

Besov 空間の基礎と積公式について¹
 Navier-Stokes 方程式の定常問題への応用²
 Besov 空間における熱半群の $L^p - L^q$ 型評価について³

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1 Introduction to the Besov space and Leibniz rule

Let us recall homogeneous and inhomogeneous Besov spaces. For that purpose, we first introduce the Littlewood-Paley decomposition of functions defined on \mathbb{R}^n in terms with the partition $\{\varphi_j\}_{j=-\infty}^{\infty}$ of unity in the Fourier variables. We take $\phi \in C_0^\infty(\mathbb{R}^n)$ in such a way that $\text{supp } \phi = \{\xi \in \mathbb{R}^n; \frac{1}{2} \leq |\xi| \leq 2\}$ satisfying $\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1$ for all $\xi \neq 0$. The functions φ_j are defined as $\mathcal{F}\varphi_j(\xi) = \phi(2^{-j}\xi)$, $j \in \mathbb{Z}$, where \mathcal{F} denotes the Fourier transform. Let ψ be as $\mathcal{F}\psi(\xi) = 1 - \sum_{j=1}^{\infty} \phi(2^{-j}\xi)$. For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, the homogeneous Besov space $\dot{B}_{p,q}^s$ is defined by $\dot{B}_{p,q}^s \equiv \{f \in \mathcal{S}'/\mathcal{P}; \|f\|_{\dot{B}_{p,q}^s} < \infty\}$ with the seminorm

$$\|f\|_{\dot{B}_{p,q}^s} = \begin{cases} \left\{ \sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_j * f\|_{L^p})^q \right\}^{\frac{1}{q}} & \text{for } 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{sj} \|\varphi_j * f\|_{L^p} & \text{for } q = \infty, \end{cases}$$

where \mathcal{P} is the set of polynomials in \mathbb{R}^n . We also define the corresponding inhomogeneous Besov space $B_{p,q}^s$ by $B_{p,q}^s \equiv \{f \in \mathcal{S}'; \|f\|_{B_{p,q}^s} < \infty\}$ with the norm

$$\|f\|_{B_{p,q}^s} = \begin{cases} \|\psi * f\|_{L^p} + \left\{ \sum_{j=0}^{\infty} (2^{sj} \|\varphi_j * f\|_{L^p})^q \right\}^{\frac{1}{q}} & \text{for } 1 \leq q < \infty, \\ \|\psi * f\|_{L^p} + \sup_{j \in \mathbb{N}} 2^{sj} \|\varphi_j * f\|_{L^p} & \text{for } q = \infty. \end{cases}$$

For more precise, see e.g., Bergh–Löfström [2]. The following lemma is a fundamental property of Besov spaces.

Proposition 1.1 (i) If $q_1 \leq q_2$, then it holds that $\dot{B}_{p,q_1}^s \subset B_{p,q_2}^s$ for all $1 \leq p \leq \infty$ and $s \in \mathbb{R}$.
 (ii) It holds the continuous continuous embedding

$$\dot{B}_{p,1}^s \subset \dot{H}_p^s \subset \dot{B}_{p,\infty}^s$$

for all $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, where

$$\dot{H}_p^s \equiv \{f \in \mathcal{S}'/\mathcal{P}; \|f\|_{\dot{H}_p^s} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} < \infty\}.$$

¹清水扇丈氏（京都大）と金子健太氏（早稲田大）との共同研究 [8].

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³清水扇丈氏（京都大）と岡田晃氏（京都大）との共同研究 [10], [11], [12].

If $s_0 \neq s_1$, we have

$$(\dot{H}_p^{s_0}, \dot{H}_p^{s_1})_{\theta, q} = \dot{B}_{p, q}^s \quad \text{for } 1 \leq p, q \leq \infty \text{ and } 0 < \theta < 1,$$

where $s = (1 - \theta)s_0 + s_1\theta$.

(iii) If $s > 0$, then it we have that

$$B_{p, q}^s = L^p \cap \dot{B}_{p, q}^s$$

for all $1 \leq p, q \leq \infty$.

We next consider the embedding theorem.

Proposition 1.2 *Let $1 \leq p \leq p_1 \leq \infty$, and $s_1, s_2 \in \mathbb{R}$ satisfy*

$$\frac{n}{p} - s = \frac{n}{p_1} - s_1.$$

Let $1 \leq q \leq q_1 \leq \infty$. Then it holds that

$$B_{p, q}^s \subset B_{p_1, q_1}^{s_1}, \quad \dot{B}_{p, q}^s \subset \dot{B}_{p_1, q_1}^{s_1}.$$

Finally, we consider the Leibnitz rule in the homogeneous Besov space.

Lemma 1.1 ([8, Proposition 2.2]) (i) *Let $1 \leq p, q \leq \infty$, $s > 0$, $\alpha > 0$ and $\beta > 0$. Assume that $1 \leq p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leq \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2}$. If $f \in \dot{B}_{p_1, q}^{s+\alpha} \cap \dot{B}_{\tilde{p}_1, \infty}^{-\beta}$ and $g \in \dot{B}_{p_2, \infty}^{-\alpha} \cap \dot{B}_{\tilde{p}_2, q}^{s+\beta}$, then we have $fg \in \dot{B}_{p, q}^s$ with the estimate*

$$\|fg\|_{\dot{B}_{p, q}^s} \leq C(\|f\|_{\dot{B}_{p_1, q}^{s+\alpha}} \|g\|_{\dot{B}_{p_2, \infty}^{-\alpha}} + \|f\|_{\dot{B}_{\tilde{p}_1, \infty}^{-\beta}} \|g\|_{\dot{B}_{\tilde{p}_2, q}^{s+\beta}}) \tag{1.1}$$

where $C = C(p, p_1, p_2, \tilde{p}_1, \tilde{p}_2, q, s, \alpha, \beta)$.

(ii) *Let $1 \leq p, q \leq \infty$ and $s > 0$. Assume that $1 \leq p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leq \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2}$. If $f \in \dot{B}_{p_1, q}^s \cap L^{\tilde{p}_1}$ and $g \in L^{p_2} \cap \dot{B}_{\tilde{p}_2, q}^s$, then we have $fg \in \dot{B}_{p, q}^s$ with the estimate*

$$\|fg\|_{\dot{B}_{p, q}^s} \leq C(\|f\|_{\dot{B}_{p_1, q}^s} \|g\|_{L^{p_2}} + \|f\|_{L^{\tilde{p}_1}} \|g\|_{\dot{B}_{\tilde{p}_2, q}^s}) \tag{1.2}$$

where $C = C(p, p_1, p_2, \tilde{p}_1, \tilde{p}_2, q, s)$.

Proof. (i) We make use of the following paraproduct formula of fg due to Bony [3]. Our method is related to Christ-Weinstein [4, Proposition 3.3] and Kozono-Shimada [7, Lemma 2.1].

$$\begin{aligned} f \cdot g &= \sum_{k=-\infty}^{\infty} (\varphi_k * f)(P_k g) + \sum_{k=-\infty}^{\infty} (P_k f)(\varphi_k * g) + \sum_{k=-\infty}^{\infty} \sum_{|l-k| \leq 2} (\varphi_k * f)(\varphi_l * g) \\ &=: h_1 + h_2 + h_3, \end{aligned} \tag{1.3}$$

where $P_k g = \sum_{l=-\infty}^{k-3} \varphi_l * g$. We first consider the case $1 \leq q < \infty$. Since

$$\begin{aligned} \text{supp } \mathcal{F}((\varphi_k * f)(P_k g)) &\subset \{\xi \in \mathbb{R}^n; \in 2^{k-2} \leq |\xi| \leq 2^{k+2}\}, \\ \text{supp } \mathcal{F}\varphi_j &= \{\xi \in \mathbb{R}^n; 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \end{aligned}$$

we have that

$$\begin{aligned} \|h_1\|_{\dot{B}_{p,q}^s} &= \left\{ \sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_j * h_1\|_{L^p})^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{k=-\infty}^{\infty} \varphi_j * ((\varphi_k * f)(P_k g)) \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{|k-j|\leq 2} \varphi_j * ((\varphi_k * f)(P_k g)) \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Since $\varphi_j(x) = 2^{jn}(\mathcal{F}^{-1}\phi)(2^jx)$ for all $j \in \mathbb{Z}$, it holds by the Hausdorff-Young and the Hölder inequalities that

