Asymptotic structure of steady flow around a two-dimensional rotating body

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1 Introduction and the main result

Consider the steady Navier-Stokes system in the frame attached to a rotating rigid body in 2D with constant angular velocity $a \in \mathbb{R} \setminus \{0\}$. By a simple transformation it is given by

$$-\Delta u - a(x^\perp \cdot \nabla u - u^\perp) + \nabla p + u \cdot \nabla u = f, \qquad \text{div } u = 0 \tag{1.1}$$

in Ω being an exterior domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, where $x^\perp = (-x_2, x_1)^\top$. For the linearized system

$$-\Delta u - a(x^\perp \cdot \nabla u - u^\perp) + \nabla p = f, \qquad \text{div } u = 0, \tag{1.2}$$

it was discovered first by Hishida [14] that the oscillation of the body leads to the resolution of the Stokes paradox and that the leading term of decaying solutions of (1.2) in Ω subject to

$$\int_{\partial\Omega} \nu \cdot u \, d\sigma = 0 \tag{1.3}$$

is given by

$$\left(\int_{\partial\Omega} y^\perp \cdot \{(T(u,p) + au \otimes y^\perp)\nu\} d\sigma + \int_\Omega y^\perp \cdot f \, dy \right) \frac{x^\perp}{4\pi |x|^2}$$

provided that the force $f(x)$ decays sufficiently fast, where $T(u,p) = \nabla u + (\nabla u)^\top - p\mathbb{I}$ is the Cauchy stress tensor and ν denotes the outer unit normal to $\partial\Omega$. This tells us that the rate of decay is controlled by the torque (not by the force). The case of general flux condition $\beta := \int_{\partial\Omega} \nu \cdot u \, d\sigma \neq 0$ can be easily reduced to the one mentioned above by subtracting the flux carrier $\beta\frac{-x}{2\pi|x|^2}$, which becomes the other part of the leading term. The proof of [14] relies upon a detailed analysis of the fundamental solution tensor associated with (1.2).

Later on, Higaki, Maekawa and Nakahara [12] established a nice estimate of the remainder of the asymptotic representation mentioned above with less singular behavior with respect to $|a|$ and applied it to the nonlinear problem (1.1). Roughly speaking, their theorem asserts that if $|a|$ is small and the decaying force $f(x)$ (of divergence form) is also small compared to some rate of $|a|$ (which is almost $|a|^{1/2}$), problem (1.1) in Ω subject to no-slip condition $u|_{\partial\Omega} = ax^\perp$ admits a unique solution $u(x)$ which possesses the same leading profile as above. Indeed, the pair

$$U(x) = \frac{cx^\perp}{|x|^2}, \qquad P(x) = \frac{-c^2}{2|x|^2} \qquad (c \in \mathbb{R}) \tag{1.4}$$

is a self-similar Navier-Stokes flow in $\mathbb{R}^2 \setminus \{0\}$ and it also solves (1.1) with $f = 0$ in $\mathbb{R}^2 \setminus \{0\}$ since $x^\perp \cdot \nabla U = U^\perp$. Thus the asymptotic structure of the solution constructed in [12] is reasonable because their solution is a scale-critical one so that nonlinearity is balanced with the linear part. Given solutions to (1.1) in Ω which decay like $O(|x|^{-1})$ without specifying a boundary condition except (1.3), it would be interesting to ask whether they exhibit the same asymptotic structure (no matter how they are constructed).

The first aim of this paper is to provide a different proof (considerably shorter proof) of the resolution of the Stokes paradox than the previous one [14]. The strategy is to go back to the time-periodic regime and to split the solution into two parts; one is the steady part, the other is the purely oscillatory one. This idea is developed in terms of a time-periodic fundamental solution introduced by Kyed [15]. Our procedure yields a useful estimate of our own for the linearized system (1.2) in the whole plane $\mathbb{R}^2$, see Theorem 2.1, when the torque of $f = f_0 + \text{div}\, F$ with $F = (F_{ij})$ vanishes, that is,

$$\int_{\mathbb{R}^2} y^\perp \cdot f_0 \, dy + \int_{\mathbb{R}^2} (F_{12} - F_{21}) \, dy = 0. \tag{1.5}$$

The point is that the leading term comes only from the steady part, while the singular behavior with respect to $|a|$ arises only from the purely oscillatory part.

