-(-u)_+\sigma_{t_1,t}$ is an admissible test function in (D2), since $\partial_t(|u|^{q-1}u) \in L^2(\Omega_\infty)$ by (D1) and, $-(-u)_+\sigma_{t_1,t}$ is in $L^{q+1}(\Omega \times (t_1, t))$. Thus, we have

$$\int_{\Omega_{t_1,t}} \partial_\tau(|u|^{q-1}(-u))(-u)_+\sigma_{t_1,t} \, dz \quad + \quad \int_{\Omega_{t_1,t}} |\nabla u|^{p-2}\nabla(-u) \cdot \nabla\left((-u)_+\sigma_{t_1,t}\right) \, dz$$
$$= \int_{\Omega_{t_1,t}} \lambda(\tau)|u|^{q-1}(-u)(-u)_+\sigma_{t_1,t} \, dz.$$

and thus,

$$\frac{q}{q+1}\int_\Omega (-u(t))_+^{q+1}\,dx \le \int_0^t \lambda(\tau)\int_\Omega (-u(\tau))_+^{q+1}\,dx\,d\tau.$$

From the Gronwall lemma it follows that

$$\int_\Omega (-u(t))_+^{q+1}\,dx \le 0,$$

since by (D4), $(-u(t))_+ \to 0$ in $L^{q+1}(\Omega)$ as $t \searrow 0$. Therefore we have $-u(x,t) \le 0$ for $(x,t) \in \Omega_\infty$ and the claim is actually verified. □

Here, we recall that $\lambda(t)$ is explicitly computed as follows:

Proposition 7 *Let u be a nonnegative weak solution to (1.1). Then*

$$\lambda(t) = \|\nabla u(t)\|^p_{L^p(\Omega)}, \quad t \in [0,\infty).$$

The proof is done as in Introduction, where we note that $\partial_t(|u|^{q-1}u) \in L^2(\Omega_\infty)$ by (D1) and $u \in L^\infty(0,\infty; L^{q+1}(\Omega))$ by (D1) and the Sobolev embedding of $W_0^{1,p}(\Omega)$ into $L^{q+1}(\Omega)$.

We next derive the boundedness of the p-Sobolev flow (1.1).

Proposition 8 (Boundedness) *Let u be a nonnegative weak solution of the p-Sobolev flow (1.1). Then u is bounded from above in Ω_∞*

$$\|u(t)\|_{L^\infty(\Omega)} \le e^{ct/q}\|u_0\|_{L^\infty(\Omega)},$$

with $c := \sup\limits_{0<t<\infty} \|\nabla u(t)\|^p_{L^p(\Omega)}$.

Proof. Let u be a nonnegative weak solution of (1.1). Since $u \in L^\infty(0,\infty; W_0^{1,p}(\Omega))$, $\|\nabla u(t)\|^p_{L^p(\Omega)} \in L^\infty(0,\infty)$ and thus, by Proposition 7, $\lambda(t) = \|\nabla u(t)\|^p_{L^p(\Omega)} \in L^\infty(0,\infty)$. Thus, the weak solution u to (1.1) is the weak subsolution, satisfying in the weak sense

$$\partial_t u^q - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \le cu^q, \quad c := \sup_{0<t<\infty} \lambda(t).$$

We will follow the similar argument as in [1]. Set $M := \|u_0\|_{L^\infty(\Omega)}$ and, for a small $\delta > 0$, let us define the Lipschitz truncated function $\phi_\delta(u)$ by

$$\phi_\delta(u) := \min\left\{1, \frac{(e^{-ct/q}|u| - M)_+}{\delta}\right\},$$

where the support of ϕ_δ is $\{|u| > e^{ct/q}M\}$, $\phi_\delta(u) \in L^\infty(\Omega_T)$ and $\phi_\delta(0) = 0$ and, $\phi_\delta(u) \in L^\infty(0,T; W_0^{1,p}(\Omega))$ since $\phi_\delta(u)$ is Lipschitz on u. Let $0 < t_1 < t \le T$ and $\sigma_{t_1,t}$ be the same time cut-off function as in the proof of Lemma 6. The function $e^{-ct}\sigma_{t_1,t}\phi_\delta(u)$ is an admissible test function in (D2). Thus, we have

$$\int_{\Omega_{t_1,t}} \partial_t(e^{-ct}|u|^q)\sigma_{t_1,t}\phi_\delta(u)\,dz + \int_{\Omega_{t_1,t}} |\nabla u|^{p-2}\nabla u \cdot \nabla\left(e^{-ct}\sigma_{t_1,t}\phi_\delta(u)\right)\,dz \le 0. \tag{3.3}$$

The first term of (3.3) is computed as

$$\int_{\Omega_{t_1,t}} \partial_t(e^{-ct}|u|^q)\min\left\{1, \frac{(e^{-ct/q}|u| - M)_+}{\delta}\right\}\sigma_{t_1,t}\,dz. \tag{3.4}$$

Since, on the support of ϕ_δ, $\{|u| > e^{ct/q}M\}$,

$$\nabla u \cdot \nabla \phi_\delta(u) = \frac{1}{\delta}\chi_{\{e^{ct/q}M<|u|\le e^{ct/q}(M+\delta)\}}|\nabla u|^2,$$

the integrand in second term of (3.3) is estimated from below as

$$\frac{|\nabla u|^p}{\delta}\chi_{\{e^{ct/q}M<|u|\le e^{ct/q}(M+\delta)\}}\sigma_{t_1,t}e^{-ct} \ge 0. \tag{3.5}$$

From (3.3), (3.4) and (3.5), we obtain

$$\int_{\Omega_{t_1,t}} \partial_t(e^{-ct}|u|^q)\min\left\{1,\ \frac{(e^{-ct/q}|u|-M)_+}{\delta}\right\}\sigma_{t_1,t}\,dz \le 0. \tag{3.6}$$

Since by (D1) $\partial_t(|u|^{q-1}u) = \partial_t|u|^q \in L^2(\Omega)$ it holds that $\partial_t(e^{-ct}|u|^q) \in L^2(\Omega_T)$. Taking the limit as $\delta \searrow 0$ in (3.6), by the Lebesgue dominant convergence theorem, we have that

$$\int_{\Omega_{t_1,t}} \partial_t(e^{-ct}|u|^q)\chi_{\{|u|>e^{ct/q}M\}}\,dz \le 0;$$
$$\int_{\Omega_{t_1,t}} \partial_t(e^{-ct}|u|^q - M^q)_+dxdt \le 0.$$

