Boundedness of the commutators of fractional integral operators on mixed Morrey spaces

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Abstract

We give a necessary and sufficient condition for the boundedness of the commutators of fractional integral operators on mixed Morrey spaces. Furthermore, we show the sharp maximal inequality of the mixed Morrey norm. This inequality is one of the key inequalities of the main theorem.

1 Introduction

The aim of this paper is to obtain a necessary and sufficient condition for the boundedness of the commutators generated by BMO and the fractional integral operator I_α on mixed Morrey spaces defined in [15]. Let $1 \le q \le p < \infty$. Define the *Morrey norm* $\|\cdot\|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup\left\{ |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \ : \ Q \text{ is a cube in } \mathbb{R}^n \right\}$$

for a measurable function f. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_{\mathcal{M}^p_q}$ is finite. The mixed Morrey space $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ is the function space which combines mixed Lebesgue spaces [4] and Morrey spaces [13].

Definition 1.1. [15] Let $\vec{q} = (q_1, \ldots, q_n) \in (0,\infty]^n$ and $p \in (0,\infty]$ satisfy

$$\sum_{j=1}^n \frac{1}{q_j} \ge \frac{n}{p}.$$

We define the *mixed Morrey space* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ to be the set of all $f \in L^0(\mathbb{R}^n)$ satisfying the following norm $\|\cdot\|_{\mathcal{M}^p_{\vec{q}}}$ is finite:

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \equiv \sup\left\{ |Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|f\chi_Q\|_{\vec{q}} \ : \ Q \text{ is a cube in } \mathbb{R}^n \right\},$$

where

$$\|f\|_{\vec{q}} \equiv \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1,\ldots,x_n)|^{q_1} \mathrm{d}x_1\right)^{\frac{q_2}{q_1}} \mathrm{d}x_2\right)^{\frac{q_3}{q_2}} \cdots \mathrm{d}x_n\right)^{\frac{1}{q_n}}.$$

Remark 1.2. (i) When each $q_i = q$, then $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n)$.

(ii) $0 < \vec{q} \leq \vec{r} \leq \infty$, $0 < p < \infty$, $\dfrac{1}{r_1} + \cdots + \dfrac{1}{r_n} \geq \dfrac{n}{p}$. Then,

$$\mathcal{M}^p_{\vec{r}}(\mathbb{R}^n) \subset \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n).$$

Let $0 < \alpha < n$. Define the fractional integral operator I_α of order α by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y, \quad x \in \mathbb{R}^n$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as long as the right-hand side makes sense. The commutator $[a, I_\alpha]$ is given by

$$[a, I_\alpha](f)(x) \equiv \int_{\mathbb{R}^n} \frac{a(x)-a(y)}{|x-y|^{n-\alpha}} f(y)\mathrm{d}y, \quad x \in \mathbb{R}^n$$

as long as the integral makes sense.

The following is our main theorem.

Theorem 1.3. *Let $0 < \alpha < n$ and $1 < p < \dfrac{n}{\alpha}$. Assume that*

$$\frac{n}{p} \leq \sum_{j=1}^n \frac{1}{q_j}, \quad \frac{n}{r} \leq \sum_{j=1}^n \frac{1}{s_j}, \quad \frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q_j}{p} = \frac{s_j}{r} \quad (j = 1, \ldots, n).$$

Then, the following conditions are equivalent:

(a) $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(b) $[b, I_\alpha]$ *is bounded from* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *to* $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

(c) $[b, I_\alpha]$ *is bounded from* $\widetilde{\mathcal{M}}^p_{\vec{q}}(\mathbb{R}^n)$ *to* $\mathcal{M}^r_{\vec{s}}(\mathbb{R}^n)$.

Here, $\widetilde{\mathcal{M}}^p_{\vec{q}}(\mathbb{R}^n)$ *is the* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$*-closure of* $C^\infty_c(\mathbb{R}^n)$.