By making use of estimate mentioned above (Theorem 2.1), the second aim is to give an affirmative answer (however, in the small) to the question raised above.

Theorem 1.1. *Let* $a \in \mathbb{R} \setminus \{0\}$. *Given* $\delta \in (0, 1/2)$ *and* $R > e$ *satisfying* $\mathbb{R}^2 \setminus \Omega \subset B_R$, *there are positive constants* $\gamma_1 = \gamma_1(\delta)$ *(independent of* R*) and* $\gamma_2 = \gamma_2(\delta, R)$ *such that the following holds: For every solution (smooth solution for simplicity)* $\{u, p\} \in H^1_{loc}(\overline{\Omega}) \times L^2_{loc}(\overline{\Omega})$ *to* (1.1) *in* Ω *with* $f \in L^2_{loc}(\overline{\Omega})$ *subject to* (1.3) *which satisfies*

$$\begin{cases} (1 + |a|^{-\delta/2}) \sup_{|x|\geq R} |x||u(x)| \leq \gamma_1, \\ (1 + |a|^{-(\delta+1/2)}) \sup_{|x|\geq R} |x|^{3+\delta}(\log|x|)|f(x)| \leq \gamma_1, \\ (|a| + |a|^{-(\delta+1/2)})(|M| + N_R) \leq \gamma_2, \end{cases} \tag{1.6}$$

we have the asymptotic representation

$$u(x) = M\frac{x^\perp}{4\pi|x|^2} + O(|x|^{-(1+\delta)}) \qquad as\ |x| \to \infty, \tag{1.7}$$

where

$$M := \int_{\partial\Omega} y^\perp \cdot \{(T(u,p) + au \otimes y^\perp - u \otimes u)\nu\}d\sigma + \int_\Omega y^\perp \cdot f\, dy \tag{1.8}$$

and

$$N_R := \|\{u, \nabla u, \nabla^2 u, p\}\|_{L^\infty(A_R)}, \quad A_R = \{x \in \mathbb{R}^2;\ R < |x| < 2R\}. \tag{1.9}$$

Note that the boundary integral in (1.8) is understood as $\langle y^\perp, (\cdots)\nu\rangle_{\partial\Omega}$ since $(\cdots)\nu \in H^{-1/2}(\partial\Omega) := H^{1/2}(\partial\Omega)^*$ by the normal trace theorem on account of the assumptions on the regularity of $\{u,p\}$ anf f up to $\partial\Omega$. Consider (1.1) subject to no-slip condition $u|_{\partial\Omega} = ax^\perp$, then $|M| + N_R$ as well as $\sup_{|x|\geq R}|x||u(x)|$ are controlled by $|a|$ and f. Since $\delta + 1/2 < 1$, (1.6) could be accomplished when $|a|$ and f are small enough.

In the next section we provide the linear theory for (1.2). Section 3 is devoted to the proof of Theorem 1.1.

2 Linear theory

In this section we develop the linear theory for the whole plane problem. We begin with introducing the function space

$$X_{\alpha,\beta}(\mathbb{R}^2) := \{f \in L^\infty(\mathbb{R}^2);\ [f]_{\alpha,\beta} < \infty\},$$

which is a Banach space endowed with norm

$$[f]_{\alpha,\beta} := \sup_{x\in\mathbb{R}^2}(1+|x|)^\alpha\big(\log(e+|x|)\big)^\beta|f(x)|,$$

where $\alpha > 0$ and $\beta \geq 0$. The spaces $X_{\alpha,\beta}(\mathbb{R}^2)^2$ and $X_{\alpha,\beta}(\mathbb{R}^2)^{2\times 2}$ of vector and tensor functions, respectively, are abbreviated to $X_{\alpha,\beta}$ for notational simplicity. The same abbreviation is also used for some other function spaces.