By (D3)

$$\int_\Omega (e^{-ct_1}|u(t_1)|^q - M^q)_+dx \le \int_\Omega (|u(t_1)|^q - M^q)_+dx \to 0$$

as $t_1 \searrow 0$. Hence, pass to the limit as $t_1 \searrow 0$ in (3.7) to have

$$\int_\Omega (e^{-ct}|u(t)|^q - M^q)_+dx \le 0$$

if and only if $|u(t)| \le e^{ct/q}M$ in $\Omega \times [0,T]$, and we arrive at the assertion. □

3.2 Energy equality

Here, we derive the energy equality for a weak solution to (1.1). Firstly, we need the existence of $\partial_t u$ in $L^2(\Omega_\infty)$.

Lemma 9 *Let u be a nonnegative solution to (1.1). Then there exists $\partial_t u$ in a weak sense, such that $\partial_t u \in L^2(\Omega_\infty)$.*

The proof of this lemma is by a Lipschitz approximation with the non-negativity of a solution and integrability that $\partial_t u^q$ in (D1).

By using the lemma above and Proposition 8 we have the following energy equality:

Proposition 10 (Energy equality) *Let u be a nonnegative solution to (1.1). Then the following estimates hold true:*

$$q\int_{\Omega_{0,t}} u^{q-1}(\partial_t u)^2\,dz + \frac{1}{p}\lambda(t) = \frac{1}{p}\lambda(0), \quad t \in [0,\infty).$$

In particular,

$$\lambda(t) \le \lambda(0), \quad t \in [0,\infty). \tag{3.8}$$

Proof. The function $\sigma_{t_1,t}\partial_t u$ is an admissible test function in (D2) by a usual regularization, Proposition 8 and Lemma 9. By a test function $\sigma_{t_1,t}\partial_t u$ in (D2), we have

$$\begin{aligned}
&\int_{\Omega_{t_1,t}} \partial_t(u^q)\sigma_{t_1,t}\partial_t u\, dz + \int_{\Omega_{t_1,t}} |\nabla u|^{p-2}\nabla u\cdot\nabla\left(\sigma_{t_1,t}\partial_t u\right) dz \\
&= \int_{\Omega_{t_1,t}} \lambda(t)u^q\sigma_{t_1,t}\partial_t u\, dz.
\end{aligned}\tag{3.9}$$

Note that the integral on the right hand side in (3.9) is finite by Proposition 8 and Lemma 9. Using the Lebesgue dominated convergence theorem with Proposition 8 and Lemma 9, the first term on the left hand side of (3.9) is computed as

$$\begin{aligned}
\int_{\Omega_{t_1,t}} \partial_t u^q\sigma_{t_1,t}\partial_t u\, dz &= q\int_{\Omega_{t_1,t}} u^{q-1}(\partial_t u)^2\,\sigma_{t_1,t}dz \\
&\to q\int_{\Omega_{t_1,t}} u^{q-1}(\partial_t u)^2\, dz \quad \text{as} \quad \delta\searrow 0.
\end{aligned}\tag{3.10}$$

The second term on the left hand side of (3.9) is treated as

$$\begin{aligned}
\int_{\Omega_{t_1,t}} |\nabla u|^{p-2}\nabla u\cdot\nabla\left(\sigma_{t_1,t}\partial_t u\right) dz &= \int_{\Omega_{t_1,t}} |\nabla u|^{p-2}\nabla u\cdot\partial_t\nabla u\sigma_{t_1,t}\, dz \\
&= \int_{\Omega_{t_1,t}} \partial_t\left(\frac{1}{p}|\nabla u|^p\right)\sigma_{t_1,t}\, dz \\
&= \left.\int_\Omega \frac{1}{p}|\nabla u|^p\,\sigma_{t_1,t}\, dx\right|_{t_1}^{t} - \int_{\Omega_{t_1,t}} \frac{1}{p}|\nabla u|^p\partial_t\sigma_{t_1,t}\, dz \\
&\to \int_\Omega \frac{1}{p}|\nabla u(t)|^p\, dx - \int_\Omega \frac{1}{p}|\nabla u(t_1)|^p\, dx \quad \text{as} \quad \delta\searrow 0 \\
&= \frac{1}{p}\lambda(t) - \frac{1}{p}\lambda(t_1),
\end{aligned}\tag{3.11}$$

where the manipulation in the second and third lines is justified by a usual regularization. By the volume conservation $\int_\Omega u(t)^{q+1} = 1,\ t\geq 0$, the right hand side of (3.9) is calculated as

$$\begin{aligned}
\int_{\Omega_{t_1,t}} \lambda(t)u^q\sigma_{t_1,t}\partial_t u\, dz &= \int_{t_1}^{t} \lambda(t)\sigma_{t_1,t}\frac{d}{dt}\left(\int_\Omega \frac{u^{q+1}}{q+1}\, dx\right) dt \\
&= 0.
\end{aligned}\tag{3.12}$$

From (3.10), (3.11) and (3.12), it follows that

$$q\int_{\Omega_{t_1,t}} u^{q-1}(\partial_t u)^2\, dz + \frac{1}{p}\lambda(t) - \frac{1}{p}\lambda(t_1) = 0.$$

Passing to the limit as $t_1\searrow 0$, we have the desired result. □

According to Proposition 7 and (3.8), Proposition 8 concerning the boundedness is quantitatively written as follows:

Proposition 11 *Let u be a nonnegative weak solution to (1.1) and put $\lambda_0 := \lambda(0)$. Then*

$$\|u(t)\|_{L^\infty(\Omega)} \leq e^{\lambda_0 t/q}\|u_0\|_{L^\infty(\Omega)}, \quad t\in[0,\infty).$$

4 Expansion of positivity

In this section, we will establish the expansion of positivity of a nonnegative solution of the p-Sobolev flow (1.1). A nonnegative solution of the $p-$Sobolev flow is a supersolution, satisfying in the weak sense

$$\partial_t u^q - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq 0, \tag{4.1}$$

since $u \geq 0$ by Lemma 6 and $\lambda(t) \geq 0$ for any $t \in [0, \infty)$ by Proposition 7
We make local estimates to show the expansion on space-time of positivity of a weak super solution in (4.1). For any positive numbers ρ, τ and any point $z_0 = (x_0, t_0) \in \Omega_T$, a local parabolic cylinder of radius ρ and height τ with vertex at z_0 is denoted by

$$Q(\tau, \rho)(z_0) := B_\rho(x_0) \times (t_0 - \tau, t_0).$$

For brevity, we write $Q(\tau, \rho)$ as $Q(\tau, \rho)(0)$. Following the argument in [6] (also see [5] and [24]), we proceed our local estimates.

