CONVERGENCE THEOREMS FOR GENERALIZED HYBRID-TYPE SEQUENCES AND SOME NONLINEAR MAPPINGS

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1. Introduction

Throughout this paper, we denote a real Hilbert space by H, and its inner product and norm by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty subset of H. A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For a mapping $T : C \to H$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [21] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

In 1975, Baillon [4] proved the following first nonlinear ergodic theorem in a Hilbert space (see also [19]). Kohsaka and Takahashi [9], and Takahashi [20] introduced the following nonlinear mappings. A mapping $T : C \to H$ is called nonspreading [9] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [20] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems for such mappings (see also [6, 10, 23]). In general, nonspreading and hybrid mappings are not continuous mappings. Kocourek, Takahashi and Yao [7] introduced a more broad class of nonlinear mappings than the class of λ-hybrid mappings in Hilbert spaces (see also [1]). A mapping $T : C \to E$ is called generalized hybrid [7] if there are real numbers α, β such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

2010 *Mathematics Subject Classification.* Primary 47H09, 47H10.

Key words and phrases. Fixed point, iteration, nonexpansive mapping, hybrid mapping, generalized hybrid mapping, convergence theorem.

for all $x, y \in C$. The nonlinear ergodic theorem by Baillon [4] for nonexpansive mapping has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao [7]. Takahashi and Takeuchi [21] proved a nonlinear ergodic theorem of Baillon's type without convexity for generalized hybrid mappings by using the concept of attractive points. Maruyama, Takahashi and Yao [14] defined a broad class of nonlinear mapping called 2-generalized hybrid which contains generalized hybrid mappings in Hilbert spaces. Let C be a nonempty subset of H. A mapping $T : C \to C$ is said to be 2-generalized hybrid [14] if there exist real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\begin{aligned} \alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 \end{aligned} \tag{1.1}$$

for all $x, y \in C$. Kondo and Takahashi [11] introduced the following class of nonlinear mapping which covers 2-generalized hybrid mappings in Hilbert spaces. A mapping $T : C \to C$ is said to be normally 2-generalized hybrid [11] if there exist real numbers $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2$ such that $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\begin{aligned} \alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ + \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \leq 0 \end{aligned} \tag{1.2}$$

for all $x, y \in C$.

On the other hand, Rouhani [16] introduced the notion of generalized hybrid sequences in Hilbert spaces and proved a nonlinear ergodic theorem of Baillon's type for the sequences. Djafari Rouhani [17] also introduced the notion of 2-generalized hybrid sequences in Hilbert spaces. Djafari Rouhani [17] proved a nonlinear ergodic theorem of Baillon's type for the sequences. Furthermore, Djafari Rouhani [15] introduced the concept of absolute fixed points for nonexpansive mappings. He studied an extension of nonexpansive mappings and proved the existence of absolute fixed points of nonexpansive mappings. He [16] proved the existence of absolute fixed points of hybrid mappings and some fixed point theorems. He [17] proved the existence of absolute fixed points of generalized hybrid mappings and some fixed point theorems.

In this paper, motivated by Baillon [4], Hojo [5], Kondo and Takahashi [11] and Djafari Rouhani[16, 17], we study a broad class of sequences which covers nonexpansive sequences, generalized hybrid sequences [16] and 2-generalized hybrid sequences [17]. Then, we obtain nonlinear ergodic theorems for the sequences by using the idea of attractive points. We also get weak convergence theorems for weakly asymptotically regular sequences. Furthermore, we obtain the existence of absolute fixed points of normally 2-generalized hybrid mappings in Hilbert spaces. We also get some fixed point theorems.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

Let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \geq 0$$

for all $y \in C$. See [19] for more details. The following result is well-known; see [19].

Lemma 2.1. *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$.