Recall that $\mathrm{BMO}(\mathbb{R}^n)$ is the John–Nirenberg space. That is, $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space, modulo constants, with the norm $\|\cdot\|_{\mathrm{BMO}}$ defined by

$$\|b\|_{\mathrm{BMO}} \equiv \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| \mathrm{d}x,$$

where

$$b_Q \equiv \frac{1}{|Q|}\int_Q b(y)\mathrm{d}y$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Throughout the paper, we use the following notation. The letters $\vec{p}, \vec{q}, \vec{r}, \dots$ will denote n-tuples of the numbers in $[0,\infty]$ $(n \geq 1)$, $\vec{p} = (p_1, \dots, p_n), \vec{q} = (q_1, \dots, q_n), \vec{r} = (r_1, \dots, r_n)$. By definiton, the inequality, for example, $0 < \vec{p} < \infty$ means that $0 < p_i < \infty$ for each i. Furthermore, for $\vec{p} = (p_1, \dots, p_n)$ and $r \in \mathbb{R}$, let

$$\frac{1}{\vec{p}} = \left(\frac{1}{p_1}, \dots, \frac{1}{p_n}\right), \quad \frac{\vec{p}}{r} = \left(\frac{p_1}{r}, \dots, \frac{p_n}{r}\right), \quad \vec{p}' = (p_1', \dots, p_n'),$$

where $p_j' = \frac{p_j}{p_j - 1}$ is the conjugate exponent of p_j. Let $Q = Q(x, r)$ be a cube having center x and radius r, whose sides are parallel to the cordinate axes and $\mathcal{Q}$ denote the set of all cubes in $\mathbb{R}^n$. $|Q|$ denotes the volume of the cube Q and $\ell(Q)$ denotes the side length of the cube Q. By $A \lesssim B$, we denote that $A \leq CB$ for some constant $C > 0$, and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

We showed that the Hardy–Littlewood maximal operator M and the fractional integral operator I_α are bounded on mixed Morrey spaces [15]:

Theorem 1.4. [15] *Let $1 < \vec{q} < \infty$ and $1 < p < \infty$ satisfy $\dfrac{n}{p} \leq \displaystyle\sum_{j=1}^{n} \frac{1}{q_j}$. Then*

$$\|Mf\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}}$$

for all $f \in L^0(\mathbb{R}^n)$.

Theorem 1.5. [15] *Let $0 < \alpha < n, 1 < \vec{q}, \vec{s} < \infty$ and $1 < p, r < \infty$. Assume that*

$$\frac{n}{p} \leq \sum_{j=1}^{n} \frac{1}{q_j}, \quad \frac{n}{r} \leq \sum_{j=1}^{n} \frac{1}{s_j}, \quad \frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{\vec{q}}{p} = \frac{\vec{s}}{r}.$$

Then, for $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$,

$$\|I_\alpha f\|_{\mathcal{M}^r_{\vec{s}}} \lesssim \|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

The above two theorems extend the classical case. Chiarenza and Frasca [5] showed the boundedness of the Hardy–Littlewood maximal operator M on $\mathcal{M}^p_q(\mathbb{R}^n)$ for $1 < q \leq p < \infty$. Adams [1] pointed out that I_α is bounded from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ whenever $1 < q \leq p < \infty$, $1 < t \leq s < \infty$, $\dfrac{q}{p} = \dfrac{t}{s}$ and $\dfrac{1}{s} = \dfrac{1}{p} - \dfrac{\alpha}{n}$.

We compare our results with the classical ones. Usually, when we handle commtators, the sharp maximal operator is a useful tool as was done in [6, 8, 18]. The sharp maximal operator, defined in [7], is a good operator to control singularity of the integral operators. To control the sharp maximal operator, we use the "so-called" good λ-inequality described in [21]. However, the layer cake formula, which is also described in [21], is not available in the mixed-norm setting. So, we need a new device. We make use of the dyadic local sharp maximal function defined in [12] together with a key formula [12, Theorem 2.2]. What is new in this paper is to investigate the duality of mixed Morrey spaces. In [2, 17, 19] a duality formula is obtained. In particular in [16] Rosenthal and Schmeisser applied this formula to the boundedness of operators acting on Morrey spaces. By borrowing their ideas [2, 16, 17, 19] we will obtain a new characterization of the boundedness of the commutators.