4.1 Local energy estimates

We present the local energy estimates, called Caccioppoli type estimates, which have a crucial role in De Giorgi's method (see Section 4).

Let K be a subset compactly contained in Ω, and $0 < t_1 < t_2 \leq T$. Here we use the notation $K_{t_1,t_2} = K \times (t_1, t_2)$. Let ζ be a smooth function such that $0 \leq \zeta \leq 1$ and $\zeta = 0$ outside K_{t_1,t_2}. By use of $A^+(k, u)$ and $A^-(k, u)$, the local energy inequality can be derived.

Lemma 12 *Let $k \geq 0$. Let u be a nonnegative weak supersolution in (4.1). Then there exists a positive constant C depending only on p, n such that*

$$\begin{aligned}
&\sup_{t_1<t<t_2} \int_{K\times\{t\}} A^-(k, u)\zeta^p \, dx + \int_{K_{t_1,t_2}} |\nabla (k-u)_+ \zeta|^p dz \\
&\qquad \leq C \int_{K\times\{t_1\}} A^-(k, u)\zeta^p \, dx + C \int_{K_{t_1,t_2}} (k-u)_+^p |\nabla \zeta|^p \, dz \\
&\qquad\quad + C \int_{K_{t_1,t_2}} A^-(k, u)\zeta^{p-1}|\zeta_t| \, dz.
\end{aligned} \tag{4.2}$$

Proof. Since $\partial_t u^q \in L^2(\Omega_T)$ by (D1) and the nonnegativity of u in Ω_T, we choose a test function φ as $-(k-u)_+\zeta^p$ in (D2) to have

$$-\int_{K_{t_1,t}} \partial_t u^q (k-u)_+ \zeta^p \, dz - \int_{K_{t_1,t}} |\nabla u|^{p-2}\nabla u \cdot \nabla((k-u)_+\zeta^p) dz \leq 0. \tag{4.3}$$

By the formula (2.5), the first term of (4.3) is computed as

$$\begin{aligned}
-\int_{K_{t_1,t}} \partial_t u^q (k-u)_+ \zeta^p \, dz &= \int_{K_{t_1,t}} \partial_t A^-(k, u)\zeta^p \, dz \\
&= \int_K A^-(k, u)\zeta^p \, dx\Big|_{t_1}^{t} - p\int_{K_{t_1,t}} A^-(k, u)\zeta^{p-1}|\zeta_t| \, dz.
\end{aligned} \tag{4.4}$$

By use of Young's inequality, the second term of (4.3) is estimated from below by

$$\frac{1}{2}\int_{K_{t_1,t}} |\nabla (k-u)_+|^p \zeta^p \, dz - C\int_{K_{t_1,t}} (k-u)_+^p |\nabla \zeta|^p \, dz. \tag{4.5}$$

Gathering (4.3), (4.4) and (4.5), we obtain, for any $t \in (t_1, t_2)$,

$$\begin{aligned}\int_{K\times\{t\}} A^-(k,u)\zeta^p\,dx &+ \int_{K_{t_1,t}} |\nabla(k-u)_+|^p\zeta^p\,dz \\ &\le C\int_{K\times\{t_1\}} A^-(k,u)\zeta^p\,dx + C\int_{K_{t_1,t}} A^-(k,u)\zeta^{p-1}|\zeta_t|\,dz \\ &\quad +C\int_{K_{t_1,t}} (k-u)_+|\nabla\zeta|^p\,dz.\end{aligned}$$

Thus, we arrive at the conclusion. □

The so-called Caccioppoli type estimate follows from Lemma 12.

Proposition 13 (Caccioppoli type estimate) *Let $k \ge 0$. Let u be a nonnegative weak supersolution in (4.1). Then, there exists a positive constant C depending only on p, n such that*

$$\begin{aligned}&\sup_{t_1<t<t_2}\int_{K\times\{t\}} (k-u)_+^{q+1}\zeta^p\,dx + \int_{K_{t_1,t_2}} |\nabla(k-u)_+\zeta|^p dz \\ &\le C\int_{K\times\{t_1\}} k^{q-1}(k-u)_+^2\zeta^p\,dx + C\int_{K_{t_1,t_2}} (k-u)_+^p|\nabla\zeta|^p\,dz \\ &\qquad +C\int_{K_{t_1,t_2}} k^{q-1}(k-u)_+^2|\zeta_t|\,dz.\end{aligned} \tag{4.6}$$

Proof. We estimate $A^-(k,u) = q\int_u^k (k-\eta)_+\eta^{q-1}\,d\eta$ defined as in (2.4). The lower boundedness is obtained as follows:
Case 1 $(u \ge k/2)$: Since $\eta \ge k-\eta \ge 0$ for $\frac{k}{2} \le u \le \eta \le k$, it holds that

$$A^-(k,u) \ge q\int_u^k (k-\eta)^q\,d\eta = \frac{q}{q+1}(k-u)^{q+1}. \tag{4.7}$$