$$\|\varphi_j * ((\varphi_k * f)(P_k g))\|_{L^p} \leq \|\varphi_j\|_{L^1} \|(\varphi_k * f)(P_k g)\|_{L^p} \leq \|\mathcal{F}^{-1}\phi\|_{L^1} \|\varphi_k * f\|_{L^{p_1}} \|P_k g\|_{L^{p_2}}$$

for all $j, k \in \mathbb{Z}$. Hence it follows from the Minkowski inequality that

$$\begin{aligned} \|h_1\|_{\dot{B}_{p,q}^s} &\leq C \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{|k-j|\leq 2} \|\varphi_k * f\|_{L^{p_1}} \|P_k g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\ &= C \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{|l|\leq 2} \|\varphi_{j+l} * f\|_{L^{p_1}} \|P_{j+l} g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\ &\leq C \sum_{|l|\leq 2} \left\{ \sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_{j+l} * f\|_{L^{p_1}} \|P_{j+l} g\|_{L^{p_2}})^q \right\}^{\frac{1}{q}} \\ &= C \sum_{|l|\leq 2} \left\{ \sum_{i=-\infty}^{\infty} (2^{si} 2^{-sl} \|\varphi_i * f\|_{L^{p_1}} \|P_i g\|_{L^{p_2}})^q \right\}^{\frac{1}{q}} \\ &= C \sum_{|l|\leq 2} 2^{-sl} \left\{ \sum_{i=-\infty}^{\infty} \left(2^{(s+\alpha)i} \|\varphi_i * f\|_{L^{p_1}} 2^{-\alpha i} \left\| \sum_{k=-\infty}^{i-3} \varphi_k * g_k \right\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\ &\leq C \left\{ \sum_{i=-\infty}^{\infty} \left(2^{(s+\alpha)i} \|\varphi_i * f\|_{L^{p_1}} \sum_{k=-\infty}^{i-3} 2^{-\alpha k} \|\varphi_k * g\|_{L^{p_2}} 2^{-\alpha(i-k)} \right)^q \right\}^{\frac{1}{q}} \\ &\leq C \sup_{k \in \mathbb{Z}} 2^{-\alpha k} \|\varphi_k * g\|_{L^{p_2}} \left\{ \sum_{i=-\infty}^{\infty} \left(2^{(s+\alpha)i} \|\varphi_i * f\|_{L^{p_1}} \sum_{l=3}^{\infty} 2^{-\alpha l} \right)^q \right\}^{\frac{1}{q}} \\ &= C \|g\|_{\dot{B}_{p_2,\infty}^{-\alpha}} \|f\|_{\dot{B}_{p_1,q}^{s+\alpha}}, \end{aligned} \tag{1.4}$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$. In the above estimate it should be noted that $\sum_{l=3}^{\infty} 2^{-\alpha l} < \infty$ since $\alpha > 0$. In the case $q = \infty$, we see similarly to (1.4) that

$$\|h_1\|_{\dot{B}_{p,\infty}^s} \leq C \sup_{k \in \mathbb{Z}} 2^{-\alpha k} \|\varphi_k * g\|_{L^{p_2}} \sup_{i \in \mathbb{Z}} 2^{(s+\alpha)i} \|\varphi_i * f\|_{L^{p_1}} \sum_{l=3}^{\infty} 2^{-\alpha l} = C \|g\|_{\dot{B}_{p_2,\infty}^{-\alpha}} \|f\|_{\dot{B}_{p_1,\infty}^{s+\alpha}},$$

with $C = C(n, p, p_1, p_2, s, \alpha)$, from which and (1.4) it follows that

$$\|h_1\|_{\dot{B}_{p,q}^s} \leq C \|g\|_{\dot{B}_{p_2,\infty}^{-\alpha}} \|f\|_{\dot{B}_{p_1,q}^{s+\alpha}} \quad \text{for all } 1 \leq q \leq \infty, \tag{1.5}$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$.

Replacing the role of f by g , we obtain similarly to (1.4) and (1.5) that

$$\|h_2\|_{\dot{B}_{p,q}^s} \leq C \|f\|_{\dot{B}_{\tilde{p}_1,\infty}^{-\beta}} \|g\|_{\dot{B}_{\tilde{p}_2}^{s+\beta}} \quad \text{for all } 1 \leq q \leq \infty, \tag{1.6}$$

where $C = C(n, p, \tilde{p}_1, \tilde{p}_2, q, s, \beta)$.

Next we treat h_3 in $\dot{B}_{p,q}^s$. Let us consider the case $1 \leq q < \infty$. Since

$$\text{supp } \mathcal{F}((\varphi_k * f)(\varphi_l * g)) \subset \{\xi \in \mathbb{R}^n; |\xi| \leq 2^{\max\{k,l\}+2}\},$$

we have that

$$\begin{aligned} \|h_3\|_{\dot{B}_{p,q}^s} &= \left\{ \sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_j * h_3\|_{L^p})^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{k=-\infty}^{\infty} \sum_{|l-k| \leq 2} \varphi_j * (\varphi_k * f)(\varphi_l * g) \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{\max\{k,l\} \geq j-2} \sum_{|l-k| \leq 2} \varphi_j * (\varphi_k * f)(\varphi_l * g) \right\|_{L^p} \right)^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \left\| \sum_{r \geq -4} \sum_{|t| \leq 2} \|\varphi_j * (\varphi_{j+r} * f)(\varphi_{j+r+t} * g)\|_{L^p} \right\| \right)^q \right\}^{\frac{1}{q}} \\ &\leq \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{r \geq -4} \sum_{|t| \leq 2} \|\varphi_j * (\varphi_{j+r} * f)(\varphi_{j+r+t} * g)\|_{L^p} \right)^q \right\}^{\frac{1}{q}}. \end{aligned}$$

By the Hausdorff-Young and the Hölder inequalities, it holds that

$$\begin{aligned} \|\varphi_j * (\varphi_{j+r} * f)(\varphi_{j+r+t} * g)\|_{L^p} &\leq \|\varphi_j\|_{L^1} \|(\varphi_{j+r} * f)(\varphi_{j+r+t} * g)\|_{L^p} \\ &\leq \|\mathcal{F}^{-1}\phi\|_{L^1} \|\varphi_{j+r} * f\|_{L^{p_1}} \|\varphi_{j+r+t} * g\|_{L^{p_2}} \end{aligned}$$

for all $j, r, t \in \mathbb{Z}$. Hence it follows from the Minkowski inequality that

$$\begin{aligned}
 & \|h_3\|_{\dot{B}_{p,q}^s} \\
 & \leq C \left\{ \sum_{j=-\infty}^{\infty} \left(2^{sj} \sum_{r \geq -4} \sum_{|t| \leq 2} \|\varphi_{j+r} * f\|_{L^{p_1}} \|\varphi_{j+r+t} * g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
 & \leq C \sum_{r \geq -4} \sum_{|t| \leq 2} \left\{ \sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_{j+r} * f\|_{L^{p_1}} \|\varphi_{j+r+t} * g\|_{L^{p_2}})^q \right\}^{\frac{1}{q}} \\
 & = C \sum_{r \geq -4} 2^{-sr} \sum_{|t| \leq 2} 2^{\alpha t} \left\{ \sum_{j=-\infty}^{\infty} \left(2^{(s+\alpha)(j+r)} \|\varphi_{j+r} * f\|_{L^{p_1}} 2^{-\alpha(j+r+t)} \|\varphi_{j+r+t} * g\|_{L^{p_2}} \right)^q \right\}^{\frac{1}{q}} \\
 & \leq C \sup_{l \in \mathbb{Z}} 2^{-\alpha l} \|\varphi_l * g\|_{L^{p_2}} \sum_{r \geq -4} 2^{-sr} \sum_{|t| \leq 2} 2^{\alpha t} \left\{ \sum_{k=-\infty}^{\infty} (2^{(s+\alpha)k} \|\varphi_k * f\|_{L^{p_1}})^q \right\}^{\frac{1}{q}} \\
 & = C \|g\|_{\dot{B}_{p_2, \infty}^{-\alpha}} \|f\|_{\dot{B}_{p_1, q}^{s+\alpha}}, \tag{1.7}
 \end{aligned}$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$. In the above estimate it should be noted that $\sum_{r \geq -4} 2^{-sr} < \infty$ since $s > 0$. In case $q = \infty$, similarly to (1.7), we have that