2 Predual of mixed Morrey spaces

In the proof of the main theorem, we use the duality of mixed Morrey spaces and its predual. So in this section, we introduce the predual spaces of mixed Morrey spaces. Here we follow the idea of Zorko [22].

Definition 2.1. Let $1 \le p < \infty$ and $\dfrac{n}{p} \le \displaystyle\sum_{j=1}^{n} \frac{1}{q_j}$. A measurable function A is said to be a $(p, \vec{q})$-block if there exists a cube Q that supports A such that

$$\|A\|_{\vec{q}} \le |Q|^{\frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right) - \frac{1}{p}}.$$

Definition 2.2. Let $1 \le p < \infty$ and $\dfrac{n}{p} \le \displaystyle\sum_{j=1}^{n} \frac{1}{q_j}$. Define the function space $\mathcal{H}^{p}_{\vec{q}}(\mathbb{R}^n)$ as the set of all $f \in L^p(\mathbb{R}^n)$ for which f is realized as the sum $f = \displaystyle\sum_{j=0}^{\infty} \lambda_j A_j$ with some $\lambda = \{\lambda_j\}_{j \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0)$ and a sequence $\{A_j\}_{j \in \mathbb{N}_0}$ of $(p, \vec{q})$-blocks, where the convergence takes place in $L^p(\mathbb{R}^n)$. Define the norm $\|f\|_{\mathcal{H}^{p}_{\vec{q}}}$ for $f \in \mathcal{H}^{p}_{\vec{q}}(\mathbb{R}^n)$ as

$$\|f\|_{\mathcal{H}^{p}_{\vec{q}}} \equiv \inf_{\lambda} \|\lambda\|_{\ell^1},$$

where $\lambda = \{\lambda_j\}_{j \in \mathbb{N}_0}$ runs over all the above expressions

We shall give one simple property of the function space $\mathcal{H}^{p}_{\vec{q}}(\mathbb{R}^n)$. By this lemma, we can regard the element of $L^{\vec{q}}_{c}(\mathbb{R}^n)$ as a $(p', \vec{q'})$-block modulo multiplicative constants.

Lemma 2.3. *Let* $1 \le p < \infty$ *and* $\dfrac{n}{p} \le \displaystyle\sum_{j=1}^{n} \frac{1}{q_j}$. *Let* A *be an* $L^{\vec{q'}}(\mathbb{R}^n)$ *function supported on a cube* Q. *Then,*

$$\|A\|_{\mathcal{H}^{p'}_{\vec{q'}}} \le \|A\|_{\vec{q'}} |Q|^{\frac{1}{n}\left(\sum_{j=1}^{n} \frac{1}{q_j}\right) - \frac{1}{p}}. \tag{1}$$

We can extend the result by Zorko [22] to mixed Morrey spaces.

Theorem 2.4. *Suppose that* $1 < p < \infty$ *and* $\dfrac{n}{p} \le \displaystyle\sum_{j=1}^{n} \frac{1}{q_j}$*. Then,* $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$ *is a predual space of the mixed Morrey space* $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$*. More precisely,*

(i) *Any* $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *defines a continuous functional* L_f *by:*

$$L_f : \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n) \ni g \longmapsto \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \in \mathbb{C}$$

on $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$.

(ii) *Conversely, every continuous functional* L *on* $\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)$ *can be realized with* $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$.