Theorem 2.1. *Let $a \in \mathbb{R}\setminus\{0\}$ and $\delta \in (0,1)$. Suppose that the external force is decomposed as $f = f_0 + \operatorname{div} F$ with $f_0 \in X_{3+\delta,1}$ and $F \in X_{2+\delta,0}$. If (1.5) is fulfilled, then problem (1.2) in the whole plane $\mathbb{R}^2$ admits a unique solution $u \in X_{1+\delta,0}$ (together with the associated pressure) subject to*

$$[u]_{1+\delta,0} \leq C_*\big\{(1+|a|^{-(1+\delta)/2})[f_0]_{3+\delta,1} + (1+|a|^{-\delta/2})[F]_{2+\delta,0}\big\} \tag{2.1}$$

with some constant $C_* = C_*(\delta) > 0$ independent of $a \in \mathbb{R}\setminus\{0\}$ and f.

Remark 2.1. *If in particular f_0 is compactly supported, the singular behavior $|a|^{-(1+\delta)/2}$ in (2.1) for $a \to 0$ has been deduced first by [12, Theorem 3.1 (i)]. For the external force of divergence form, the singular behavior $|a|^{-\delta/2}$ for $a \to 0$ is not explicitly found in [12, Theorem 3.1 (ii)], however, it is hidden there. Note that one cannot have the case $\delta = 0$.*

Let us give the proof of Theorem 2.1, at least its outline as well as the idea, however, without precise computations. First of all, the solution to (1.2) in $\mathbb{R}^2$ is unique within the class of tempered distributions up to additive (specified) polynomials and, therefore, within $X_{1+\delta,0}$ by [14, Lemma 5.3.5]. Since $f_0 \in X_{3+\delta,1} \subset L^q(\mathbb{R}^2)$ and $F \in X_{2+\delta,0} \subset L^q(\mathbb{R}^2)$ for every $q \in (1,\infty)$, the argument from [6] and [13] gives us a solution (it works in 2D as well, see also [9] and [10]). It is also reperesented as the volume potential of f in terms of the associated fundamental solution if f satisfies an appropriate condition, see [14, Proposition 5.3.2] and [12, Theorem 3.1].

Let $a > 0$ and set

$$Q_a(t) = \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix}.$$

By a simple transformation

$$v(y,t) = Q_a(t)u(Q_a(t)^\top y), \qquad q(y,t) = p(Q_a(t)^\top y),$$

$$g(y,t) = Q_a(t)f(Q_a(t)^\top y),$$

one pulls back from (1.2) in the frame attached to the body to

$$\partial_t v - \Delta_y v + \nabla_y q = g, \qquad \operatorname{div}_y v = 0$$

in $\mathbb{R}^2_y \times \mathbb{T}_{2\pi/a}$, where $\mathbb{T}_{\mathcal{T}} = \mathbb{R}/(\mathcal{T}\mathbb{Z})$ for $\mathcal{T} > 0$. Following the idea of Kyed [15] as well as Galdi [8], we split the time-periodic Stokes flow $v(y,t)$ into two parts:

$$v(y,t) = v_s(y) + v_{po}(y,t),$$

where

$$v_s(y) := \frac{1}{2\pi/a}\int_0^{2\pi/a} v(y,\tau)\,d\tau = \frac{1}{2\pi}\int_0^{2\pi} Q_1(\tau)u(Q_1(\tau)^\top y)\,d\tau$$

is the steady part and does not depend on a, while the other part $v_{po}(y,t)$ is called the purely oscillatory part since

$$\int_0^{2\pi/a} v_{po}(y,\tau)\,d\tau = 0. \tag{2.2}$$

Correspondingly to the splitting above, we have

$$u(x) = u_s(x) + u_{po}(x)$$

with

$$u_s(x) := Q_a(t)^\top v_s(Q_a(t)x) = v_s(x),$$

which depends on neither a nor t. Therefore, the dependence of $u(x)$ on a is determined only by the one of $u_{po}(x)$.