A sequence $\{x_n\}$ in H is said to be generalized hybrid [16] if there exist real numbers α, β such that

$$\alpha\|x_{i+1} - x_{j+1}\|^2 + (1-\alpha)\|x_i - x_{j+1}\|^2 \leq \beta\|x_{i+1} - x_j\|^2 + (1-\beta)\|x_i - x_j\|^2$$

for all $i, j \in \mathbb{N}$. A sequence $\{x_n\}$ in H is said to be 2-generalized hybrid [17] if there exist real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\begin{aligned} &\alpha_1\|x_{i+2} - x_{j+1}\|^2 + \alpha_2\|x_{i+1} - x_{j+1}\|^2 + (1-\alpha_1-\alpha_2)\|x_i - x_{j+1}\|^2 \\ &\quad \leq \beta_1\|x_{i+2} - x_j\|^2 + \beta_2\|x_{i+1} - x_j\|^2 + (1-\beta_1-\beta_2)\|x_i - x_j\|^2 \end{aligned} \tag{2.1}$$

for all $i, j \geq 0$. Such a sequence $\{x_n\}$ is said to be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid sequence. We note that the class of $(0, \alpha, 0, \beta)$-generalized hybrid sequences is the class of generalized hybrid sequences.

Let $\{x_n\}$ be a sequence in H. We use the following notations:

$$F_1 = \{q \in H : \text{the sequence } \{\|x_n - q\|\} \text{ is nonincreasing}\};$$

$$F_\ell = \{q \in H : \lim_{n\to\infty} \|x_n - q\| \text{ exists}\}.$$

Lemma 2.2. *Let $\{x_n\}$ be a sequence in H. Then, F_1 and F_ℓ are closed convex subset of H.*

Using a mean, we obtain the following results (see [18]): Let $\{x_n\}$ be a bounded sequence in H and μ be a mean on ℓ^∞. Then, there exists a unique point $z_0 \in H$ $\overline{\text{co}}\{x_n : n \in \mathbb{N}\}$, where $\overline{\text{co}}A$ is the closure of convex hull of A such that

$$(\mu)_n \langle x_n, z\rangle = \langle z_0, z\rangle \quad \forall z \in H.$$

We call such a unique point $z_0 \in H$ the mean values of $\{x_n\}$ for μ.

3. Nonlinear mean ergodic theorems

In this section, motivated by Baillon [4], Hojo [5], Kondo and Takahashi [11] and Djafari Rouhani [16, 17], we introduced a broad class of sequences which covers nonexpansive sequences, generalized hybrid sequences [16] and 2-generalized hybrid sequences [17]. Then, we obtain a strong convergence theorem and a nonlinear mean ergodic theorem for normally 2-generalized hybrid sequences in a Hilbert space (see [2]). In 1975, Baillon [4] proved the following first nonlinear ergodic theorem in a Hilbert space (see also [19]):

Theorem 3.1 ([4])**.** *Let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$, $S_n x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ converges weakly to a fixed point of T.*

A sequence $\{x_n\}$ in H is said to be normally 2-generalized hybrid if there exist real number $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\begin{aligned} 0 \geq\ & \alpha_2\|x_{i+2} - x_{j+1}\|^2 + \alpha_1\|x_{i+1} - x_{j+1}\|^2 + \alpha_0\|x_i - x_{j+1}\|^2 \\ & + \beta_2\|x_{i+2} - x_j\|^2 + \beta_1\|x_{i+1} - x_j\|^2 + \beta_0\|x_i - x_j\|^2 \end{aligned} \tag{3.1}$$

for all $i, j \in \mathbb{Z}^+$. We call such a sequence an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$-normally 2-generalized hybrid sequence. We note that the class of $(1-\alpha, -(1-\beta), \alpha, -\beta, 0, 0)$ -normally 2-generalized hybrid sequences is the class of generalized hybrid sequences (see [16]).

As in the proof of [13, Theorem 4], we have the following theorem (see also [12, 22]).