(iii) *The correspondence*

$$\tau \,:\, \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) \ni f \longmapsto L_f \in \left(\mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n)\right)^*$$

is an isomorphism. Furthermore,

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : g \in \mathcal{H}^{p'}_{\vec{q}'}(\mathbb{R}^n), \|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} = 1\right\} \tag{2}$$

and

$$\|g\|_{\mathcal{H}^{p'}_{\vec{q}'}} = \max\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n), \|f\|_{\mathcal{M}^p_{\vec{q}}} = 1\right\}. \tag{3}$$

In the proof of Theorem 2.4, we obtain the following duality inequality:

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| \le \|f\|_{\mathcal{M}^p_{\vec{q}}}\|g\|_{\mathcal{H}^{p'}_{\vec{q}'}}. \tag{4}$$

Example 2.5. Let $1 < p < \infty$, $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$. Then, we have

$$\|\chi_Q\|_{\mathcal{M}^p_{\vec{q}}} = |Q|^{\frac{1}{p}}, \quad \|\chi_Q\|_{\mathcal{H}^{p'}_{\vec{q}'}} = |Q|^{\frac{1}{p'}}.$$

3 Sharp maximal inequality

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Sharp maximal operaor is defined by

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|\mathrm{d}y, \quad (x \in \mathbb{R}^n).$$

To show the main theorem, we need the following sharp maximal inequality:

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \left\|f^{\#}\right\|_{\mathcal{M}^p_{\vec{q}}}.$$

To prove this inequality, we use the dyadic local sharp maximal operator $M^{\#,d}_{\lambda;Q_0}$.

Definition 3.1. Let $f \in L^0(\mathbb{R}^n)$ and $Q \in \mathcal{Q}$.

1. The *decreasing rearrangement* of f on $\mathbb{R}^n$ is defined by

$$f^*(t) \equiv |\{\rho > 0 \, : \, \mu_f(\rho) > t\}| \quad (0 < t < \infty),$$

where $\mu_f(\rho) = |\{x \in \mathbb{R}^n \, : \, |f(x)| > \rho\}|$.

2. The *local mean oscillation* of f on Q is defined by

$$\omega_\lambda(f\,;\,Q) \equiv \inf_{c \in \mathbb{C}} \left((f - c)\chi_Q\right)^* (\lambda|Q|) \quad (0 < \lambda < 2^{-1}).$$

3. Assume that the function f is real-valued. the *median* of f over Q, denoted by $m_f(Q)$, is a real number satisfying

$$|\{x \in Q \, : \, |f(x)| > m_f(Q)\}| \, , \quad |\{x \in Q \, : \, |f(x)| < m_f(Q)\}| \le \frac{1}{2}|Q|.$$

Denote $\mathcal{D}(Q_0)$ the set of all dyadic cubes with respect to the cube Q_0. For $0 < \lambda < 2^{-1}$ and $Q_0 \in \mathcal{Q}$, the *dyadic local sharp maximal operator* $M^{\#,d}_{\lambda;Q_0}$ is defined by

$$M^{\#,d}_{\lambda;Q_0} f(x) \equiv \sup_{Q \in \mathcal{D}(Q_0)} \omega_\lambda(f\,;\,Q)\chi_Q(x), \quad x \in \mathbb{R}^n, f \in L^0(\mathbb{R}^n).$$

Moreover, we use the following sharp maximal operator:

$$M^{\#,d}_{\lambda} f(x) \equiv \sup_{Q_0 \in \mathcal{Q}} \sup_{Q \in \mathcal{D}(Q_0)} \omega_\lambda(f\,;\,Q)\chi_Q(x), \quad x \in \mathbb{R}^n, f \in L^0(\mathbb{R}^n).$$

Jawerth and Torchinsky proved a pointwise equivalence between these two types of the sharp maximal operators : in [11]

$$M\left[M^{\#,d}_{\lambda} f\right](x) \sim f^{\#}(x) \quad (x \in \mathbb{R}^n) \tag{5}$$

for sufficiently small λ.