$$\begin{aligned}
 \|h_3\|_{\dot{B}_{p, \infty}^s} & \leq C \sup_{l \in \mathbb{Z}} 2^{-\alpha l} \|\varphi_l * g\|_{L^{p_2}} \sum_{r \geq -4} 2^{-sr} \sum_{|t| \leq 2} 2^{\alpha t} \sup_{k \in \mathbb{Z}} 2^{(s+\alpha)k} \|\varphi_k * f\|_{L^{p_1}} \\
 & = C \|g\|_{\dot{B}_{p_2, \infty}^{-\alpha}} \|f\|_{\dot{B}_{p_1, \infty}^{s+\alpha}}
 \end{aligned}$$

with $C = C(n, p, p_1, p_2, s, \alpha)$, from which and (1.7) it follows that

$$\|h_3\|_{\dot{B}_{p,q}^s} \leq C \|g\|_{\dot{B}_{p_2, \infty}^{-\alpha}} \|f\|_{\dot{B}_{p_1, q}^{s+\alpha}} \quad \text{for all } 1 \leq q \leq \infty, \tag{1.8}$$

where $C = C(n, p, p_1, p_2, q, s, \alpha)$. Now the desired estimate (1.1) is a consequence of (1.5), (1.6) and (1.8).

(ii) We also make use of the paraproduct formula (1.3). Let us first consider the case $1 \leq q < \infty$. In the same way as in (1.4), we have

$$\begin{aligned}
 \|h_1\|_{\dot{B}_{p,q}^s} & \leq C \sum_{|l| \leq 2} 2^{-sl} \left\{ \sum_{i=-\infty}^{\infty} (2^{si} \|\varphi_i * f\|_{L^{p_1}} \|P_i g\|_{L^{p_2}})^q \right\}^{\frac{1}{q}} \\
 & \leq C \sup_{i \in \mathbb{Z}} \|P_i g\|_{L^{p_2}} \left\{ \sum_{i=-\infty}^{\infty} (2^{si} \|\varphi_i * f\|_{L^{p_1}})^q \right\}^{\frac{1}{q}}. \tag{1.9}
 \end{aligned}$$

It should be noticed that

$$\sum_{l=-\infty}^k \varphi_l(x) = 2^{kn} \psi(2^k x) = \psi_{2^{-k}}(x), \quad \forall k \in \mathbb{Z},$$

where $f_\varepsilon(x) = \varepsilon^{-n}f(x/\varepsilon)$ for $\varepsilon > 0$. Hence we have $\|\sum_{l=-\infty}^k \varphi_l\|_{L^1} = \|\psi\|_{L^1}$ for all $k \in \mathbb{Z}$, and it holds that

$$\|P_i g\|_{L^{p_2}} = \left\| \sum_{l=-\infty}^{i-3} \varphi_l * g \right\|_{L^{p_2}} = \|\psi_{2^{i-3}} * g\|_{L^{p_2}} \leq \|\psi\|_{L^1} \|g\|_{L^{p_2}} \quad \text{for all } i \in \mathbb{Z},$$

from which and (1.9) it follows that

$$\|h_1\|_{\dot{B}_{p,q}^s} \leq C \|g\|_{L^{p_2}} \|f\|_{\dot{B}_{p_1,q}^s}, \tag{1.10}$$

where $C = C(n, p, p_1, p_2, q, s)$.

In case $q = \infty$, we have that

$$\|h_1\|_{\dot{B}_{p,\infty}^s} \leq C \sup_{i \in \mathbb{Z}} \|P_i g\|_{L^{p_2}} \sup_{i \in \mathbb{Z}} 2^{si} \|\varphi_i * f\|_{L^{p_1}} \leq C \|g\|_{L^{p_2}} \|f\|_{\dot{B}_{p_1,\infty}^s},$$

from which and (1.10) it follows that

$$\|h_1\|_{\dot{B}_{p,q}^s} \leq C \|g\|_{L^{p_2}} \|f\|_{\dot{B}_{p_1,q}^s} \quad \text{for all } 1 \leq q \leq \infty, \tag{1.11}$$

where $C = C(n, p, p_1, p_2, q, s)$.

Replacing the role of f by g , we have similarly to (1.11) that

$$\|h_2\|_{\dot{B}_{p,q}^s} \leq C \|f\|_{L^{p_1}} \|g\|_{\dot{B}_{p_1,q}^s} \quad \text{for all } 1 \leq q \leq \infty, \tag{1.12}$$

where $C = C(n, p, \tilde{p}_1, \tilde{p}_2, q, s)$.

Concerning the estimate of h_3 in $\dot{B}_{p,q}^s$ for $1 \leq q < \infty$, we have similarly to (1.7) that

$$\begin{aligned} \|h_3\|_{\dot{B}_{p,q}^s} &\leq C \sum_{r \geq -4} 2^{-sr} \sum_{|t| \leq 2} \left\{ \sum_{j=-\infty}^{\infty} (2^{s(j+r)} \|\varphi_{j+r} * f\|_{L^{p_1}} \|\varphi_{j+r+t} * g\|_{L^{p_2}})^q \right\}^{\frac{1}{q}} \\ &\leq C \sup_{i \in \mathbb{Z}} \|\varphi_i * g\|_{L^{p_2}} \sum_{r \geq -4} 2^{-sr} \left\{ \sum_{i=-\infty}^{\infty} (2^{si} \|\varphi_i * f\|_{L^{p_1}})^q \right\}^{\frac{1}{q}} \\ &\leq C \|g\|_{L^{p_2}} \|f\|_{\dot{B}_{p_1,q}^s}, \end{aligned} \tag{1.13}$$

where $C = C(n, p, p_1, p_2, q, s)$.

In case $q = \infty$, we have similarly to the above that

$$\begin{aligned} \|h_3\|_{\dot{B}_{p,\infty}^s} &\leq C \sum_{r \geq -4} 2^{-sr} \sum_{|t| \leq 2} \sup_{j \in \mathbb{Z}} 2^{s(j+r)} \|\varphi_{j+r} * f\|_{L^{p_1}} \|\varphi_{j+r+t} * g\|_{L^{p_2}} \\ &\leq C \sup_{i \in \mathbb{Z}} \|\varphi_i * g\|_{L^{p_2}} \sup_{l \in \mathbb{Z}} 2^{sl} \|\varphi_l * f\|_{L^{p_1}} \sum_{r \geq -4} 2^{-sr} \\ &\leq C \|g\|_{L^{p_2}} \|f\|_{\dot{B}_{p_1,\infty}^s}, \end{aligned}$$

from which and (1.13) it follows that

$$\|h_3\|_{\dot{B}_{p,q}^s} \leq C \|g\|_{L^{p_2}} \|f\|_{\dot{B}_{p_1,q}^s} \quad \text{for all } 1 \leq q \leq \infty, \tag{1.14}$$

where $C = C(n, p, p_1, p_2, s)$. Now the desired estimate (1.2) is a consequence of (1.11), (1.12) and (1.14). This proves Lemma 1.1. \blacksquare

2 Application to the stationary Navier-Stokes equations

Let us consider the stationary Navier-Stokes equation in \mathbb{R}^n for $n \geq 3$;

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla \pi = f, \\ \operatorname{div} u = 0, \end{cases} \tag{NS}$$

where $u = u(x) = (u^1(x), \dots, u^n(x))$ and $\pi = \pi(x)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, respectively, while $f = f(x) = (f^1(x), \dots, f^n(x))$ denotes the given external force. we rewrite (NS) to the generalized form by means of the abstract setting of the functional analysis. Let P be the projection operator from L^p onto the solenoidal space $L^p_\sigma \equiv \{u \in L^p; \operatorname{div} u = 0\}$. It is known that P has the expression $P = \{P_{jk}\}_{1 \leq j, k \leq n}$ with $P_{jk} = \delta_{jk} + R_j R_k$, $j, k = 1, \dots, n$, where δ_{jk} denotes the Kronecker symbol and $R_k = \frac{\partial}{\partial x_k} (-\Delta)^{-\frac{1}{2}}$ denotes the Riesz transform. Since R_k , $k = 1, 2, \dots, n$ is a bounded operator in L^p for $1 < p < \infty$, P is also bounded from L^p onto L^p_σ for $1 < p < \infty$. However, P is unbounded in L^p for $p = 1$ and for $p = \infty$. On the other hand, we have