Theorem 3.2 ([2])**.** *Let $\{x_n\}$ be an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$ -normally 2-generalized hybrid sequence in H. Assume that $\{x_n\}$ is bounded. Then, $\{Px_n\}$ converges strongly to some $v \in H$, where P is the metric projection from H onto F_1.*

Using the idea of attractive points and Theorem 3.2, we obtain a nonlinear ergodic theorem for normally 2-generalized hybrid sequences (see also [4, 5, 11, 16, 17]).

Theorem 3.3 ([2]). *Let $\{x_n\}$ be an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$-normally 2-generalized hybrid sequence in H. Assume $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$ and $\alpha_2+\alpha_1+\alpha_0 > 0$. Then, the following are equivalent.*

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$;
(iii) $\{x_n\}$ *is bounded in* H;
(iv) $\left\{\frac{1}{n}\sum_{k=0}^{n-1} x_k\right\}$ *converges weakly to an element* $w \in H$.

Moreover, in this case $w = \lim_{n\to\infty} Px_n \in F_1$, *where* P *is the metric projection of* H *onto* F_1.

Remark 3.4. *In Theorem 3.3, we obtain that* $q = w\text{-}\lim_{n\to\infty} S_n$ *is the asymptotic center of* $\{x_n\}$ *(see, for instance* [19]*).*

By Theorem 3.3, we get the following nonlinear ergodic theorem by Djafari Rouhani [16] for generalized hybrid sequences.

Theorem 3.5 ([16]). *Let $\{x_n\}$ be a generalized hybrid sequence in H. Then, the following are equivalent.*

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$;
(iii) $\{x_n\}$ *is bounded in* H;
(iv) $\left\{\frac{1}{n}\sum_{k=1}^{n-1} x_k\right\}$ *converges weakly to some* $w \in H$.

Moreover, in this case $w = \lim_{n\to\infty} Px_n \in F_1$.

By Theorem 3.3, we get the following nonlinear ergodic theorem by Djafari Rouhani [17] for 2-generalized hybrid sequences.

Theorem 3.6 ([17]). *Let $\{x_n\}$ be a 2-generalized hybrid sequence in H. Then, the following are equivalent.*

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$;
(iii) $\{x_n\}$ *is bounded in* H;
(iv) $\left\{\frac{1}{n}\sum_{k=1}^{n-1} x_k\right\}$ *converges weakly to some* $w \in H$.

Moreover, in this case $w = \lim_{n\to\infty} Px_n \in F_1$.

4. Weak convergence theorems

In this section, using the idea of Theorem 3.3, we get weak convergence theorems for weakly asymptotically regular sequences (see [2]).

Theorem 4.1 ([2])**.** *Let $\{x_n\}$ be a normally 2-generalized hybrid sequence in H. Assume $\sum_{n=0}^{2}(\alpha_n+\beta_n) \geq 0$ and $\alpha_2+\alpha_1+\alpha_0>0$.*
Let P be the metric projection from H onto F_1. And suppose that $\{x_n\}$ is weakly asymptotically regular, i.e.,

$$x_{n+1}-x_n \rightharpoonup 0.$$

Then, the following are equivalent.

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$*;*
(iii) $\{x_n\}$ *is bounded in* H*;*
(iv) $\{x_k\}$ *converges weakly to some* $u \in H$.

Moreover, in this case $u = \lim_{n\to\infty} Px_n \in F_1$.

Remark 4.2. *In Theorem 3.3, we have that $u = w\text{-}\lim_{n\to\infty} x_n$ is the asymptotic center of $\{x_n\}$ (see, for instance* [19]*).*

By Theorem 4.1, we also get the following weak convergence theorem by Djafari Rouhani [16] for generalized hybrid sequences.

Theorem 4.3 ([16])**.** *Let $\{x_n\}$ be a generalized hybrid sequence in H. Suppose that $\{x_n\}$ is weakly asymptotically regular, i.e.,*

$$x_{n+1}-x_n \rightharpoonup 0.$$

Then, the following are equivalent.

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$*;*
(iii) $\{x_n\}$ *is bounded in* H*;*
(iv) $\{x_k\}$ *converges weakly to some* $u \in H$.