Case 2 $(u \le k/2)$: Since $0 \le k-\eta \le \eta$ for $\frac{k}{2} \le \eta \le k$, it holds that

$$\begin{aligned}A^-(k,u) &= q\int_u^{k/2}(k-\eta)_+\eta^{q-1}\,d\eta + q\int_{k/2}^k (k-\eta)_+\eta^{q-1}\,d\eta \\ &\ge q\int_{k/2}^k (k-\eta)^q\,d\eta = \frac{q}{q+1}\left(\frac{k}{2}\right)^{q+1} \\ &\ge \frac{q}{q+1}\frac{1}{2^{q+1}}(k-u)^{q+1},\end{aligned} \tag{4.8}$$

where, in the last line, $k > k-u \ge 0$ since $0 \le u \le k/2$. Also, the upper boundedness follows from

$$\begin{aligned}A^-(k,u) &= q\int_0^{(k-u)_+}(k-\eta)^{q-1}\eta\,d\eta \\ &\le qk^{q-1}\int_0^{(k-u)_+}\eta\,d\eta = qk^{q-1}\frac{(k-u)_+^2}{2}.\end{aligned} \tag{4.9}$$

From Lemma 12, (4.7), (4.8) and (4.9), we obtain the conclusion. □

4.2 Positivity estimates

Proposition 14 *Let u be a nonnegative weak supersolution in (4.1). Let $B_\rho(x_0) \subset \Omega$ with center $x_0 \in \Omega$ and radius $\rho > 0$, and $t_0 \in (0, T]$. Suppose that*

$$|B_\rho(x_0) \cap \{u(t_0) \geq L\}| \geq \alpha |B_\rho| \tag{4.10}$$

holds for some $L > 0$ and $\alpha \in (0, 1]$. Then there exists positive numbers δ, $\varepsilon \in (0, 1)$ depending only on p, n and α and independent of L such that

$$|B_\rho(x_0) \cap \{u(t) \geq \varepsilon L\}| \geq \frac{\alpha}{2} |B_\rho| \tag{4.11}$$

for all $t \in [t_0,\, t_0 + \delta L^{q+1-p} \rho^p]$.

The proof is omitted (refer to [6]).

The following estimate is crucial for the positivity of a solution.

Lemma 15 *Let u be a nonnegative weak supersolution in (4.1). Let $Q_{4\rho}(z_0) := B_{4\rho}(x_0) \times (t_0, t_0 + \delta L^{q+1-p} \rho^p) \subset \Omega_T$, where δ is selected in Proposition 14. Then for any $\nu \in (0, 1)$ there exists a positive number ε_ν depending only on p, n, α, δ and ν such that*

$$|Q_{4\rho}(z_0) \cap \{u < \varepsilon_\nu L\}| < \nu |Q_{4\rho}|.$$

Proof. We may assume $z_0 = 0$ as before. By Proposition 14, there exist positive numbers δ, $\varepsilon \in (0, 1)$ such that, for all $t \in [0,\, \delta L^{q+1-p} \rho^p]$,

$$|B_{4\rho} \cap \{u(t) \geq \varepsilon L\}| \geq \frac{\alpha}{2} 4^{-n} |B_{4\rho}| \tag{4.12}$$

Set $\theta = \delta L^{q+1-p}$ and let $\zeta = \zeta(x)$ be a piecewise smooth cutoff function satisfying $0 \leq \zeta \leq 1$, $\zeta = 0$ oustside $B_{8\rho}$, $\zeta = 1$ in $B_{4\rho}$, and $|\nabla \zeta| \leq (4\rho)^{-1}$. Let $k_j = \frac{1}{2^j} \varepsilon L$ $(j = 0, 1, \ldots)$. Applying the Caccioppoli type inequality (4.6) for the truncated solution $(k_j - u)_+$ over $Q_{4\rho}$ with the level k_j, we obtain

$$\begin{aligned}
\int_{Q_{4\rho}} |\nabla (k_j - u)_+|^p \zeta^p \, dz &\leq \int_{B_{8\rho} \times \{t=0\}} k_j^{q-1} (k_j - u)_+^2 \zeta^p \, dx + C \int_{Q_{8\rho}} (k_j - u)_+^p |\nabla \zeta|^p \, dz \\
&\leq C \left(k_j^{q+1} |B_{8\rho}| + k_j^p |Q_{8\rho}| (4\rho)^{-p} \right) \\
&\leq C k_j^p L^{q+1-p} |B_{8\rho}| \left(1 + 2^{-p} \delta\right) \\
&\leq C \frac{k_j^p}{\delta \rho^p} |Q_{8\rho}| = C \frac{k_j^p}{\delta \rho^p} |Q_{4\rho}|,
\end{aligned} \tag{4.13}$$

where the constant C depends only on n, p and independent of ρ, L. By De Giorgi's inequality in Proposition 2 to $k = k_{j+1}$ and $l = k_j$, we have, for all t, $0 \leq t \leq \delta L^{q+1-p} \rho^p = \theta \rho^p$,

$$(k_j - k_{j+1}) |A_{j+1}(t)| \leq \frac{C \rho^{n+1}}{|B_{4\rho} \setminus A_j(t)|} \int_{B_{4\rho} \cap \{k_{j+1} < u(t) < k_j\}} |\nabla u(t)| \, dx, \tag{4.14}$$

where let $A_j(t) := B_{4\rho} \cap \{u(t) < k_j\}$. By (4.12), it holds that

$$|B_{4\rho} \setminus A_j(t)| \geq \frac{\alpha}{2} 4^{-n} |B_{4\rho}|. \tag{4.15}$$

Combining (4.15) and (4.14), we have that

$$\begin{aligned}
\frac{k_j}{2} |A_{j+1}(t)| &\leq \frac{\rho^{n+1}}{|B_{4\rho} \setminus A_j(t)|} \int_{B_{4\rho} \cap \{k_{j+1} < u(t) < k_j\}} |\nabla u(t)| \, dx \\
&\leq \frac{C}{\alpha} \rho \int_{B_{4\rho} \cap \{k_{j+1} < u(t) < k_j\}} |\nabla u(t)| \, dx.
\end{aligned} \tag{4.16}$$