Proposition 2.1 *P is bounded in the homogeneous Besov space $\dot{B}^s_{p,q}$ for all $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$.*

Proof. For the proof, it suffices to show that the Riesz transforms R_k ($k = 1, 2, \dots, n$) are bounded in $\dot{B}^s_{p,q}$ for all $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. It should be noted by the Hausdorff-Young inequality that

$$\begin{aligned} \|\varphi_j * R_k f\|_{L^p} &= \left\| \sum_{l=j-1}^{j+1} \varphi_l * R_k(\varphi_j * f) \right\|_{L^p} \\ &\leq \sum_{l=j-1}^{j+1} \left\| \mathcal{F}^{-1} \left(\hat{\varphi}_l(\xi) \frac{i\xi_k}{|\xi|} \right) * \varphi_j * f \right\|_{L^p} \\ &\leq 3 \|\Phi_k\|_{L^1} \|\varphi_j * f\|_{L^p}, \quad k = 1, \dots, n \end{aligned}$$

for all $1 \leq p \leq \infty$ and for all $j \in \mathbb{Z}$ with $\Phi_k = \mathcal{F}^{-1}(\phi(\xi) \frac{i\xi_k}{|\xi|})$ in L^1 , from which we see that R_k , $k = 1, \dots, n$ is bounded in $\dot{B}^s_{p,q}$ even for $p = 1$ and $p = \infty$. This proves Proposition 2.1. ■

Since we need to find the solution u of (NS) with $\operatorname{div} u = 0$, let us introduce the space $\dot{B}^s_{p,q} \equiv PB^s_{p,q}$. Since $Pu = u$, $P(\nabla \pi) = 0$ and since P commutes with $-\Delta$, application of P to both sides of (NS) yields that $-\Delta u + P(u \cdot \nabla u) = Pf$. Since $\operatorname{div} u = 0$, it holds that $u \cdot \nabla u = \nabla \cdot u \otimes u$, and hence we see that u can be expressed by

$$\begin{aligned} u &= (-\Delta)^{-1} P(u \cdot \nabla u) + (-\Delta)^{-1} Pf \\ &= P(-\Delta)^{-1} \nabla \cdot (u \otimes u) + P(-\Delta)^{-1} f \\ &= K(u \otimes u) + P(-\Delta)^{-1} f, \end{aligned} \tag{E}$$

where $K \equiv P(-\Delta)^{-1} \nabla \cdot$ may be regarded as the Fourier multiplier with the differential order -1 . More precisely, $Kg = (Kg_1, \dots, Kg_n)$ has an expression

$$Kg_j(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sum_{k,l=1}^n \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) \frac{1}{|\xi|^2} i\xi_l \mathcal{F}g_{kl}(\xi) d\xi, \quad j = 1, \dots, n$$

for $n \times n$ tensors $g = (g_{kl})_{1 \leq k, l \leq n}$. Then we have the following proposition.

Proposition 2.2 ([8, Proposition 1.1]) *Let $1 \leq p \leq p_0$ and $-\infty < s_0 \leq s + 1 < \infty$ satisfy $s_0 - n/p_0 - 1 = s - n/p$. Let $1 \leq q \leq \infty$. K is a bounded operator from $\dot{B}_{p,q}^s$ to $\dot{B}_{p_0,q}^{s_0}$ with the estimate*

$$\|Kg\|_{\dot{B}_{p_0,q}^{s_0}} \leq C\|g\|_{\dot{B}_{p,q}^s}, \tag{2.1}$$

for all $g \in \dot{B}_{p,q}^s$, where $C = C(n, p, p_0, q, s, s_0)$.

Proof. Since the projection P is bounded from $\dot{B}_{p_0,q}^{s_0}$ onto $\dot{B}_{p_0,q}^{s_0}$, it suffices to show that $K' \equiv (-\Delta)^{-1}\nabla$ with the expression

$$K'g_k(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{1}{|\xi|^2} \sum_{l=1}^n i\xi_l \mathcal{F}g_{kl}(\xi) d\xi, \quad k = 1, \dots, n$$

is a bounded operator from $\dot{B}_{p,q}^s$ to $\dot{B}_{p_0,q}^{s_0}$ with such an estimate as (2.1).

Let us first consider the case $1 \leq q < \infty$. We define $1 \leq r \leq \infty$ by $1/r = 1 - (1/p - 1/p_0)$. By the Hausdorff-Young inequality, we have that

$$\begin{aligned} \|K'g\|_{\dot{B}_{p_0,q}^{s_0}} &= \left\{ \sum_{j \in \mathbb{Z}} (2^{s_0 j} \|\varphi_j * K'g\|_{L^{p_0}})^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{j \in \mathbb{Z}} (2^{s_0 j} \|\tilde{\varphi}_j * \varphi_j * K'g\|_{L^{p_0}})^q \right\}^{\frac{1}{q}} \\ &= \left\{ \sum_{j \in \mathbb{Z}} (2^{s_0 j} \|K'\tilde{\varphi}_j * \varphi_j * g\|_{L^{p_0}})^q \right\}^{\frac{1}{q}} \\ &\leq \left\{ \sum_{j \in \mathbb{Z}} (2^{s_0 j} \|K'\tilde{\varphi}_j\|_{L^r} \|\varphi_j * g\|_{L^p})^q \right\}^{\frac{1}{q}}, \end{aligned} \tag{2.2}$$

where $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. It is easy to see that

$$K'\tilde{\varphi}_j(x) = 2^{-j} 2^{jn} \Psi(2^j x) \quad \text{with} \quad \Psi \equiv \sum_{l=1}^n \mathcal{F}^{-1} \left(\frac{i\xi_l}{|\xi|^2} \sum_{k=-1}^1 \phi(2^{-k}\xi) \right),$$

which yields that

$$\|K'\tilde{\varphi}_j\|_{L^r} \leq 2^{-j} 2^{n(1-\frac{1}{r})j} \|\Psi\|_{L^r} \leq C 2^{-j+n(\frac{1}{p}-\frac{1}{p_0})j} = C 2^{(s-s_0)j},$$

where $C = C(n, p, p_0)$ is independent of $j \in \mathbb{Z}$. Notice that $\Psi \in \mathcal{S}$ because $\text{supp } \sum_{k=-1}^1 \phi(2^{-k}\xi) \subset \{\xi \in \mathbb{R}^n; 2^{-2} \leq |\xi| \leq 2^2\}$. Hence it follows from (2.2) that

$$\|K'g\|_{\dot{B}_{p_0,q}^{s_0}} \leq C \left\{ \sum_{j \in \mathbb{Z}} (2^{s_j} \|\varphi_j * g\|_{L^p})^q \right\}^{\frac{1}{q}} = C\|g\|_{\dot{B}_{p,q}^s},$$

where $C = C(n, p, p_0, q, s, s_0)$. In case $q = \infty$, the proof is quite similar to the above, so we may omit it. This proves Proposition 2.2. ■

Our main result in this section now reads as follows.

Theorem 2.1 ([8, Theorem 1.2]) *Let $n \geq 3$. For every $1 \leq p < n$ and $1 \leq q \leq \infty$ there is a constant $\delta = \delta(n, p, q) > 0$ such that if $f \in \dot{B}_{p,q}^{-3+\frac{n}{p}}$ satisfies $\|f\|_{\dot{B}_{p,q}^{-3+\frac{n}{p}}} < \delta$, then there exists a solution $u \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ of (E). Moreover, there exists a constant $\eta = \eta(n, p, q) > 0$ such that if u and v are two solutions of (E) in the class $\dot{B}_{p,q}^{-1+\frac{n}{p}}$ satisfying $\|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq \eta$, $\|v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq \eta$, then it holds that $u \equiv v$.*

In the case $n/2 < p < n$, a similar result to Theorem 2.1 has been obtained by Cunanan-Okabe-Tsutsui [5]. An immediate consequence of the above theorem is the existence of self-similar solutions.