It is immediately seen that

$$-\Delta u_s + \nabla p_s = f_s, \qquad \operatorname{div} u_s = 0$$

in $\mathbb{R}^2$, where

$$p_s(x) = \frac{1}{2\pi}\int_0^{2\pi} p(Q_1(\tau)^\top x)\,d\tau, \qquad f_s(x) = \frac{1}{2\pi}\int_0^{2\pi} Q_1(\tau)f(Q_1(\tau)^\top x)\,d\tau.$$

With this particular form of f_s at hand, the Taylor expansion of the Stokes fundamental solution

$$E(x) = \frac{1}{4\pi}\left[\left(\log\frac{1}{|x|}\right)\mathbb{I} + \frac{x\otimes x}{|x|^2}\right]$$

implies that each component of $u_s(x) = (u_{s,1}(x), u_{s,2}(x))^\top$ is represented as

$$\begin{aligned} u_{s,l}(x) &= E_{lj}(x) \int_{\mathbb{R}^2} f_{s,j}\, dy + \partial_k E_{lj}(x) \int_{\mathbb{R}^2} (-y_k) f_{s,j}\, dy + \mathcal{R}_l(x) \\ &= \frac{(x^\perp)_l}{4\pi|x|^2} \int_{\mathbb{R}^2} y^\perp \cdot f\, dy + \mathcal{R}_l(x) \end{aligned}$$

in $\mathbb{R}^2 \setminus \{0\}$ as long as f decays sufficiently fast, where $\mathcal{R}_l(x)$ denotes the remainder term for $l = 1, 2$. Here, the summation is implicitly taken over all repeated indices. The resolution of the Stokes paradox follows from $\int f_s\, dy = 0$ since the purely oscillatory part decays even faster on account of (2.2) as is clarified later. Under the assumptions of Theorem 2.1, we have

$$u_s(x) = \frac{x^\perp}{4\pi|x|^2} \left(\int_{\mathbb{R}^2} y^\perp \cdot f_0\, dy + \int_{\mathbb{R}^2} (F_{12} - F_{21})\, dy \right) + \mathcal{R}(x) \tag{2.3}$$

with the remainder $\mathcal{R}(x) = (\mathcal{R}_1(x), \mathcal{R}_2(x))^\top$ enjoying

$$\sup_{|x|\geq 1} |x|^{1+\delta} |\mathcal{R}(x)| \leq C \left([f_0]_{3+\delta,1} + [F]_{2+\delta,0} \right), \tag{2.4}$$

so that (2.3) is actually the asymptotic representation of the steady part for $|x| \to \infty$.

On the other hand, it is easily seen that

$$\sup_{|x|<1} |u_s(x)| \leq C \left([f_0]_{3+\delta,1} + [F]_{2+\delta,0} \right). \tag{2.5}$$

Since (1.5) leads to $u_s(x) = \mathcal{R}(x)$, combining (2.4) with (2.5) yields

$$[u_s]_{1+\delta,0} \leq C \left([f_0]_{3+\delta,1} + [F]_{2+\delta,0} \right). \tag{2.6}$$

Let us recall that the constant $C > 0$ is independent of a since u_s itself does not depend on a.

By (2.2) the purely oscillatory part $v_{po}(y, t)$ can be represented as the Fourier series

$$v_{po}(y, t) = \sum_{k \in \mathbb{Z}\setminus\{0\}} V_{po}(y, k) e^{iakt} \tag{2.7}$$

with the coefficients

$$V_{po}(y, k) := \frac{1}{2\pi/a} \int_0^{2\pi/a} v_{po}(y, \tau)\, e^{-iak\tau}\, d\tau,$$

where $i = \sqrt{-1}$. Note that $V_{po}(y, k)$ may be regarded as the Stokes resolvent with resolvent parameter iak and possesses a fine decay property at spatial infinity. This latter thing can be justified as follows even after taking the summation over $k \in \mathbb{Z} \setminus \{0\}$ by use of (the purely oscillatory part of) the fundamental solution for time-periodic problems introduced by Kyed [15].

Given $\mathcal{T} > 0$, we set

$$\Gamma_{\mathcal{T}}^{\perp}(x,t) := \sum_{k\in\mathbb{Z}\setminus\{0\}} G\left(x, i\frac{2\pi}{\mathcal{T}}k\right) e^{i\frac{2\pi}{\mathcal{T}}kt}, \tag{2.8}$$

where

$$G(x,\lambda) := \mathcal{F}_{\mathbb{R}^2}^{-1}\left[\frac{\mathbb{I} - \frac{\xi\otimes\xi}{|\xi|^2}}{\lambda + |\xi|^2}\right](x)$$

is the fundamental solution of the Stokes resolvent with resolvent parameter $\lambda \in \mathbb{C}\setminus(-\infty, 0]$. Here and in what follows, $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. Then (2.7) is rewritten as

$$v_{po}(y,t) = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}}\int_{\mathbb{R}^2} \Gamma_{\mathcal{T}}^{\perp}(y-z, t-s)g(z,s)\,dz\,ds \tag{2.9}$$

with $\mathcal{T} = 2\pi/a$. Several fine decay properties of $\Gamma_{\mathcal{T}}^{\perp}(x,t)$ for $|x| \to \infty$ due to (2.2) have been studied in [15] and [5], however, there are two important issues to be developed here.