By Theorem 4.1, we also get the following weak convergence theorem by Djafari Rouhani [17] for 2-generalized hybrid sequences.

Theorem 4.4 ([17])**.** *Let $\{x_n\}$ be a 2-generalized hybrid sequence in H. Suppose that $\{x_n\}$ is weakly asymptotically regular, i.e.,*

$$x_{n+1}-x_n \rightharpoonup 0.$$

Then, the following are equivalent.

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$;
(iii) $\{x_n\}$ *is bounded in* H*;*
(iv) $\{x_k\}$ *converges weakly to some* $u \in H$.

By Theorem 4.1, we also get the following weak convergence theorem for normally 2-generalized hybrid mappings (see [2, 3]).

Theorem 4.5. *Let* C *be a nonempty subset of* H *and let* T *be a normally 2-generalized hybrid mapping of* C *into itself. Suppose that* T *is weakly asymptotically regular, i.e.,*

$$T^{n+1} - T^n x \rightharpoonup 0$$

for each $x \in C$*. Then, the following are equivalent.*

(i) $F_1 \neq \emptyset$;
(ii) $F_\ell \neq \emptyset$;
(iii) $A(T) \neq \emptyset$;
(iv) $\{T^n x\}$ *is bounded in* H *for each* $x \in C$.
(v) $\{T^n z\}$ *is bounded in* H *for some* $z \in C$.
(vi) $\{T^n x\}$ *converges weakly to an element* $v \in H$.

Moreover, in this case $v = \lim_{n\to\infty} Px_n \in A(T)$*, where* P *is the metric projection of* H *onto* F_1.

5. Absolute fixed points

In this section, using the concepts of attractive points, we establish the existence of absolute fixed points of normally 2-generalized hybrid mappings (see also [15, 16, 17]). The concept of absolute fixed points was introduced by Djafari Rouhani [15] (see also [16, 17]). Let C be a nonempty subset of H and let T be a normally 2-generalized hybrid mapping of C into itself. A point $p \in H$ is said to be an absolute fixed point of T if there exists a normally 2-generalized hybrid extension S of T from $C \cup \{p\}$ to $C \cup \{p\}$ such that $Sp = p$, and if p is a fixed point of every normally 2-generalized hybrid extension of T to the union of C and a subset of H containing p.

Kondo and Takahashi [11] proved the following nonlinear ergodic theorem of Baillon's type ([4]) (see also [2, 4]).

Theorem 5.1 ([11])**.** *Let* C *be a nonempty subset of* H *and let* T *be a normally 2-generalized hybrid mapping of* C *into itself. Assume that* $\{T^n z\}$ *is bounded for some* $z \in D$*. Let* $P_{A(T)}$ *be the metric projection of* H *onto* $A(T)$*. Then,* *for each* $x \in C$, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} T^k x$ *converges weakly to* $u \in A(T)$, *where* $u = \lim_{n\to\infty} P_{A(T)} T^n x$.

By Theorem 5.1, we have the following (see [3]).

Proposition 5.2 ([3]). *Let* C *be a nonempty subset of* H *and let* T *be a normally* 2*-generalized hybrid mapping of* C *into itself. Assume that* $\{T^n z\}$ *is bounded for some* $z \in C$. *Let* $x \in C$ *and let*

$$u = \text{w-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} T^k x.$$

Then, for each $y \in C$ *and* $n \in \mathbb{Z}^+$,

$$\|T^{n+1}y - u\| \leq \|T^n y - u\|$$

holds. Thus, $u \in F_1$.

Theorem 5.3 ([3]). *Let* C *be a nonempty subset of* H *and let* T *be an* $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$ *-normally* 2*-generalized hybrid mapping of* C *into itself Assume that* $\{T^n z\}$ *is bounded for some* $z \in C$. *Let* $x \in C$ *and let*

$$u = \text{w-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} T^k x.$$

Let M *be a nonempty subset of* H *such that* $M \supset C \cup \{u\}$. *Assume that* S *is a normally* 2*-generalized hybrid extension of* T *to* M. *Then, we have* $Su = u$.