Corollary 2.1 ([8, Corollary 1.3]) *Let $n \geq 3$. Let $1 \leq p < n$ and $q = \infty$. If $f \in \dot{B}_{p,\infty}^{-3+\frac{n}{p}}$ is a homogeneous function with degree -3 , i.e., $f(\lambda x) = \lambda^{-3}f(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$ and if f satisfies $\|f\|_{\dot{B}_{p,\infty}^{-3+\frac{n}{p}}} < \delta$, then the solution u given by Theorem 2.1 is a homogeneous function with degree -1 , i.e., $u(\lambda x) = \lambda^{-1}u(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$, which means that u may be regarded as a self-similar solution of (NS).*

The following lemma of the bilinear estimate plays an important role for the proof of our main theorem.

Lemma 2.1 ([8, Lemma 2.3]) *Let $n \geq 3$ and let $1 \leq p < n$, $1 \leq q \leq \infty$. For $u, v \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ we have $K(u \otimes v) \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ with the estimate*

$$\|K(u \otimes v)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq C\|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}\|v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}, \tag{2.3}$$

where $C = C(n, p, q)$.

Proof. Taking $p = p_0$, $s = -2 + n/p$ in Proposition 2.2, we have that $s_0 = -1 + n/p$, and so it holds that

$$\|K(u \otimes v)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq C\|u \otimes v\|_{\dot{B}_{p,q}^{-2+\frac{n}{p}}}, \tag{2.4}$$

where $C = C(n, p, q)$.

Let us first consider the case for $n \geq 3$ and $1 \leq p < n/2$. Take p_1 and p_2 in such a way

$$p_1 = p, \quad n < p_2, \quad p' \equiv \frac{p}{p-1} \leq p_2.$$

We define p_0 and s_0 by

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_0 = \frac{n}{p_0} - 2. \tag{2.5}$$

Since $1 \leq p < n/2$, we have that

$$1 \leq p_0 \leq p, \quad 0 < s_0, \quad \left(-2 + \frac{n}{p}\right) - \frac{n}{p} = s_0 - \frac{n}{p_0}. \tag{2.6}$$

It should be noted that the above (2.6) yields $1 \leq p_0 \leq p < n/2$, which necessarily implies that $n \geq 3$. Hence it follows from Proposition 1.2 that

$$\|u \otimes v\|_{\dot{B}_{p,q}^{-2+\frac{n}{p}}} \leq C \|u \otimes v\|_{\dot{B}_{p_0,q}^{s_0}}. \tag{2.7}$$

Since $n < p_2$, we have $\alpha \equiv 1 - n/p_2 > 0$, and we have by Lemma 1.1 (i) that

$$\|u \otimes v\|_{\dot{B}_{p_0,q}^{s_0}} \leq C (\|u\|_{\dot{B}_{p_1,q}^{s_0+\alpha}} \|v\|_{\dot{B}_{p_2,q}^{-\alpha}} + \|u\|_{\dot{B}_{p_2,q}^{-\alpha}} \|v\|_{\dot{B}_{p_1,q}^{s_0+\alpha}}). \tag{2.8}$$

Since $p = p_1, p < p_2$ and since

$$s_0 + \alpha - \frac{n}{p_1} = -1 = \left(-1 + \frac{n}{p}\right) - \frac{n}{p}, \quad -\alpha - \frac{n}{p_2} = -1 = \left(-1 + \frac{n}{p}\right) - \frac{n}{p},$$

it follows from Proposition 1.2 that

$$\dot{B}_{p,q}^{-1+\frac{n}{p}} \hookrightarrow \dot{B}_{p_1,q}^{s_0+\alpha}, \quad \dot{B}_{p,q}^{-1+\frac{n}{p}} \hookrightarrow \dot{B}_{p_2,q}^{-\alpha}.$$

Hence we obtain from (2.8) that

$$\|u \otimes v\|_{\dot{B}_{p_0,q}^{s_0}} \leq C \|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \|v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}. \tag{2.9}$$

Now, the desired estimate (2.3) is a consequence of (2.4), (2.7) and (2.9).

We next consider the case for $n \geq 3$ and $n/2 \leq p < n$. In such a case, we take p_1 and p_2 so that

$$p_1 = p, \quad n < p_2 < \frac{np}{2p-n}.$$

Define p_0 and s_0 by (2.5). Since

$$\begin{aligned} \frac{1}{p} < \frac{1}{p_0} &= \frac{1}{p} + \frac{1}{p_2} < \frac{2}{n} + \frac{1}{n} \leq 1, \\ s_0 = \frac{n}{p_0} - 2 &= n \left(\frac{1}{p_2} - \left(\frac{2}{n} - \frac{1}{p} \right) \right) > 0, \end{aligned}$$

we have (2.6), so it holds (2.7). Since $\alpha \equiv 1 - n/p_2 > 0$, implied by $n < p_2$, in the same way as in the above case, we obtain (2.9), which yields the desired estimate (2.3). This proves Lemma 2.1. ■

Proof of Theorem 2.1. We first prove the existence of the solution to (E). We solve (E) by the successive approximation. For that purpose, let us define the approximating solutions $\{u_j\}$ of (E) by

$$\begin{cases} u_0 = P(-\Delta)^{-1} f, \\ u_{j+1} = K(u_j \otimes u_j) + u_0, \quad j = 0, 1, \dots \end{cases} \tag{2.10}$$

Since $f \in \dot{B}_{p,q}^{-3+\frac{n}{p}}$, we see that $u_0 \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$. Assume that $u_j \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$. By Lemma 2.1, we have that $u_{j+1} \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ with the estimate

$$\|u_{j+1}\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq C\|u_j\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}^2 + \|u_0\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}, \tag{2.11}$$

where $C = C(n, p, q)$ is independent of j . By induction, it holds that $u_j \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ for all $j = 0, 1, \dots$. Taking $M_j = \|u_j\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}$, we have by (2.11) that

$$M_{j+1} \leq CM_j^2 + M_0, \quad j = 0, 1, \dots \tag{2.12}$$

By the standard argument we see from (2.12) that under the condition

$$M_0 < \frac{1}{4C}, \tag{2.13}$$

the sequence $\{M_j\}_{j=0}^\infty$ is subject to the estimate

$$M_j \leq \alpha \equiv \frac{1 - \sqrt{1 - 4CM_0}}{2C}, \quad j = 0, 1, \dots \tag{2.14}$$

Take $w_j = u_{j+1} - u_j$, and we have

$$\begin{aligned} w_j &= K(u_j \otimes u_j) - K(u_{j-1} \otimes u_{j-1}) \\ &= K(u_j \otimes w_{j-1}) + K(w_{j-1} \otimes u_{j-1}). \end{aligned}$$

Letting $L_j = \|w_j\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}$, we have similarly to (2.12) that

$$\begin{aligned} L_j &\leq C(M_j + M_{j-1})L_{j-1} \\ &\leq 2C\alpha L_{j-1}. \end{aligned}$$

Therefore, it holds that

$$L_j \leq (2C\alpha)^j L_0, \quad j = 1, 2, \dots$$

By the definition of α in (2.14), we see that

$$2C\alpha = 1 - \sqrt{1 - 4cM_0} < 1,$$

and hence it holds that

$$\sum_{j=0}^\infty L_j < \infty, \tag{2.15}$$

which implies that u_j converges to some u in $\dot{B}_{p,q}^{-1+\frac{n}{p}}$. Since

$$M_0 = \|A^{-1}Pf\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq C\|f\|_{\dot{B}_{p,q}^{-3+\frac{n}{p}}} < C\delta, \tag{2.16}$$

with $C = C(n, p)$, by taking $\delta = \delta(n, p, q)$ sufficiently small, we see from the above estimate that the condition (2.13) is fulfilled provided $\|f\|_{\dot{B}_{p,q}^{-3+\frac{n}{p}}} \leq \delta$. Now, letting $j \rightarrow \infty$ in (2.10), we see

from Lemma 2.1 that the limit $u \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ is a solutions of (E).