Integrating (4.16) in $t \in (0, \theta\rho^p)$ yields

$$\frac{k_j}{2}|A_{j+1}| \leq \frac{C}{\alpha}\rho \int_{Q_{4\rho}\cap\{k_{j+1}<u<k_j\}} |\nabla u|\, dz, \tag{4.17}$$

where we put $|A_j| := \int_0^{\theta\rho^p} |A_j(t)|\, dt = |Q_{4\rho} \cap \{u(t) < k_j\}|$. By use of Hölder's inequality, (4.13) and (4.17), we have

$$\begin{aligned} \frac{k_j}{2}|A_{j+1}| &\leq \frac{C}{\alpha}\rho \left[\int_{Q_{4\rho}} |\nabla(k_j - u)_+|^p\, dz\right]^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}} \\ &\leq \frac{C}{\alpha}\rho \frac{k_j}{\delta^{\frac{1}{p}}\rho}|Q_{4\rho}|^{\frac{1}{p}} (|A_j| - |A_{j+1}|)^{\frac{p-1}{p}} \\ &= \frac{C}{\alpha\delta^{\frac{1}{p}}} k_j |Q_{4\rho}|^{\frac{1}{p}} (|A_j| - |A_{j+1}|)^{\frac{p-1}{p}} \end{aligned} \tag{4.18}$$

and thus,

$$|A_{j+1}|^{\frac{p}{p-1}} \leq \left(\frac{C}{\alpha\delta^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} |Q_{4\rho}|^{\frac{1}{p-1}} (|A_j| - |A_{j+1}|)\,. \tag{4.19}$$

Let $J \in N$ be determined later. Summing (4.19) over $j = 0, 1, \ldots, J-1$, we obtain

$$J|A_J|^{\frac{p}{p-1}} \leq \left(\frac{C}{\alpha\delta^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} |Q_{4\rho}|^{\frac{p}{p-1}}. \tag{4.20}$$

Indeed, by use of $|A_0| \geq |A_j| \geq |A_J|$ for $j \in \{0, 1, \ldots, J\}$, we find that

$$\sum_{j=0}^{J-1} |A_{j+1}|^{\frac{p}{p-1}} \geq J|A_J|^{\frac{p}{p-1}}; \quad \sum_{j=0}^{J-1} (|A_j| - |A_{j+1}|) \leq |A_0| \leq |Q_{4\rho}|.$$

Therefore, from (4.20), it follows that

$$|A_J| \leq \frac{1}{J^{\frac{p-1}{p}}} \left(\frac{C}{\alpha\delta^{\frac{1}{p}}}\right) |Q_{4\rho}|. \tag{4.21}$$

Thus, for any $\nu \in (0, 1)$, we choose sufficiently large $J \in N$ satisfying

$$\frac{1}{J^{\frac{p-1}{p}}} \left(\frac{C}{\alpha\delta^{\frac{1}{p}}}\right) \leq \nu \iff J \geq \left(\frac{C}{\nu\alpha\delta^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}. \tag{4.22}$$

Here we note that J depends only on p, n, α, δ and ν. We finally take $\varepsilon_\nu = \dfrac{\varepsilon}{2^J}$ and then (4.21) yields that

$$\frac{|Q_{4\rho} \cap \{u < \varepsilon_\nu L\}|}{|Q_{4\rho}|} < \nu,$$

which is the desired assertion. □

Remark 16 *Noting that the parameters δ and ε in the proof of Proposition 14 are sufficiently small, we can choose ε such that*

$$\varepsilon = \left(\frac{\delta}{2^I}\right)^{\frac{1}{q+1-p}} \tag{4.23}$$

for some large positive integer I. In the proof of Lemma 15 and the choice of k_j, we also choose k_j as follows:

$$k_j = \frac{\varepsilon L}{2^{\frac{j}{q+1-p}}} \quad \text{for} \quad j = 0, 1, \ldots, J. \tag{4.24}$$

Under such choice as above we note that $k_J = \left(\frac{\delta}{2^{I+J}}\right)^{\frac{1}{q+1-p}} L$ *and obtain that* $\frac{\delta L^{q+1-p}\rho^p}{(k_J)^{q+1-p}\rho^p} = 2^{J+I}$*, which is a positive integer. Following a similar argument to [6, p.76], we next divide $Q_{4\rho}(z_0)$ into finitely many subcylinders. For any $\nu \in (0,1)$, let J be determined in (4.22). We divide $Q_{4\rho}(z_0)$ along time direction into parabolic cylinders of number $s_0 := 2^{I+J}$ with each time-length $k_J^{q+1-p}\rho^p$, and set*

$$Q^{(\ell)} := B_{4\rho}(x_0) \times \left(t_0 + \ell k_J^{q+1-p}\rho^p,\ t_0 + (\ell+1)k_J^{q+1-p}\rho^p\right)$$

for $\ell = 0, 1, \ldots, s_0 - 1$. Then there exists a $Q^{(\ell)}$ such that

$$|Q^{(\ell)} \cap \{u < k_J\}| < \nu |Q^{(\ell)}|. \tag{4.25}$$

Under the preparation above, the positivity of a solution in (4.1) is obtained in a small interval.

Theorem 17 (Expansion of local positivity) *Let u be a nonnegative weak supersolution in (4.1). Let $B_\rho(x_0) \subset \Omega$ with center $x_0 \in \Omega$ and radius $\rho > 0$, and $t_0 \in (0,T]$. Suppose that (4.10). Under (4.25) there exists a positive number $\eta < 1$ such that*

$$u \geq \eta L \quad \text{a.e.} \quad B_{2\rho}(x_0) \times \left(t_0 + \left(\ell + \frac{1}{2}\right) k_J^{q+1-p}\rho^p,\ t_0 + (\ell+1)k_J^{q+1-p}\rho^p\right). \tag{4.26}$$

Proof. Hereafter we fix the parameters ρ, ℓ and k_J. By translation we may assume to shift $(x_0, t_0 + (\ell+1)k_J^{q+1-p}\rho^p)$ to the origin and thus, $Q^{(l)}$ is transformed to $B_{4\rho}(x_0) \times (-k_J^{q+1-p}\rho^p, 0)$. For $m = 0, 1, 2, \ldots$, let

$$\tau_m = \frac{\rho^p}{2}\left(1 + \frac{1}{2^m}\right), \quad \rho_m = 2\rho\left(1 + \frac{1}{2^m}\right); \quad B_m := B_{4\rho_m}, \quad Q_m := B_m \times (-\theta\tau_m, 0),$$

where $\theta := k_J^{q+1-p}$, and also set

$$\kappa_m := \left(\frac{1}{2} + \frac{1}{2^{m+1}}\right) k_J.$$

Then, we have

$$\begin{aligned} &\rho^p = \tau_0 \geq \tau_m \searrow \tau_\infty = \rho^p/2, \quad 4\rho = \rho_0 \geq \rho_m \searrow \rho_\infty = 2\rho; \\ &Q_0 = Q^{(\ell)} \supset Q_m \searrow Q_\infty = B_{2\rho} \times (0, \theta\rho^p/2); \\ &k_J = \kappa_0 \geq \kappa_m \searrow \kappa_\infty = k_J/2. \end{aligned}$$