First, estimates of $\Gamma_{\mathcal{T}}^{\perp}(x,t)$ with faster decay rate involve more growing rate for $\mathcal{T}(= 2\pi/a) \to \infty$, that is, more singular behavior for $a \to 0$ as the price. A point of our analysis is to deduce the singular behavior in (2.1) with respect to the angular velocity as less as possible. Given $\delta \in (0,1)$, we intend to find a reasonable singular behavior to get the decay of $v_{po}(y,t)$ like $O(|y|^{-(1+\delta)})$ uniformly in t. To this end, the scaling property

$$\Gamma_{\mathcal{T}}^{\perp}(x,t) = \Gamma_1^{\perp}\left(\frac{x}{\sqrt{\mathcal{T}}}, \frac{t}{\mathcal{T}}\right) \tag{2.10}$$

plays a key role.

The second issue is the singular behavior of $\Gamma_{\mathcal{T}}^{\perp}(x,t)$ for $x \to 0$, which has not been studied in [15], [5]. Since we are concerned with 2D problem, it should be $O(\log|x|^{-1})$ (uniformly in t), otherwise, $\Gamma_{\mathcal{T}}^{\perp}(x,t)$ cannot be the purely oscillatory part of the fundamental solution for the time-periodic problem. For later use, it is convenient to adopt the following estimate: For every $\mu \in (0,1)$ and $q \in (1, 1/(1-\mu))$, there is a constant $C = C(\mu,q) > 0$ such that

$$\|\Gamma_1^{\perp}(x,\cdot)\|_{L^q(\mathbb{T}_1)} \le C|x|^{-2\mu}, \qquad \forall x \in \mathbb{R}^2 \setminus \{0\}. \tag{2.11}$$

Since this provides us simultaneously with both estimates at large distance and around the origin, the singular behavior $(x \to 0)$ as well as the decay rate ($|x| \to \infty$) is no longer optimal, nevertheless, (2.11) is useful for our purpose. In fact, once we have that, we are able to deduce (2.1) as follows.

By (2.9) and (2.10) we have

$$v_{po}(y,t) = \int_0^1 \int_{\mathbb{R}^2} \Gamma_1^{\perp}\left(\frac{y-z}{\sqrt{\mathcal{T}}}, \frac{t}{\mathcal{T}} - s\right) g(z, \mathcal{T}s)\, dz\, ds,$$

to which one applies (2.11) to obtain

$$\begin{aligned} |v_{po}(y,t)| &\le [f]_{2+\delta,0} \int_{\mathbb{R}^2} \left\| \Gamma_1^{\perp}\left(\frac{y-z}{\sqrt{\mathcal{T}}}, \cdot\right)\right\|_{L^q(\mathbb{T}_1)} (1+|z|)^{-(2+\delta)}\, dz \\ &\le C\mathcal{T}^{\mu}[f]_{2+\delta,0} \int_{\mathbb{R}^2} |y-z|^{-2\mu}(1+|z|)^{-(2+\delta)}\, dz \end{aligned}$$

with $\mathcal{T} = 2\pi/a$ provided that $(1-\mu)q < 1$ and that $f \in X_{2+\delta,0}$; here, we are discussing the case $f = f_0$, $F = 0$ and the assumption $f \in X_{3+\delta,1}$ in Theorem 2.1 is too much although it is needed for the steady part, see (2.4). Given $\delta \in (0,1)$, we choose $\mu = (1+\delta)/2$ and fix $q \in (1, 2/(1-\delta))$ in the estimate above to find that