We next consider the uniqueness. Let $u \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ and $v \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ be the solutions of (E) such that $\|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq \eta$, $\|v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \leq \eta$. It follows from Lemma 2.1 that

$$\begin{aligned} \|u - v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} &= \|K(u \otimes (u - v)) + K((u - v) \otimes v)\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \\ &\leq C(\|u\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} + \|v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}})\|u - v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}} \\ &\leq 2C\eta\|u - v\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}. \end{aligned}$$

By taking $\eta > 0$ sufficiently small to satisfy $2C\eta < 1$, we obtain $u - v = 0$. This completes the proof of Theorem 2.1. ■

3 $L^p - L^q$ estimates of the Stokes semigroup in Besov spaces

We first investigate the behavior of the heat semigroup in the homogeneous Besov spaces.

Proposition 3.1 (i) *Let $1 \leq p \leq q \leq \infty$, $1 \leq r \leq \infty$ and $s_0 \leq s_1$. It holds that*

$$\|e^{t\Delta} a\|_{\dot{B}_{q,r}^{s_1}} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}(s_1-s_0)} \|a\|_{\dot{B}_{p,r}^{s_0}}$$

for all $a \in \dot{B}_{p,r}^{s_0}$ and all $0 < t < \infty$ with a constant $C = C(n, p, q, r, s_0, s_1)$.

(ii) *Let $s_0 < s_1$, $1 \leq p \leq \infty$. It holds that*

$$\|e^{t\Delta} a\|_{\dot{B}_{p,1}^{s_1}} \leq Ct^{-\frac{1}{2}(s_1-s_0)} \|a\|_{\dot{B}_{p,\infty}^{s_0}}$$

for all $a \in \dot{B}_{p,\infty}^{s_0}$ and for all $0 < t < \infty$ with a constant $C = C(n, p, s_0, s_1)$.

(iii) *Let $s_0 < s_1$ and $1 \leq p \leq q \leq \infty$. It holds that*

$$\|e^{t\Delta} a\|_{\dot{B}_{q,1}^{s_1}} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}(s_1-s_0)} \|a\|_{\dot{B}_{p,\infty}^{s_0}}$$

for all $a \in \dot{B}_{p,\infty}^{s_0}$ and for all $0 < t < \infty$ with a constant $C = C(n, p, q, s_0, s_1)$.

For the proof, see [10, Lemma 2.2] and [9, Lemma 2.2].

The following theorem characterizes the class of the initial data a in the homogeneous Besov space in the case that $e^{t\Delta} a$ belongs to the Serrin class in the generalized Lorentz space in time.

Theorem 3.1 ([12, Lemma 2.1]) (i) *Let $n < p < \infty$ and $1 \leq q \leq \infty$. For $a \in \dot{B}_{p,q}^{-1+\frac{n}{p}}$ it holds that $e^{t\Delta} a \in L^{\alpha,q}(0, \infty; \dot{B}_{r,1}^0)$ for all $p \leq r \leq \infty$ and $2 \leq \alpha < \infty$ satisfying $\frac{2}{\alpha} + \frac{n}{r} = 1$ with the estimate*

$$\left\| \|e^{t\Delta} a\|_{\dot{B}_{r,1}^0} \right\|_{L^{\alpha,q}(0,\infty)} \leq C \|a\|_{\dot{B}_{p,q}^{-1+\frac{n}{p}}}, \tag{3.1}$$

where $C = C(n, p, q, r)$. In particular, if $a \in \dot{B}_{p,s}^{-1+\frac{n}{p}}$ for $\frac{2}{s} + \frac{n}{p} = 1$ with $n < p < \infty$, then it holds that $e^{t\Delta} a \in L^s(0, \infty; \dot{B}_{p,1}^0)$.

(ii) Assume that $a \in \mathcal{S}'$ satisfies

$$e^{t\Delta}a \in L^{\alpha,q}(0, \infty; L^r).$$

for $n < r \leq \infty$ and $2 \leq \alpha < \infty$ with $\frac{2}{\alpha} + \frac{n}{r} = 1$ and for $1 < q \leq \infty$. Then it holds that $a \in \dot{B}_{r,q}^{-1+\frac{n}{r}}$ with the estimate

$$\|a\|_{\dot{B}_{r,q}^{-1+\frac{n}{r}}} \leq C \|e^{t\Delta}a\|_{L^{\alpha,q}(0,\infty;L^r)}, \tag{3.2}$$

where $C = C(n, r, q)$.

Proof. The special case when $q = \alpha$ was proved by [1, Theorem 2.34]. Here we give another proof based on the real interpolation.

(i) We take p_0, p_1 and $0 < \theta < 1$ in such a way that

$$n < p_0 < p < p_1 \leq \infty, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

For every $a \in \dot{B}_{p_i,\infty}^{-1+\frac{n}{p_i}}$, $i = 0, 1$, it follows from Proposition 3.1(iii) that

$$\|e^{t\Delta}a\|_{\dot{B}_{r,1}^0} \leq C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}(0-(-1+\frac{n}{p_i}))} \|a\|_{\dot{B}_{p_i,\infty}^{-1+\frac{n}{p_i}}} = C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{r}+\frac{1}{n}-\frac{1}{p_i})} \|a\|_{\dot{B}_{p_i,\infty}^{-1+\frac{n}{p_i}}}, \quad i = 0, 1. \tag{3.3}$$

Let us define α_0 and α_1 in such a way that

$$\frac{1}{\alpha_i} = \frac{n}{2} \left(\frac{1}{p} - \frac{1}{r} + \frac{1}{n} - \frac{1}{p_i} \right), \quad i = 0, 1. \tag{3.4}$$

Since $n < p_0 < p < p_1$ and since $p \leq r \leq \infty$, we easily verify that $1 < \alpha_i < \infty$ for $i = 0, 1$, and obtain from (3.3) that the mappings

$$\dot{B}_{p_i,\infty}^{-1+\frac{n}{p_i}} \ni a \mapsto \|e^{t\Delta}a\|_{\dot{B}_{r,1}^0} \in L^{\alpha_i,\infty}(0, \infty), \quad i = 0, 1.$$

are bounded sub-additive operators. Here $L^{\alpha_i,q}(0, \infty)$ denotes the Lorentz space on $(0, \infty)$ (see, e.g., Bergh-Löfström [2, Chapter 5]). Then it follows from the real interpolation theorem that

$$(\dot{B}_{p_0,\infty}^{-1+\frac{n}{p_0}}, \dot{B}_{p_1,\infty}^{-1+\frac{n}{p_1}})_{\theta,q} \ni a \mapsto \|e^{t\Delta}a\|_{\dot{B}_{r,1}^0} \in (L^{\alpha_0,\infty}(0, \infty), L^{\alpha_1,\infty}(0, \infty))_{\theta,q} \tag{3.5}$$

is also bounded sub-additive. Since

$$(\dot{B}_{p_0,\infty}^{-1+\frac{n}{p_0}}, \dot{B}_{p_1,\infty}^{-1+\frac{n}{p_1}})_{\theta,q} = \dot{B}_{p,q}^{-1+\frac{n}{p}}, \quad (L^{\alpha_0,\infty}(0, \infty), L^{\alpha_1,\infty}(0, \infty))_{\theta,q} = L^{\alpha,q}(0, \infty)$$

with α defined by

$$\frac{1}{\alpha} = \frac{1-\theta}{\alpha_0} + \frac{\theta}{\alpha_1} = -\frac{n}{2r} + \frac{1}{2},$$

we conclude from (3.5) that

$$\dot{B}_{p,q}^{-1+\frac{n}{p}} \ni a \mapsto \|e^{t\Delta}a\|_{\dot{B}_{r,1}^0} \in L^{\alpha,q}(0, \infty)$$

is a bounded sub-additive operator for $\frac{2}{\alpha} + \frac{n}{r} = 1$ with $p \leq r \leq \infty$, which implies (3.1). This proves (i).