The cutoff function ζ is taken of the form $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$, where $\zeta_i\,(i = 1,2)$ are Lipschitz functions such that $\zeta_1 = 1$ in B_{m+1}, $\zeta_1 = 0$ in $\mathbb{R}^n \setminus B_m$ and $|\nabla\zeta_1| \leq 1/(\rho_m - \rho_{m+1}) = 2^{m+2}/\rho$ and, $\zeta_2 = 0$ for $t \leq -\theta\tau_m$, $\zeta_2 = 1$ for $t > -\theta\tau_{m+1}$ and $0 \leq \zeta_{2,t} \leq 1/\theta(\tau_m - \tau_{m+1}) \leq 2^{p(m+2)}/\theta\rho^p$. Applying the local energy inequality (4.6) over B_m and

Q_m to the truncated solution $(\kappa_m - u)_+$ and above ζ, we obtain

$$
\begin{aligned}
&\sup_{-\theta\tau_m<t<0} \int_{B_m} (\kappa_m - u(t))_+^{q+1}\zeta^p\,dx + \int_{Q_m} |\nabla(\kappa_m - u)_+|^p\zeta^p\,dz \\
&\le C\int_{Q_m} (\kappa_m - u)_+^p|\nabla\zeta|^p\,dz + C\int_{Q_m} \kappa_m^{q-1}(\kappa_m - u)^2|\zeta_t|\,dz \\
&\le C\left(\frac{2^{m+2}}{\rho}\right)^p \kappa_m^p \int_{Q_m}\left(1+\frac{\kappa_m^{q+1-p}}{\theta}\right)\chi_{\{(\kappa_m-u)_+>0\}}\,dz \\
&\le C\left(\frac{2^{m+2}}{\rho}\right)^p \kappa_m^p \int_{Q_m}\chi_{\{(\kappa_m-u)_+>0\}}\,dz, \qquad (4.27)
\end{aligned}
$$

where we used that $\dfrac{\kappa_m^{q+1-p}}{\theta} = \left(\dfrac{\kappa_m}{k_J}\right)^{q+1-p} \le 1$. Combining Proposition 3 and (4.27), we have

$$
\begin{aligned}
&\int_{Q_m} |(\kappa_m - u)_+\zeta|^{q+1}\,dz = \int_{Q_m} |(\kappa_m - u)_+\zeta|^{p\frac{n+q+1}{n}}\,dz \\
&\le C\Big(\int_{Q_m} |\nabla[(\kappa_m - u)_+\zeta]|^p\,dz\Big)\Big(_{-\theta\tau_m<t<0}\int_{B_m} |(\kappa_m - u(t))_+\zeta|^{q+1}\,dz\Big)^{\frac{p}{n}} \\
&\le C\left(\frac{2^{m+2}}{\rho}\right)^{p(1+\frac{p}{n})} k_J^{p(1+\frac{p}{n})}\Big(\int_{Q_m}\chi_{\{(\kappa_m-u)_+>0\}}\,dz\Big)^{1+\frac{p}{n}}, \qquad (4.28)
\end{aligned}
$$

where $q+1 = \frac{p(n+q+1)}{n}$ in the second line.

The left hand side of (4.28) is estimates from below as

$$
\begin{aligned}
\int_{Q_m} [(\kappa_m - u)_+\zeta]^{q+1}\,dz &\ge \int_{Q_m} [(\kappa_m - u)_+\zeta]^{q+1}\chi_{\{(\kappa_{m+1}-u)_+>0\}}\,dz \\
&\ge |\kappa_m - \kappa_{m+1}|^{q+1}\int_{Q_{m+1}}\chi_{\{(\kappa_{m+1}-u)_+>0\}}\,dz \\
&= \left(\frac{k_J}{2^{m+2}}\right)^{q+1}\int_{Q_{m+1}}\chi_{\{(\kappa_{m+1}-u)_+>0\}}\,dz. \qquad (4.29)
\end{aligned}
$$

Hence, by (4.28) and (4.29), we have

$$
\int_{Q_{m+1}}\chi_{\{(\kappa_{m+1}-u)_+>0\}}\,dz \le \frac{C[2^{p(1+\frac{p}{n})+q+1}]^m}{\rho^{p(1+\frac{p}{n})}} k_J^{p(1+\frac{p}{n})-(q+1)}\Big(\int_{Q_m}\chi_{\{(\kappa_m-u)_+>0\}}\,dz\Big)^{1+\frac{p}{n}}, \qquad (4.30)
$$

where we compute

$$
C\Big(\frac{k_J}{2^{m+2}}\Big)^{-(q+1)}\left(\frac{2^{m+1}}{\rho}\right)^{p(1+\frac{p}{n})} k_J^{p(1+\frac{p}{n})} = C\frac{[2^{p(1+\frac{p}{n})+q+1}]^m}{\rho^{p(1+\frac{p}{n})}} k_J^{p(1+\frac{p}{n})-(q+1)}.
$$

Dividing the both side of (4.30) by $|Q_{m+1}|>0$, we have

$$
\frac{1}{|Q_{m+1}|}\int_{Q_{m+1}}\chi_{\{(\kappa_{m+1}-u)_+>0\}}\,dz \le C[2^{p(1+\frac{p}{n})+q+1}]^m\left(\frac{1}{|Q_m|}\int_{Q_m}\chi_{\{(\kappa_m-u)_+>0\}}\,dz\right)^{1+\frac{p}{n}}, \qquad (4.31)
$$

where

$$
\frac{|Q_m|^{1+\frac{p}{n}}}{|Q_{m+1}|} \le C\rho^{p(1+\frac{p}{n})}(k_J^{q+1-p})^{\frac{p}{n}}
$$

and $p(1+\frac{p}{n})-(q+1)+(q+1-p)\frac{p}{n}=0$ are used.