Theorem 3.2. *Let $0 < \vec{q} < \infty$, $0 < p < \infty$ satisfy $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$. Then, for any $f \in L^0(\mathbb{R}^n)$ satisfying $Mf \in \mathcal{M}^{p_0}_{\vec{q_0}}(\mathbb{R}^n)$ for some $\vec{q_0} = (q_{0,1}, \ldots, q_{0,n}) \in (0,\infty)^n$ and $0 < p_0 < \infty$ with*

$$\frac{n}{p_0} \le \sum_{j=1}^{n} \frac{1}{q_{0,j}},$$

we have

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \sim \left\|M^{\#,d}_{\lambda} f\right\|_{\mathcal{M}^p_{\vec{q}}} \lesssim \left\|f^{\#}\right\|_{\mathcal{M}^p_{\vec{q}}}.$$

One of the key theorem is the following one, which represent a norm equivalence similar to [14, 18].

Theorem 3.3. *Let* $0 < \vec{q} < \infty$ *and* $0 < p, s < \infty$ *satisfying*

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}, \quad s \le \min(q_1, \ldots, q_n, p).$$

Then, for all $f \in L^0(\mathbb{R}^n)$*, it holds that*

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \sim \left\| M^{\#,d}_\lambda f \right\|_{\mathcal{M}^p_{\vec{q}}} + \|f\|_{\mathcal{M}^p_s}.$$

The term $\|f\|_{\mathcal{M}^p_s}$ in Theorem 3.3 is an auxiliary one although this explains how Morrey spaces can be used to control operators acting on Lebesgue spaces. We can remove this term under a resonable condition using the idea by Fujii [9].

Theorem 3.4. *Let* $0 < \vec{q} < \infty$ *and* $0 < p < \infty$ *satisfying* $\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}$*. Assume that* $f \in L^0(\mathbb{R}^n)$ *satisfies*

$$m_f(2^\ell Q) \to 0 \quad (\ell \to \infty)$$

for any $Q \in \mathcal{Q}$ *and for some medians* $\{m_f(2^\ell Q)\}_{\ell \in \mathbb{N}_0}$*. Then, we have*

$$\|f\|_{\mathcal{M}^p_{\vec{q}}} \sim \left\| M^{\#,d}_\lambda f \right\|_{\mathcal{M}^p_{\vec{q}}} \le \left\| M^{\#,d}_\lambda f \right\|_{\mathcal{M}^p_{\vec{q}}}.$$

The condition proposed by Fujii [9] can be verified as follows:

Lemma 3.5. *Let* $f \in L^0(\mathbb{R}^n)$*. Assume that* $Mf \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *for some* $0 < \vec{q} < \infty$ *and* $0 < p < \infty$ *satisfying*

$$\frac{n}{p} \le \sum_{j=1}^{n} \frac{1}{q_j}.$$

Then, for any $Q \in \mathcal{Q}$ *and any medians* $\{m_f(2^\ell Q)\}_{\ell \in \mathbb{N}_0}$*, it holds that*

$$\lim_{\ell \to \infty} m_f(2^\ell Q) = 0.$$

Finally, we evaluate the sharp maximal function of the commutator $[b, I_\alpha]f$. The following estimate is also important to show the main theorem.

Lemma 3.6. *Let* $0 < \alpha < n$ *and* $1 < \eta < \infty$*. Then,*

$$([b, I_\alpha]f)^{\#}(x) \lesssim \|b\|_{\mathrm{BMO}} \left(M^{(\eta)}[I_\alpha f](x) + M^{(\eta)}_{\eta\alpha} f(x) \right)$$

for all $b \in \mathrm{BMO}(\mathbb{R}^n)$*,* $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$*. Here,*

$$M^{(\eta)}_{\eta\alpha} f(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\frac{\eta\alpha}{n}}} \int_Q |f(y)|^\eta \mathrm{d}y \right)^{\frac{1}{\eta}}.$$

Note that a similar lemma to this is proved in [3, 20]. Shirai showed this estimate for $f \in C_c^\infty(\mathbb{R}^n)$ [20, Lemma 4.2], and Arai and Nakai showed a similar estimate for the element of generalized Campanato spaces and generalized Morrey spaces [3, Proposition 5.2], respectively.