(ii) Let us first consider the case $1 < q < \infty$. We make use of the following characterization of the equivalent norm of the homogeneous Besov space $\dot{B}_{r',q'}^{1-\frac{n}{r}}$ due to Triebel [13]:

$$\|\varphi\|_{\dot{B}_{r',q'}^{1-\frac{n}{r}}} \simeq \left\{ \int_0^\infty (t^{1-\frac{1}{2}(1-\frac{n}{r})}) \|(-\Delta)e^{t\Delta}\varphi\|_{L^{r'}}^{q'} \frac{dt}{t} \right\}^{\frac{1}{q'}}, \tag{3.6}$$

where we have used the relation $\frac{2}{\alpha} + \frac{n}{r} = 1$ with $1 - \frac{n}{r} = \frac{2}{\alpha} > 0$.

For $a \in \mathcal{S}'$, we take a dual coupling with $\varphi \in \mathcal{S}$. Since

$$e^{t\Delta}\varphi - \varphi = \int_0^t \frac{\partial}{\partial \tau} e^{\tau\Delta}\varphi d\tau = - \int_0^t (-\Delta)e^{\tau\Delta}\varphi d\tau,$$

φ is expressed by $\varphi = e^{t\Delta}\varphi + \int_0^t (-\Delta)e^{\tau\Delta}\varphi d\tau$. We consider the coupling

$$|\langle a, \varphi \rangle| \leq |\langle a, e^{t\Delta}\varphi \rangle| + \int_0^t |\langle a, (-\Delta)e^{\tau\Delta}\varphi \rangle| d\tau =: I_1(t) + I_2(t). \tag{3.7}$$

By (3.6) and the Hölder inequality, it holds that

$$\begin{aligned} I_2(t) &\leq \int_0^t |\langle e^{\frac{\tau}{2}\Delta}a, (-\Delta)e^{\frac{\tau}{2}\Delta}\varphi \rangle| d\tau \\ &\leq \int_0^t \|e^{\frac{\tau}{2}\Delta}a\|_{L^r} \|(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\|_{L^{r'}} d\tau \\ &\leq \int_0^t \tau^{-1+\frac{1}{q'}+\frac{1}{2}(1-\frac{n}{r})} \|e^{\frac{\tau}{2}\Delta}a\|_{L^r} \tau^{1-\frac{1}{q'}-\frac{1}{2}(1-\frac{n}{r})} \|(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\|_{L^{r'}} d\tau \\ &\leq \left[\int_0^t (\tau^{-1+\frac{1}{q'}+\frac{1}{2}(1-\frac{n}{r})}) \|e^{\frac{\tau}{2}\Delta}a\|_{L^r}^q d\tau \right]^{\frac{1}{q}} \left[\int_0^t (\tau^{1-\frac{1}{q'}-\frac{1}{2}(1-\frac{n}{r})}) \|(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\|_{L^{r'}}^{q'} d\tau \right]^{\frac{1}{q'}} \\ &\leq \left[\int_0^t (\tau^{\frac{1}{2}(1-\frac{n}{r})}) \|e^{\frac{\tau}{2}\Delta}a\|_{L^r}^q \frac{d\tau}{\tau} \right]^{\frac{1}{q}} \left[\int_0^t (\tau^{1-\frac{1}{2}(1-\frac{n}{r})}) \|(-\Delta)e^{\frac{\tau}{2}\Delta}\varphi\|_{L^{r'}}^{q'} \frac{d\tau}{\tau} \right]^{\frac{1}{q'}} \\ &\leq \left[\int_0^t (\tau^{\frac{1}{\alpha}} \|e^{\tau\Delta}a\|_{L^r}^q \frac{d\tau}{\tau} \right]^{\frac{1}{q}} \|\varphi\|_{\dot{B}_{r',q'}^{1-\frac{n}{r}}}. \end{aligned}$$

Since $\{e^{\tau\Delta}\}_{\tau \geq 0}$ is a contraction semigroup in L^r , we see that $t \in (0, \infty) \rightarrow \|e^{t\Delta}a\|_{L^r}$ is a non-negative and non-increasing function. Hence it is easy to see that

$$\left[\frac{\alpha}{q} \int_0^t (\tau^{\frac{1}{\alpha}} \|e^{\tau\Delta}a\|_{L^r}^q \frac{d\tau}{\tau} \right]^{\frac{1}{q}} = \|e^{t\Delta}a\|_{L^{\alpha,q}(0,\infty;L^r)},$$

which yields that

$$I_2(t) \leq C \|e^{t\Delta}a\|_{L^{\alpha,q}(0,\infty;L^r)} \|\varphi\|_{\dot{B}_{r',q'}^{1-\frac{n}{r}}},$$

for all $0 < t < \infty$ and all $\varphi \in \mathcal{S}$ with $C = C(n, r, q)$. Since $a \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, it is easy to see that $I_1(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence, letting $t \rightarrow \infty$ in both sides of (3.7), we obtain that

$$|\langle a, \varphi \rangle| \leq C \|e^{t\Delta} a\|_{L^{\alpha,q}(0,\infty;L^r)} \|\varphi\|_{\dot{B}_{r',q}^{1-\frac{n}{r}}}$$

for all $\varphi \in \mathcal{S}$. Since $\dot{B}_{r,q}^{-1+\frac{n}{r}} = (\dot{B}_{r',q'}^{1-\frac{n}{r}})^*$ and since \mathcal{S} is dense in $\dot{B}_{r',q'}^{1-\frac{n}{r}}$, it follows from the above estimate that

$$\|a\|_{\dot{B}_{r,q}^{-1+\frac{n}{r}}} = \sup_{\varphi \in \mathcal{S}, \|\varphi\|_{\dot{B}_{r',q'}^{1-\frac{n}{r}}}=1} |\langle a, \varphi \rangle| \leq C \|e^{t\Delta} a\|_{L^{\alpha,q}(0,\infty;L^r)},$$

which implies (3.2).

Next, we consider the case $q = \infty$. Notice that $\dot{B}_{r,\infty}^{-1+\frac{n}{r}} = (\dot{B}_{r',1}^{1-\frac{n}{r}})^*$. Again by the characterization of the norm $\dot{B}_{r,1}^{1-\frac{n}{r}}$, we see that

$$\|\varphi\|_{\dot{B}_{r',1}^{1-\frac{n}{r}}} \leq \int_0^\infty t^{1-\frac{1}{2}(1-\frac{n}{r})} \|(-\Delta)e^{t\Delta}\varphi\|_{L^{r'}} \frac{dt}{t}. \tag{3.8}$$

Since

$$\|e^{t\Delta} a\|_{L^{\alpha,\infty}(0,\infty;L^r)} = \sup_{0 < t < \infty} t^{\frac{1}{\alpha}} \|e^{t\Delta} a\|_{L^r},$$

in the same manner as in the above case $1 < q < \infty$, we see easily that

$$I_2(t) \leq C \|e^{t\Delta} a\|_{L^{\alpha,\infty}(0,\infty;L^r)} \|\varphi\|_{\dot{B}_{r',1}^{1-\frac{n}{r}}},$$

for all $0 < t < \infty$ and all $\varphi \in \mathcal{S}$ with $C = C(n, r)$, from which, as in the same way as the above case, we obtain the desired estimate. ■

We next consider the maximal regularity theorem for the heat equation in the homogeneous Besov space. To this end, let us first consider the homogeneous heat equation.

Proposition 3.2 ([11, Lemma 2.1]) *Let $1 < p < \infty$, $1 < \alpha < \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. Assume that $1 \leq r \leq p$ satisfies*

$$\frac{n}{p} \leq \frac{n}{r} < \frac{2}{\alpha} + \frac{n}{p}.$$

For $a \in \dot{B}_{r,q}^k$ with $k = 2 + n/r - (2/\alpha + n/p - s)$, it holds that

$$\Delta e^{t\Delta} a \in L^{\alpha,q}(0, \infty; \dot{B}_{p,1}^s)$$

with the estimate

$$\|\|\Delta e^{t\Delta} a\|_{\dot{B}_{p,1}^s}\|_{L^{\alpha,q}(0,\infty)} \leq C \|a\|_{\dot{B}_{r,q}^k},$$

where $C = C(n, p, \alpha, s, q)$.