Letting $Y_m := \int_{Q_m} \chi_{\{(\kappa_m-u)_+>0\}}\, dz/|Q_m|$, the above inequality (4.31) is rewritten as

$$Y_m \le Cb^m Y_m^{1+\frac{p}{n}}, \quad m=0,1,\dots,$$

where $b := 2^{p(1+\frac{p}{n})+q+1}$. From Lemma 4, we find that, if the initial value Y_0 satisfies

$$Y_0 \le C^{-(\frac{n}{p})} b^{-(\frac{n}{p})^2} =: \nu_0 \tag{4.32}$$

then

$$Y_m \to 0 \quad \text{as} \quad m \to \infty. \tag{4.33}$$

Eq.(4.25) is equivalent to (4.32) by taking $\nu = \nu_0$, and then (4.33) leads to the conclusion (4.26) by putting $\eta = \frac{k_J}{2L} < 1$. □

If a solution is positive at an initial time, the positivity of a solution may expand from the initial time into some positive time-interval, without any "waiting time". This follows from a minor modification of the proof of Theorem 17 above.

Corollary 18 *Let u be a nonnegative weak supersolution in (4.1). Suppose that $u(t_0) > 0$ almost everywhere in $B_{4\rho}(x_0) \subset \Omega$. Then there exist positive numbers η_0 and τ_0 such that*

$$u \ge \eta_0 \quad \textit{a.e.} \quad \textit{in} \quad B_{2\rho}(x_0) \times (t_0, t_0+\tau_0).$$

The positivity of a solution may also hold true even on a *non-convex* domain. Here we note that the De Giorgi's inequality is valid only on a *convex* domain. Let Ω' be a subdomain contained compactly in Ω. We use Theorem 17 and a method of chain of finitely many balls as the so-called *Harnack chain* used in Harnack's inequality for harmonic functions, [7, Theorem 11, pp.32–33] and have the following theorem. Here we use the special choice of parameters, as explained before Theorem 17.

Theorem 19 *Let u be a nonnegative weak supersolution in (4.1). Let Ω' be a subdomain contained compactly in Ω. Let $t_0 \in (0,T]$. Suppose that*

$$|\Omega' \cap \{u(t_0) \ge L\}| \ge \alpha |\Omega'| \tag{4.34}$$

holds for some $L > 0$ and $\alpha \in (0,1]$. Then, there exist positive integer $N = N(\Omega')$ and positive real number families $\{\delta_m\}_{m=0}^N$, $\{\eta_m\}_{m=1}^{N+1} \subset (0,1)$, $\{J_m\}_{m=0}^N$, $\{I_m\}_{m=0}^N \subset N$ depending on p, n, α and independent of L, a time $t_N > t_0$ such that

$$u \ge \eta_{N+1} L \quad \textit{a.e. in}\ \Omega' \times \left(t_N + \left(k+\frac{1}{2}\right) \frac{\delta_N (\eta_N L)^{q+1-p}}{2^{J_N+I_N}} \rho^p,\ t_N + (k+1) \frac{\delta_N (\eta_N L)^{q+1-p}}{2^{J_N+I_N}} \rho^p \right)$$

for some $k \in \left\{0,1,\dots,2^{J_N+I_N}-1\right\}$, where t_N is written as

$$t_N = t_0 + \sum_{m=1}^{N} \left(\ell + \frac{3}{4}\right) \frac{\delta_{m-1}(\eta_{m-1}L)^{q+1-p}}{2^{J_{m-1}+I_{m-1}}} \rho^p$$

for some $\ell \in \{0,1,\dots,2^{J_{m-1}+I_{m-1}}\}$.

As mentioned in Corollary 18, if a solution in (4.1) is positive almost everywhere in Ω' at some time t_0, its positivity expands in space-time without "waiting time".

Corollary 20 *Let u be a nonnegative weak supersolution in (4.1). Let Ω' be a subdomain contained compactly in Ω. Suppose that $u(t_0) > 0$ almost everywhere in Ω' for some $t_0 \in [0, T)$. Then there exist positive numbers η_0 and τ_0 such that*

$$u \geq \eta_0 \quad a.e. \quad in \quad \Omega' \times (t_0, t_0 + \tau_0).$$

In general, the solution to the fast diffusion equation of the same type nonlinearity as the equation of (1.1) may vanish at a finite time. However, under the volume constraint as in (1.1), the solution may positively expand in all of times. This is actually the assertion of the following proposition.

Proposition 21 (Interior positivity by the volume constraint) *Let Ω' be a subdomain compactly contained in Ω and very close to Ω. Let T be any positive number and Assume that $u_0 > 0$ in Ω. Let u be a nonnegative weak solution of (1.1). Then there exists a positive constant $\bar{\eta}$ such that*

$$u(x,t) \geq \bar{\eta} \quad in \quad \Omega' \times [0, T].$$

Proof. By the volume constraint and Proposition 11, letting $M := e^{\lambda_0 T/q} \|u_0\|_{L^\infty(\Omega)}$, we have, for a positive number $L < M$ and any $t \in [0, T]$

$$\begin{aligned} 1 &= \int_\Omega u^{q+1}(t)\,dx \\ &= \int_{\Omega' \cap \{u(t) \geq L\}} u^{q+1}(t)\,dx + \int_{\Omega' \cap \{u(t) < L\}} u^{q+1}(t)\,dx + \int_{\Omega \setminus \Omega'} u^{q+1}(t)\,dx \\ &\leq M^{q+1} |\Omega' \cap \{u(t) \geq L\}| + L^{q+1} ||\Omega'| + M^{q+1} |\Omega \setminus \Omega'|\,; \end{aligned}$$

and thus,

$$\frac{1 - L^{q+1}|\Omega'| - M^{q+1}|\Omega \setminus \Omega'|}{M^{q+1}} \leq |\Omega' \cap \{u(t) \geq L\}|.$$