Adding that C is closed and convex, we can obtain the following fixed point theorem ([11]) by Theorem 5.3 (see also [11, 17, 21]).

Theorem 5.4 ([3]). *Let* C *be a nonempty closed and convex subset* H *and let* T *be a normally* 2*-generalized hybrid mapping of* C *into itself. Assume that* $\{T^n z\}$ *is bounded for some* $z \in C$. *Then, we have* $F(T) \neq \emptyset$.

In our next lemma, we give a sufficient condition for a normally 2-generalized hybrid mapping of C into itself with a bounded orbit, to have a normally 2-generalized hybrid extension to $C \cup \{u\}$, where $u = \text{w-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1} T^i x$.

Lemma 5.5 ([3]). *Let* C *be a nonempty subset* H *and let* T *be an* $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$ *-normally* 2*-generalized hybrid mapping of* C *into itself. Assume that* $\{T^n z\}$

is bounded for some $z \in C$. Let $x \in C$ and let

$$u = \text{w-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1} T^i x.$$

Then, the mapping $S : C\cup\{u\} \to C\cup\{u\}$ defined as $Sz = Tz$ for all $z \in C$, and $Su = u$ is an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$ -normally 2-generalized hybrid mapping of $C \cup \{u\}$ into itself, if either $\alpha_2 + \beta_2 \geq 0$, $\alpha_1 + \beta_1 \geq 0$ and $\sum_{n=0}^{2}(\alpha_n + \beta_n) = 0$, or $\alpha_2 + \beta_2 < 0$, $\alpha_1 + \beta_1 < 0$, $\sum_{n=0}^{2}(\alpha_n + \beta_n) = 0$ and the orbit $\{T^k y\}$ of every $y \in C$ lies on the sphere centered at y, with the radius $\|y - u\|$.

By Theorem 5.3 and Lemma 5.5, we get the existence of absolute fixed points of normally 2-generalized hybrid mappings.

Theorem 5.6 ([3]). *Let C be a nonempty subset H and let T be an $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$ -normally 2-generalized hybrid mapping of C into itself. Assume that $\{T^n z\}$ is bounded for some $z \in C$. Let $x \in C$ and let*

$$u = \text{w-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} T^k x.$$

Then, u is an absolute fixed point of T if either $(\alpha_2 + \beta_2) \geq 0$, $(\alpha_1 + \beta_1) \geq 0$ and $\sum_{n=0}^{2}(\alpha_n + \beta_n) = 0$, or $\alpha_2 + \beta_2 < 0$, $\alpha_1 + \beta_1 < 0$, $\sum_{n=0}^{2}(\alpha_n + \beta_n) = 0$ and the orbit $\{T^k y\}$ of every $y \in C$ lies on the sphere centered at y, with the radius $\|y - c\|$.

6. Fixed point theorems

In this section, motivated by [16, 17], we obtain some fixed point theorems for normally 2-generalized hybrid mappings defined on nonconvex domains in H.

Let C be a nonempty subset of H. We say that C is Chebyshev with respect to its convex closure, if for any $y \in \overline{\text{co}}\, C$, there is a unique $q \in C$ such that

$$\|y - q\| = \inf\{\|y - z\| : z \in C\}$$

(see [16, 17]).

Theorem 6.1 ([3]). *Let C be a nonempty subset of H and let T be a normally 2-generalized hybrid mapping of C into itself. Then, T has a fixed* *point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$, and for any $y \in \overline{\mathrm{co}}\,\{T^n x : n \in \mathbb{Z}^+\}$, there is a unique $p \in C$ such that $\|y-p\| = \inf\{\|y-z\| : z \in C\}$.*

As a directed consequence of Theorem 6.1, we obtain the following theorem.

Theorem 6.2 ([3]). *Let C be a nonempty subset of H which is Chebyshev with respect its convex closure. Let T be a normally 2-generalized hybrid mapping of C into itself. Then, T has a fixed point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

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