4 Proof of main theorem

(a) $\Rightarrow$ (b): Let $1 < \eta < \min(s_1, \dots, s_n, r)$ and $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. Put $s_0 = \min(s_1, \dots, s_n)$ and $q_0 = \min(q_1, \dots, q_n)$. Since $f \in \mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_0}(\mathbb{R}^n)$, $[b, I_\alpha]f \in \mathcal{M}^r_{s_0}(\mathbb{R}^n) = \mathcal{M}^r_{(\underbrace{s_0, \dots, s_0}_{n \text{ times}})}(\mathbb{R}^n)$.

Then, we see that $M([b, I_\alpha]f) \in \mathcal{M}^r_{(\underbrace{s_0, \dots, s_0}_{n \text{ times}})}(\mathbb{R}^n)$. Thus, the assumption of Theorem 3.2 is satisfied. Then, by virtue of Theorem 3.2 and Lemma 3.6, we have

$$\begin{aligned}
\|[b, I_\alpha](f)\|_{\mathcal{M}^r_{\vec{s}}} &\lesssim \left\|([b, I_\alpha]f)^\#\right\|_{\mathcal{M}^r_{\vec{s}}} \\
&\lesssim \|b\|_{\mathrm{BMO}} \left\|M^{(\eta)}[I_\alpha f] + M^{(\eta)}_{\eta\alpha} f\right\|_{\mathcal{M}^r_{\vec{s}}} \\
&\le \|b\|_{\mathrm{BMO}} \left\{\|M^{(\eta)}[I_\alpha f]\|_{\mathcal{M}^r_{\vec{s}}} + \left\|M^{(\eta)}_{\eta\alpha} f\right\|_{\mathcal{M}^r_{\vec{s}}}\right\} \\
&\lesssim \|b\|_{\mathrm{BMO}} \left\{\|I_\alpha(|f|)\|_{\mathcal{M}^r_{\vec{s}}} + \|I_{\eta\alpha}(|f|^\eta)\|^{\frac{1}{\eta}}_{\mathcal{M}^{\frac{r}{\eta}}_{\frac{\vec{s}}{\eta}}}\right\}.
\end{aligned}$$

Using Theorem 1.5, we conclude

$$\|[b, I_\alpha](f)\|_{\mathcal{M}^r_{\vec{s}}} \lesssim \|b\|_{\mathrm{BMO}} \left\{\|f\|_{\mathcal{M}^p_{\vec{q}}} + \||f|^\eta\|^{\frac{1}{\eta}}_{\mathcal{M}^{\frac{p}{\eta}}_{\frac{\vec{q}}{\eta}}}\right\} = \|b\|_{\mathrm{BMO}} \|f\|_{\mathcal{M}^p_{\vec{q}}}.$$

(b) $\Rightarrow$ (c): It is clear since we only restrict the domain.

(c) $\Rightarrow$ (a): We use the same method as Janson [10]. Choose $z_0 \in \mathbb{R}^n$ such that $|z_0| = 5$. Then, since $0 \notin Q(z_0, 2)$, $|x|^{n-\alpha} \in C^\infty(Q(z_0, 2))$ for $x \in Q(z_0, 2)$. Hence, we choose a function $\varphi \in C^\infty(\mathbb{R}^n)$ which is π periodic and satisfies $\varphi(x) = |x|^{n-\alpha}$ for all $x \in Q(z_0, 2)$. Then, we can expand this function into the absolutely convergent Fourier series on $Q(z_0, 2)$;

$$|x|^{n-\alpha}\chi_{Q(z_0,2)}(x) = \sum_{m \in \mathbb{Z}^n} a_m e^{2im\cdot x}\chi_{Q(z_0,2)}(x) \tag{6}$$