Choose Ω' such that $|\Omega \setminus \Omega'| \leq \frac{1}{4M^{q+1}}$ and $L > 0$ satisfying $L^{q+1}|\Omega'| < \frac{1}{4}$. Under such choice of Ω' and L, we find that, for any $t \in (0, T]$,

$$\alpha |\Omega'| \leq |\Omega' \cap \{u(t) \geq L\}|, \tag{4.35}$$

where $\alpha := \dfrac{1}{2M^{q+1}|\Omega'|}$. By (3.8), a nonnegatvie weak solution u of (1.1) is a weak supersolution in (4.1). Thus, from Theorem 19, there exist positive integer $N = N(\Omega')$ and positive number families $\{\delta_m\}_{m=0}^N$, $\{\eta_m\}_{m=1}^{N+1} \subset (0,1)$, $\{J_m\}_{m=0}^N$, $\{I_m\}_{m=0}^N \subset N$ depending on p, n, α and independent of L, a time $t_N > t$ such that, for any $t \in [0, T]$,

$$u \geq \eta_{N+1} L \quad \text{a.e. in } \Omega' \times \mathcal{I}_N^{(k)}(t),$$

where $\mathcal{I}_N^{(k)}(t) := \left(t_N + \left(k + \frac{1}{2}\right) \frac{\delta_N (\eta_N L)^{q+1-p}}{2^{J_N + I_N}} \rho^p,\ t_N + (k+1) \frac{\delta_N (\eta_N L)^{q+1-p}}{2^{J_N + I_N}} \rho^p \right)$ for some $k \in \left\{0, 1, \ldots, 2^{J_N + I_N} - 1\right\}$, and t_N is written as

$$t_N = t + \sum_{m=1}^N \left(\ell + \frac{3}{4}\right) \frac{\delta_{m-1} (\eta_{m-1} L)^{q+1-p}}{2^{J_{m-1} + I_{m-1}}} \rho^p$$

for some $\ell \in \{0, 1, \ldots, 2^{J_{m-1} + I_{m-1}} - 1\}$. □

We finally state the positivity near the boundary for p-Sobolev flow (1.1).

Proposition 22 (Positivity near the boundary) *Suppose that $u_0 > 0$ in Ω. Let u be a nonnegative weak solution of (1.1). Then u is positive near the boundary $\partial\Omega$.*

Proof. For the doubly nonlinear equation of the same type as $p-$Sobolev flow (1.1) we have the comparison principle. Thus, the usual comparison argument near boundary can be applied (refer to [1]). □

5 Hölder and gradient Hölder continuity

Here we will study the Hölder and gradient Hölder continuity of the solution to p-Sobolev flow (1.1) with respect to space-time variable.

Suppose $u_0 > 0$ in Ω. Then by Propositions 21 and 8 (or 11), for any Ω' compactly contained in Ω, we can choose positive constants $\tilde{c}$ and M such that

$$0 < \tilde{c} \leq u \leq M \quad \text{in} \quad \Omega' \times [0, T]. \tag{5.1}$$

Note that the constant M depends only on $T, \lambda_0, \|u_0\|_{L^\infty(\Omega)}$ p and ν and, $\tilde{c}$ depends only on M, Ω', p and n. Under such positivity of a solution in the domain as in (5.1), we can rewrite the first equation of (1.1) as follows : Set $v := u^q$, which is equivalent to $u = v^{\frac{1}{q}}$ and put $g := \frac{1}{q} v^{1/q-1}$ and then, we find that the first equation of (1.1) is equivalent to

$$\partial_t v - \operatorname{div}(|\nabla v|^{p-2} g^{p-1} \nabla v) = \lambda(t) v \quad \text{in} \quad \Omega' \times [0, T] \tag{5.2}$$

and thus, v is a positive and bounded weak solution of the evolutionary p-Laplacian equation (5.2). By (5.1) g is uniformly elliptic and bounded in Ω'_∞. Then we have a local energy inequality for a local weak solution v to (5.2) (see [5]).

The following Hölder continuity is proved via using the local energy inequality and standard iterative real analysis methods. See [5, Chapter III] or [24, Section 4.4, pp.44–47] for more details.

Theorem 23 *Let v be a positive and bounded weak solution to (5.2). Then v is Hölder continuous in Ω'_T with a Hölder exponent $\beta \in (0, 1)$ on a space-time metric $|x| + |t|^{1/p}$ for any $T > 0$.*

By a positivity and boundedness as in (5.1) and a Hölder continuity in Theorem 23, we see that the coefficient g^{p-1} is Hölder continuous and thus, obtain a Hölder continuity of its spacial gradient.

Theorem 24 *Let v be a positive and bounded weak solution to (5.2). Then, there exist a positive exponent $\alpha < 1$ depending only on n, p, β and a positive constant C depending only on $n, p, \tilde{c}, M, \lambda(0), \beta, \|\nabla v\|_{L^p(\Omega'_T)}, [g]_{\beta,\Omega'_T}$ and $[v]_{\beta,\Omega'_T}$ such that ∇v is Hölder continuous in Ω'_T with an exponent α on the usual parabolic distance. Furthermore, its Hölder constant is bounded above by C, where $[f]_\beta$ denote the Hölder semi-norm of a Hölder continuous function f with a Hölder exponent β.*

By an elementary algebraic estimate and a interior positivity, boundedness and a Hölder regularity of v and its gradient ∇v in Theorems 23 and 24, we also have a Hölder regularity of the solution u and its gradient ∇u.

Theorem 25 (Hölder and Gradient Hölder continuity) *Let u be a positive and bounded weak solution to the p-Sobolev flow (1.1). Then, there exist a positive exponent $\gamma < 1$ depending only on n, p, β, α and a positive constant C depending only on $n, p, \tilde{c}, M, \lambda(0)$, β, α, $\|\nabla u\|_{L^p(\Omega'_T)}$, $[g]_{\beta,\Omega'_T}$ and $[v]_{\beta,\Omega'_T}$ such that u and ∇u is Hölder continuous in Ω'_T with an exponent γ on a parabolic metric $|x| + |t|^{1/p}$ and on the parabolic one, respectively. The Hölder constants are bounded above by C, where $[f]_\beta$ denote the Hölder semi-norm of a Hölder continuous function f with a Hölder exponent β.*

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