$$|v_{po}(y,t)| \le Ca^{-(1+\delta)/2}[f]_{2+\delta,0}(1+|y|)^{-(1+\delta)}$$

for all $y \in \mathbb{R}^2$ and $t \in \mathbb{T}_{2\pi/a}$, where the constant $C > 0$ is independent of (y,t). Since $u_{po}(x) = Q_a(t)^{\top} v_{po}(Q_a(t)x, t)$, we obtain

$$[u_{po}]_{1+\delta,0} \le Ca^{-(1+\delta)/2}[f]_{2+\delta,0},$$

which combined with (2.6) implies (2.1) when $f = f_0$, $F = 0$. The other case $f = \operatorname{div} F$, $f_0 = 0$ is discussed similarly by using

$$\|\nabla \Gamma_1^{\perp}(x,\cdot)\|_{L^q(\mathbb{T}_1)} \le C|x|^{-(1+2\mu)}, \qquad \forall x \in \mathbb{R}^2 \setminus \{0\},$$

with some constant $C = C(\mu, q) > 0$, instead of (2.11), where $\mu \in (0,1)$ and $q \in (1, 1/(1-\mu))$ are arbitrary and play the same role as above.

It remains to show (2.11). To this end, given $\mu \in (0,1)$, it is convenient to rewrite (2.8) with $\mathcal{T} = 1$ as

$$\begin{aligned} \Gamma_1^{\perp}(x,t) &= \mathcal{F}_{\mathbb{T}_1}^{-1}[(1-\delta_{\mathbb{Z}}(k))G(x, i(2\pi)k)](t) \\ &= \mathcal{F}_{\mathbb{T}_1}^{-1}\left[(1-\delta_{\mathbb{Z}}(k))|k|^{\mu} G(x, i(2\pi)k)\mathcal{F}_{\mathbb{T}_1}[h_\mu]\right](t) \end{aligned} \tag{2.12}$$

with

$$\delta_{\mathbb{Z}}(k) = \begin{cases} 0, & k \in \mathbb{Z}\setminus\{0\}, \\ 1, & k = 0, \end{cases} \qquad h_\mu(t) := \mathcal{F}_{\mathbb{T}_1}^{-1}[(1-\delta_{\mathbb{Z}}(k))|k|^{-\mu}](t).$$

Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function with $\chi(\eta) = 0$ for $|\eta| \le 1/2$ and $\chi(\eta) = 1$ for $|\eta| \ge 1$. Set

$$\phi_\mu(t) := \mathcal{F}_{\mathbb{R}}^{-1}[\chi(\eta)|\eta|^{-\mu}](t).$$

The the function $(1 - \delta_{\mathbb{Z}}(k))|k|^{-\mu}$ may be regarded as the restriction of the Fourier transform $\widehat{\phi_\mu}$ on $\mathbb{Z}$. Since $\phi_\mu(t)$ decays rapidly as $|t| \to \infty$ and since

$$\phi_\mu(t) = c_0|t|^{-1+\mu} + \psi_\mu(t)$$

with some smooth function $\psi_\mu(t)$ on $\mathbb{R}$ and a definite constant $c_0 > 0$, we find

$$h_\mu \in L^q(\mathbb{T}_1), \qquad \forall q \in \big(1, \frac{1}{1-\mu}\big), \tag{2.13}$$

by the Poisson summation formula, see [11, Example 3.1.19]. Let us also regard the symbol in (2.12) as the restriction of

$$m_x(\eta) := \chi(\eta)|\eta|^\mu G(x, i(2\pi)\eta), \qquad \eta \in \mathbb{R},$$

on $\mathbb{Z}$. The fundamental solution $G(x, i(2\pi)\eta)$ of the Stokes resolvent in 2D can be explicitly described in terms of the modified Bessel functions of the second kind (order 0/order 1), see Borchers and Varnhorn [4]. One can thus use the asymptotic behavior of those special functions, see for instance [1], to deduce

$$|m_x(\eta)| + |\eta||\partial_\eta m_x(\eta)| \le C|x|^{-2\mu}$$

for all $\eta \in \mathbb{R}$ and $x \in \mathbb{R}^2 \setminus \{0\}$, where the constant $C > 0$ is independent of η, x. This implies that $m_x(\eta)$ is a Fourier multiplier on $L^q(\mathbb{R})$ for every $q \in (1, \infty)$. By the transference principle ([11, Section 3.6.2]) we conclude that $m_x(k)$ is a Fourier multiplier on $L^q(\mathbb{T}_1)$, too, for every $q \in (1, \infty)$ with operator norm bounded by $|x|^{-2\mu}$. This together with (2.13) leads us to

$$\|\Gamma_1^\perp(x, \cdot)\|_{L^q(\mathbb{T}_1)} \le C|x|^{-2\mu}\|h_\mu\|_{L^q(\mathbb{T}_1)},$$

yielding (2.11) as long as $1 < q < 1/(1-\mu)$.