with $\sum_m |a_m| < \infty$. For any $x_0 \in \mathbb{R}^n$ and $t > 0$, let $Q = Q(x_0, t)$ and $Q' = Q(x_0 + z_0 t, t)$. Let

$$s(x) = \mathrm{sgn}\left(\int_{Q'} (b(x) - b(y))\mathrm{d}y\right).$$

If $x \in Q$ and $y \in Q'$, then $\frac{y-x}{t} \in Q(z_0, 2)$. Hence, we have

$$\begin{aligned}
&\int_Q |b(x) - b_{Q'}|\mathrm{d}x = \int_Q (b(x) - b_{Q'})\overline{s(x)}\mathrm{d}x \\
&= \frac{1}{|Q'|}\int_Q \overline{s(x)}\left(\int_{Q'} (b(x) - b(y))\mathrm{d}y\right)\mathrm{d}x \\
&= \frac{t^{n-\alpha}}{t^n}\int_Q \overline{s(x)}\left(\int_{Q'} (b(x) - b(y))|x-y|^{-n+\alpha}\left|\frac{x-y}{t}\right|^{n-\alpha}\mathrm{d}y\right)\mathrm{d}x.
\end{aligned}$$

By (6), we get

$$\begin{aligned}
&\int_Q |b(x) - b_{Q'}|\mathrm{d}x \\
&= t^{-\alpha}\sum_{m\in\mathbb{Z}^n}\int_Q \overline{s(x)}\left(\int_{Q'} (b(x) - b(y))|x-y|^{-n+\alpha}a_m e^{2im\cdot\frac{y}{t}}\mathrm{d}y\right)e^{-2im\cdot\frac{x}{t}}\mathrm{d}x \\
&\le t^{-\alpha}\sum_{m\in\mathbb{Z}^n}\left|a_m\int_{\mathbb{R}^n}\overline{s(x)}[b, I_\alpha](e^{2im\cdot\frac{\cdot}{t}}\chi_{Q'})(x)\chi_Q(x)e^{-2im\cdot\frac{x}{t}}\mathrm{d}x\right|.
\end{aligned}$$

Applying (4), we obtain

$$\begin{aligned}
\int_Q |b(x) - b_{Q'}|\mathrm{d}x &\le t^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\left\|[b, I_\alpha](e^{2im\cdot\frac{\cdot}{t}}\chi_{Q'})\right\|_{\mathcal{M}^r_{\vec{s}}}\|\chi_Q\|_{\mathcal{H}^{r'}_{\vec{s}'}} \\
&\le t^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\,\|[b, I_\alpha]\|_{\tilde{\mathcal{M}}^p_{\vec{q}}\to\mathcal{M}^r_{\vec{s}}}\|\chi_{Q'}\|_{\mathcal{M}^p_{\vec{q}}}\|\chi_Q\|_{\mathcal{H}^{r'}_{\vec{s}'}} \\
&\le t^{-\alpha}\sum_{m\in\mathbb{Z}^n}|a_m|\,\|[b, I_\alpha]\|_{\tilde{\mathcal{M}}^p_{\vec{q}}\to\mathcal{M}^r_{\vec{s}}}t^{\frac{n}{p}}\cdot t^{\frac{n}{r'}} \\
&\sim t^n\,\|[b, I_\alpha]\|_{\tilde{\mathcal{M}}^p_{\vec{q}}\to\mathcal{M}^r_{\vec{s}}}.
\end{aligned}$$

Thus, we have

$$\frac{1}{|Q|}\int_Q |b(x) - b_Q|\mathrm{d}x \le \frac{2}{|Q|}\int_Q |b(x) - b_{Q'}|\mathrm{d}x \lesssim \|[b, I_\alpha]\|_{\tilde{\mathcal{M}}^p_{\vec{q}}\to\mathcal{M}^r_{\vec{s}}}.$$

This implies that $b \in \mathrm{BMO}$.

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