Proof. Since $n/r < 2/\alpha + n/p$, we have that $k < s + 2$. Hence taking $\theta \in (0, 1)$ and $k_0 < k < k_1 < s + 2$ so that $k = (1 - \theta)k_0 + \theta k_1$. By Proposition 3.1(iii) it holds that

$$\|\Delta e^{t\Delta} a\|_{\dot{B}_{p,1}^s} = \|e^{t\Delta} a\|_{\dot{B}_{p,1}^{s+2}} \leq C t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}(s+2-k_i)} \|a\|_{\dot{B}_{r,\infty}^{k_i}}$$

for $i = 0, 1$, and hence we see that the mapping

$$a \in \dot{B}_{r,\infty}^{k_i} \mapsto \|\Delta e^{t\Delta} a\|_{\dot{B}_{p,1}^s} \in L^{\alpha_i, \infty}(0, \infty)$$

is a bounded sub-additive operator for

$$\frac{1}{\alpha_i} = \frac{n}{2} \left(\frac{1}{r} - \frac{1}{p} \right) + \frac{1}{2}(s + 2 - k_i), \quad i = 0, 1.$$

Then it follows from the real interpolation theorem that

$$a \in (\dot{B}_{r,\infty}^{k_0}, \dot{B}_{r,\infty}^{k_1})_{\theta,q} \rightarrow \|\Delta e^{t\Delta} a\|_{\dot{B}_{p,1}^s} \in (L^{\alpha_0, \infty}(0, \infty), L^{\alpha_1, \infty}(0, \infty))_{\theta,q}.$$

Since $(\dot{B}_{r,\infty}^{k_0}, \dot{B}_{r,\infty}^{k_1})_{\theta,q} = \dot{B}_{r,q}^k$ and since $(L^{\alpha_0, \infty}(0, \infty), L^{\alpha_1, \infty}(0, \infty))_{\theta,q} = L^{\alpha,q}(0, \infty)$, implied by

$$\begin{aligned} \frac{1}{\alpha} &= \frac{1-\theta}{\alpha_0} + \frac{\theta}{\alpha_1} = (1-\theta) \left(\frac{n}{2} \left(\frac{1}{r} - \frac{1}{p} \right) + \frac{1}{2}(s + 2 - k_0) \right) + \theta \left(\frac{n}{2} \left(\frac{1}{r} - \frac{1}{p} \right) + \frac{1}{2}(s + 2 - k_1) \right) \\ &= \frac{n}{2} \left(\frac{1}{r} - \frac{1}{p} \right) + \frac{1}{2}(s + 2 - (1-\theta)k_0 - \theta k_1) \\ &= \frac{n}{2} \left(\frac{1}{r} - \frac{1}{p} \right) + \frac{1}{2}(s + 2 - k), \end{aligned}$$

we conclude that the mapping

$$a \in \dot{B}_{r,q}^k \rightarrow \|\Delta e^{t\Delta} a\|_{\dot{B}_{p,1}^s} \in L^{\alpha,q}(0, \infty)$$

is a bounded sub-additive operator, which yields the desired result. This proves Proposition 3.2. ■

Now, we are in a position to state the maximal regularity theorem for the Stokes equations.

Theorem 3.2 ([11, Theorem 1]) *Let $1 < p < \infty$, $1 < \alpha < \infty$, $1 \leq \beta \leq \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. Assume that $1 \leq r \leq \infty$ satisfies*

$$\frac{n}{p} \leq \frac{n}{r} < \frac{2}{\alpha} + \frac{n}{p}. \tag{3.9}$$

For every $a \in \dot{B}_{r,q}^k$ with $k = 2 + n/r - (2/\alpha + n/p - s)$ and every $f \in L^{\alpha,q}(0, T; \dot{B}_{p,\beta}^s)$ with $0 < T \leq \infty$, there exists a unique solution u of

$$(S) \quad \begin{cases} \frac{du}{dt} - \Delta u = Pf & \text{a.e. } t \in (0, T) \text{ in } \dot{B}_{p,\beta}^s, \\ u(0) = a & \text{in } \dot{B}_{r,q}^k \end{cases}$$

in the class

$$u_t, \Delta u \in L^{\alpha,q}(0, T; \dot{B}_{p,\beta}^s).$$

Moreover, such a solution u is subject to the estimate

$$\|u_t\|_{L^{\alpha,q}(0,T;\dot{B}_{p,\beta}^s)} + \|\Delta u\|_{L^{\alpha,q}(0,T;\dot{B}_{p,\beta}^s)} \leq C(\|a\|_{\dot{B}_{r,q}^k} + \|f\|_{L^{\alpha,q}(0,T;\dot{B}_{p,\beta}^s)}), \tag{3.10}$$

where $C = C(n, p, \alpha, q, \beta, s, r)$ is a constant independent of $0 < T \leq \infty$.

Proof. Step 1. Let us first prove in case $a = 0$. By the usual maximal regularity theorem in \dot{H}_p^s for $s_0 < s < s_1 \leq k + 2$, for every $f \in L^\alpha(0, T; \dot{H}_p^{s_i})$ ($i = 0, 1$) with $0 < T \leq \infty$, there exists a unique solution u of (S) in the class

$$u_t, -\Delta u \in L^\alpha(0, T; \dot{H}_p^{s_i})$$

with the estimate

$$\|u_t\|_{L^\alpha(0,T;\dot{H}_p^{s_i})} + \|\Delta u\|_{L^\alpha(0,T;\dot{H}_p^{s_i})} \leq C\|f\|_{L^\alpha(0,T;\dot{H}_p^{s_i})}, \quad i = 0, 1,$$

where $C = C(n, p, \alpha, s_0, s_1)$ is independent of T . For the detail, see, e.g., Giga-Sohr [6, Theorem 2.1]. This implies that the mapping

$$S : f \in L^\alpha(0, T; \dot{H}_p^{s_i}) \rightarrow (u_t, -\Delta u) \in L^\alpha(0, T; \dot{H}_p^{s_i})^2, \quad i = 0, 1$$

is a bounded linear operator with its operator norm independent of T . Hence by the real interpolation, S extends a bounded operator from $L^\alpha(0, T; (\dot{H}_p^{s_0}, \dot{H}_p^{s_1})_{\theta,\beta})$ to $L^\alpha(0, T; (\dot{H}_p^{s_0}, \dot{H}_p^{s_1})_{\theta,\beta})^2$ for all $1 \leq \beta \leq \infty$.

Since $(\dot{H}_p^{s_0}, \dot{H}_p^{s_1})_{\theta,\beta} = \dot{B}_{p,\beta}^s$ with $s = (1 - \theta)s_0 + \theta s_1$, we see that

$$S : f \in L^\alpha(0, T; \dot{B}_{p,\beta}^s) \rightarrow (u_t, -\Delta u) \in L^\alpha(0, T; \dot{B}_{p,\beta}^s)^2$$

is a bounded operator with its operator norm independent of T . Taking $\alpha_0 < \alpha < \alpha_1$ and $0 < \theta < 1$ so that $1/\alpha = (1 - \theta)/\alpha_0 + \theta/\alpha_1$, we see that

$$\begin{aligned} S : f \in (L^{\alpha_0}(0, T; \dot{B}_{p,\beta}^s), L^{\alpha_1}(0, T; \dot{B}_{p,\beta}^s))_{\theta,q} \\ \rightarrow (u_t, -\Delta u) \in (L^{\alpha_0}(0, T; \dot{B}_{p,\beta}^s), L^{\alpha_1}(0, T; \dot{B}_{p,\beta}^s))_{\theta,q}^2 \end{aligned}$$

is a bounded operator with its operator norm independent of T . Since

$$(L^{\alpha_0}(0, T; \dot{B}_{p,\beta}^s), L^{\alpha_1}(0, T; \dot{B}_{p,\beta}^s))_{\theta,q} = L^{\alpha,q}(0, T; \dot{B}_{p,\beta}^s),$$

we obtain the desired result with the estimate (3.10) for $a = 0$.

Step 2. For $a \in \dot{B}_{r,q}^k$ and $f \in L^{\alpha,q}(0, T; \dot{B}_{p,\beta}^s)$, we see that

$$u(t) = e^{t\Delta}a + Sf(t), \quad 0 < t < T$$

solves (S). Since $\dot{B}_{p,1}^s \subset \dot{B}_{p,\beta}^s$, the desired result with the estimate (3.10) is a consequence of Proposition 3.2 and the argument of the above Step 1. This completes the proof of Theorem 3.2. ■

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