3 Proof of Theorem 1.1

Let us fix $\phi \in C^\infty([0, \infty))$ such that $\phi(\rho) = 1$ for $0 \le \rho \le 4/3$ and $\phi(\rho) = 0$ for $\rho \ge 5/3$. Given $R \in (e, \infty)$ satisfying $\mathbb{R}^2 \setminus \Omega \subset B_R$, we set $\varphi_R(x) = \phi(|x|/R)$ for $x \in \mathbb{R}^2$ and denote by $\mathbb{B}_{A_R}$ the Bogovskii operator which gives us a particular solution constructed by Bogovskii [2], see also [3] and [7], to the boundary value problem for the divergence equation in the annulus A_R, see (1.9), subject to the homogeneous Dirichlet boundary condition.

Given a solution $\{u, p\}$ to (1.1) with (1.3), which decays like $u(x) = O(|x|^{-1})$ as $|x| \to \infty$, we set

$$\widetilde{u} = (1-\varphi)u + \mathbb{B}[u \cdot \nabla\varphi], \qquad \widetilde{p} = (1-\varphi)p,$$
$$\widetilde{U} = (1-\varphi)U + \mathbb{B}[U \cdot \nabla\varphi], \qquad \widetilde{P} = (1-\varphi)P,$$

where $\{U, P\}$ is the candidate of the leading term given by (1.4) with $c = M/(4\pi)$ and the constant M is defined by (1.8). Here and in what follows, we abbreviate $\varphi = \varphi_R$, $A = A_R$ and $\mathbb{B} = \mathbb{B}_{A_R}$, respectively. Note that $\int_A u \cdot \nabla\varphi\, dx = 0$ follows from (1.3), while $\int_A U \cdot \nabla\varphi\, dx = \int_{|x|=R} \frac{-x}{R} \cdot U\, d\sigma = 0$. By some estimates of the Bogovskii operator (see [2], [3] and [7], in particular, dilation invariance of the constant in the L^q-estimate is needed here), we have $\widetilde{u}, \widetilde{U} \in X_{1,0}$ with

$$[\widetilde{u}]_{1,0} \leq C \sup_{|x| \geq R} |x||u(x)|, \qquad [\widetilde{U}]_{1,0} \leq C|M|, \tag{3.1}$$

where $C > 0$ is a constant independent of R.

The pair of

$$v := \widetilde{u} - \widetilde{U}, \qquad \psi := \widetilde{p} - \widetilde{P}$$

obeys

$$-\Delta v - a(x^\perp \cdot \nabla v - v^\perp) + \nabla\psi = (1-\varphi)f + g + \operatorname{div} J(v), \quad \operatorname{div} v = 0 \tag{3.2}$$

in $\mathbb{R}^2$, where

$$J(v) = -(\widetilde{u} \otimes v + v \otimes \widetilde{u}) + v \otimes v$$

and

$$g = h(u, p) - h(U, P)$$

with

$$\begin{aligned} h(u,p) = {} & 2\nabla\varphi \cdot \nabla u + (\Delta\varphi + ax^\perp \cdot \nabla\varphi)u - \Delta\mathbb{B}[u \cdot \nabla\varphi] - ax^\perp \cdot \nabla\mathbb{B}[u \cdot \nabla\varphi] \\ & + a\mathbb{B}[u \cdot \nabla\varphi]^\perp - (\nabla\varphi)p + (1-\varphi)u \cdot \nabla\{-\varphi u + \mathbb{B}[u \cdot \nabla\varphi]\} \\ & + \mathbb{B}[u \cdot \nabla\varphi] \cdot \nabla\{(1-\varphi)u + \mathbb{B}[u \cdot \nabla\varphi]\}. \end{aligned}$$

It is seen that $g \in C_0^\infty(A)$ and

$$\sup_{x \in A} |g(x)| \leq c(R)(1 + |a|)(|M| + N) \tag{3.3}$$

with some constant $c(R) > 0$ which depends on R but is independent of a, where $N = N_R$ is given by (1.9) and N as well as $|M|$ is assumed to be smaller than 1 so that $M^2 \leq |M|$, $N^2 \leq N$.

By essentially the same computation as in [14, Section 5.4] we deduce from (1.8) that

$$\int_{\mathbb{R}^2} y^\perp \cdot \{(1-\varphi)f + g\}\, dy = 0,$$

which together with symmetry $J_{12}(v) = J_{21}(v)$ enables us to reconstruct a solution $V \in X_{1+\delta,0}$ subject to

$$[V]_{1+\delta,0} \le L := 2C_*(1+|a|^{-(1+\delta)/2})[(1-\varphi)f + g]_{3+\delta,1} \tag{3.4}$$

(together with the associated pressure Ψ) under the smallness conditions (1.6) by means of the fixed point argument based on Theorem 2.1, where $C_* = C_*(\delta)$ is as in this theorem. In fact,

$$\begin{aligned} L \le\ & C(1+|a|^{-(1+\delta)/2}) \sup_{|x|\ge R} |x|^{3+\delta}(\log|x|)|f(x)| \\ & + Cc(R)R^{3+\delta}(\log R)(1+|a|^{-(1+\delta)/2})(1+|a|)(|M|+N) \end{aligned} \tag{3.5}$$

follows from (3.3) and, thereby, the conditions (1.6) with appropriate constants $\gamma_1 = \gamma_1(\delta)$, $\gamma_2 = \gamma_2(\delta, R)$ imply that $(1+|a|^{-\delta/2})L$ is sufficiently small.

Let us identify V reconstructed above with $v = \widetilde{u} - \widetilde{U}$. We set

$$w := v - V, \qquad \sigma := \psi - \Psi,$$

which obey

$$-\Delta w - a(x^\perp \cdot \nabla w - w^\perp) + \nabla\sigma = \mathrm{div}\ K(w), \qquad \mathrm{div}\ w = 0$$

in $\mathbb{R}^2$ with

$$K(w) = -(\widetilde{u} \otimes w + w \otimes \widetilde{u}) + v \otimes w + w \otimes V.$$

Since the case $\delta = 0$ is not available in Theorem 2.1, we rely on the L^q-theory; indeed, $K(w) \in L^q(\mathbb{R}^2)$ for every $q \in (1,\infty)$. Let us fix $q \in (1,2)$, then the a priori estimate obtained in [13] and [10] (where 3D case is discussed, but the argument is similar for 2D) together with the embedding relation implies that

$$\begin{aligned} \|w\|_{q_*,q} &\le C\|\nabla w\|_q \le C\|K(w)\|_q \\ &\le C\big(\|\widetilde{u}\|_{2,\infty} + \|v\|_{2,\infty} + \|V\|_{2,\infty}\big)\|w\|_{q_*,q} \\ &\le C\big([\widetilde{u}]_{1,0} + [\widetilde{U}]_{1,0} + [V]_{1+\delta,0}\big)\|w\|_{q_*,q} \end{aligned}$$

where $\|\cdot\|_{q_*,q}$ with $q_* = 2q/(2-q)$ and $\|\cdot\|_{2,\infty}$ denote the norms of the Lorentz spaces $L^{q_*,q}(\mathbb{R}^2)$ and $L^{2,\infty}(\mathbb{R}^2)$, respectively, and the Lorentz-Hölder inequality is employed. The constant C is independent of a because it turns out from a simple scaling argument that the constant in the L^q-estimate for (1.2) in $\mathbb{R}^2$ does not depend on a. We thus conclude that $v = V$, yielding (1.7), whenever $[\widetilde{u}]_{1,0} + [\widetilde{U}]_{1,0} + [V]_{1+\delta,0}$ is small enough. This latter condition can be accomplished by (1.6) (with still smaller γ_1, γ_2) on account of (3.1), (3.4) and (3.5).

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